Symmetric (31, 10, 3) Designs with Trivial Automorphism Group

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Abstract. Sixteen non-isomorphic symmetric 2-(31, 10, 3) designs with trivial full automorphism group are constructed.

Introduction

We assume familiarity with the basic notions and facts from design and coding theory (cf., e.g. [1], [2], [3], [4], [7], [13]).

The parameters 2-(31, 10, 3) are the smallest for a symmetric 2-design of non-Hadamard type for which complete enumeration has not been finished yet. One source for interest in such designs is that any 2-(31, 10, 3) design without ovals (an oval being a set of 4 points meeting each block in at most 2 points) gives rise to an extremal binary doubly-even (64, 32, 12) code [11], [12]. In [11] we showed that the only primes which can be orders of automorphisms of a 2-(31, 10, 3) design are 7, 3 and 2, and constructed all four non-isomorphic designs possessing automorphisms of order 7. One of these four designs was without ovals, thus yielding an extremal (64, 32, 12) code. Subsequently, Mathon [8] undertook a search for all designs with automorphisms of orders 3 and 2, finding altogether 38 designs, 6 among them without ovals. The six designs without ovals produce four inequivalent extremal codes (since a pair of dual designs generate a pair of equivalent codes) [5]. More recently Spence [9] found six more 2-(31, 10, 3) designs with an automorphism of order 3 missed by Mathon. Each of these six new designs does possess ovals.

In this note we summarize the results of a random search for designs with trivial automorphism group. Over a dozen of such designs have been found (eight non self-dual designs are listed in Table 2), all of them possessing ovals.

The designs

The construction method is by embedding of a given 2-(10, 3, 2) design as a derived design into a symmetric 2-(31, 10, 3) design. The search can be speeded up by using the approach from [10] exploring the binary subcode of the ternary code having as a parity check matrix the incidence matrix of the initial 2-(10, 3, 2) design. Note that not every 2-(10, 3, 2) design is embeddable into a symmetric 2-(31, 10, 3) design [6].

Non-isomorphic designs have been distinguished by an easily computable isomorphism invariant. A *spread* in a 2-(10, 3, 2) design is a set of three pairwise disjoint blocks. Given a symmetric 2-(31, 10, 3) design, we compute the number n_i of derived 2-(10, 3, 2) designs having precisely i spreads. For the designs we have encountered the number of spreads vary from 30 to 42, and the characteristics $(n_{30}, \ldots, n_{42})$ provide a very handy isomorphism invariant. We have used as starting 2-(10, 3, 2) designs some derived designs of the known 2-(31, 10, 3) designs without ovals. Although a number of new designs were found, in all cases the only symmetric designs without ovals thus obtained were the ones we have started with. The characteristics $(n_{30}, \ldots, n_{42})$ together with the number of ovals are listed in Table 1, where the dual design of a design No. i is denoted by i'. Combined with the knowledge of the possible orbit lengths for a non-trivial automorphism and comparing the number of ovals [8], [11], the data from Table 1 provides immediate evidence for the triviality of the group of the designs. The eight non self-dual designs are listed in Table 2.

Table 1. The characteristics $(n_{30}, \ldots, n_{42})$ and the number of ovals $\# O$

D	n_{30}	n_{32}	n_{34}	n_{36}	n_{38}	n_{40}	n_{42}	$\# O$
1	1	2	7	12	7	2	0	7
1′	0	3	9	10	6	3	0	7
2	0	0	4	14	9	3	1	10
2′	0	2	5	7	11	6	0	10
3	0	2	11	4	10	3	1	12
3′	0	3	7	11	4	5	1	12
4	0	2	8	5	9	6	1	4
4′	0	4	4	7	8	8	0	4
5	0	1	4	8	11	6	1	5
5′	0	2	1	11	9	8	0	5
6	0	2	5	7	11	6	0	6
6′	0	3	4	9	7	7	1	6
7	0	1	10	9	8	3	0	9
7′	0	3	6	10	10	2	0	9
8	0	3	7	6	9	5	1	7
8′	0	2	7	7	10	5	0	7

Table 2. The 16 rigid 2-(31, 10, 3) designs

1

1: 7 8 15 16 23 24 25 26 30 31
2: 1 2 11 12 19 20 23 29 30 31
3: 4 8 12 13 17 22 24 27 29 31
4: 1 4 14 15 20 21 25 27 28 31
5: 2 3 5 8 10 13 14 21 23 31

2

1: 7 8 15 16 23 24 25 26 30 31
2: 1 2 11 12 19 20 23 29 30 31
3: 4 8 12 13 17 22 24 27 29 31
4: 1 4 14 15 20 21 25 27 28 31
5: 2 3 5 8 10 13 14 21 23 31

6:	3	6	9	14	16	17	19	20	24	31
7:	5	7	9	12	18	19	21	22	25	31
8:	2	3	4	6	11	15	18	22	26	31
9:	1	7	10	11	13	16	17	18	28	31
10:	5	6	9	10	26	27	28	29	30	31
11:	1	4	5	8	9	11	12	14	16	26
12:	1	6	8	17	18	19	21	23	26	27
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23:	6	8	10	11	16	20	21	22	25	29
24:	1	6	7	9	13	14	15	22	23	29
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26:	2	4	9	16	17	21	22	23	28	30
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31:	4	8	9	10	13	15	18	19	20	30

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8:	2	3	4	6	11	15	18	22	26	31
9:	1	7	10	11	13	16	17	18	28	31
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30:	2	9	10	13	15	17	20	22	25	30
31:	1	4	5	6	7	14	17	22	23	30

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31:	1	6	7	9	11	14	20	21	22	30

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Note Added in Proof

After this manuscript had been accepted for publication the author was informed that E. Spence has undertaken a search for all non-isomorphic 2-(31,10,3) design (E. Spence, *A complete classification of symmetric (31,10,3) designs* (preprint)).

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