The 2-Packing Number of Complete Grid Graphs

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Abstract. Hare and Hare conjectured the 2-packing number of an $m \times n$ grid graph $\lceil mn/5 \rceil$ for $m, n \geq 9$. This is verified by finding the 2-packing number for grid graphs of all sizes.

The paper completely answers the question: *What is the maximum number of checkers that can be placed on an $m \times n$ checker board so there are at least two squares between each pair of checkers?* These configurations are called *2-packings* (see Figure 1) and the maximum number is the *2-packing number* of an $m \times n$ complete grid graph denoted by $a_{m,n}$.

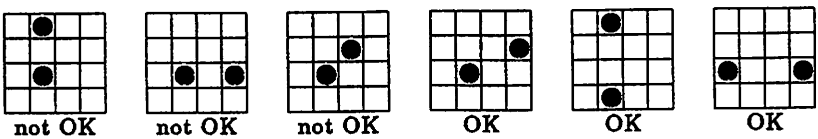

Figure 1 – Legal 2-Packings. The checkers in a 2-packing must be separated by at least two squares (horizontally or vertically).

Hare and Hare [4] developed an efficient algorithm for finding $a_{m,n}$. Based on their computer studies, they conjectured $a_{m,n} = \lceil mn/5 \rceil$ when $m, n \geq 8$, except $(m,n) = (8,10)$ and $(m,n) = (10,8)$. Here this conjecture is verified. Theorems 1 through 8 show that:

$$a_{m,n} = \begin{cases} \lceil (m+1)n/6 \rceil & \text{if } n \geq m \in \{1,2,3\} \\ \lceil 6n/7 \rceil & \text{if } n \geq m = 4 \text{ and } n \bmod 7 \neq 1 \\ \lceil 6n/7 \rceil + 1 & \text{if } n \geq m = 4 \text{ and } n \bmod 7 = 1 \\ 10 & \text{if } (m,n) = (7,7) \\ \lceil (mn+2)/5 \rceil & \text{if } n \geq m \in \{5,6,7\} \text{ and } (m,n) \neq (7,7) \\ 17 & \text{if } (m,n) = (8,10) \\ \lceil mn/5 \rceil & \text{if } n \geq m \geq 8 \text{ and } (m,n) \neq (8,10) \\ a_{n,m} & \text{if } m > n \end{cases} \quad (1)$$

1. Preliminary Calculations

Lemma 1 and 2 give lower and upper bounds for $a_{m,n}$.

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Lemma 1. *For all $m > 0$ and $n > 0$, $a_{m,n} \geq \lceil mn/5 \rceil$.*

Proof. Figure 2 partitions the squares of an $m \times n$ board into five 2-packings. Since there are mn squares, at least one set has $\lceil mn/5 \rceil$ checkers. □

Lemma 2. *For all k, m and n with $0 < k < n$ and $m > 0$,*

$$a_{m,n} \leq a_{m,n-k} + a_{m,k}.$$

Proof. A maximal 2-packing of $m \times n$ board has at most $a_{m,n-k}$ checkers in the first $n-k$ columns, and at most $a_{m,k}$ checkers in the last k columns. The result then follows. □

a	b	c	d	e	a	b	c	d	e	a	b	c
c	d	e	a	b	c	d	e	a	b	c	d	e
e	a	b	c	d	e	a	b	c	d	e	a	b
b	c	d	e	a	b	c	d	e	a	b	c	d
d	e	a	b	c	d	e	a	b	c	d	e	a
a	b	c	d	e	a	b	c	d	e	a	b	c

Figure 2– Five 2-packings of $m \times n$ board. The squares of an $m \times n$ board can be partitioned into 5 2-packings. The pattern in each row repeats every 5 squares, and each row is labelled the same as the row 5 above it. Since there are mn squares, at least one 2-packing contains $\lceil mn/5 \rceil$ checkers.

We also needed $a_{m,n}$ for 390 cases. While most are proved in Hare and Hare [4], proving the rest "by hand" would have been tedious. Instead, a computer program was used that implemented a branch and bound algorithm which bounded with Lemma 2. While not as sophisticated as the algorithm in Hare and Hare, it was quite adequate for the purposes here: finding these values in 101 cpu-seconds on a VAX 8800. Lemma 3 summarizes these results.

Lemma 3: *Equation (1) holds when $m, n \leq 18$, and when $m \leq 8$ and $n \leq 25$.*

2. The Cases where $m = 1, 2, 3, 4$

Finding $a_{1,n}$, $a_{2,n}$, $a_{3,n}$ and $a_{4,n}$ are direct applications of Lemma 2. Theorem 2 is from Hare and Hare [4], though the proof is quite different.

Theorem 1. *For all $n > 0$, $a_{1,n} = \lceil n/3 \rceil$.*

Proof. Figure 3 shows that $a_{1,n} \geq \lceil n/3 \rceil$. Lemma 3 gives the result when $n \leq 3$. For $n > 3$, assume the result holds for $n - 3$. Then by Lemma 2,

$$a_{1,n} \leq a_{1,n-3} + a_{1,3} = \lceil (n-3)/3 \rceil + 1 = \lceil n/3 \rceil.$$

The result follows by induction. □

Figure 3 – Maximal 2-packings of $1 \times n$ Board. The first n columns (the pattern repeats every 3 columns) show that $a_{1,n} \geq \lceil n/3 \rceil$. Theorem 1 shows this is maximal.

Theorem 2. *For all $n > 0$, $a_{2,n} = \lceil n/2 \rceil$.*

Proof. Figure 4 shows that $a_{2,n} \geq \lceil n/2 \rceil$. Lemma 3 shows the result holds for $n \leq 2$. For $n > 2$, assume the result holds for $n - 2$. Then by Lemma 2,

$$a_{2,n} \leq a_{2,n-2} + a_{2,2} = \lceil (n-2)/2 \rceil + 1 = \lceil n/2 \rceil.$$

The result follows by induction. □

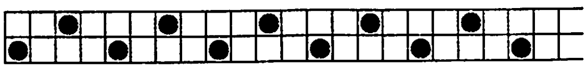

Figure 4 – Maximal 2-packings of $2 \times n$ Board. The first n columns (the pattern repeats every 4 columns) show that $a_{2,n} \geq \lceil n/2 \rceil$. Theorem 2 shows this is maximal.

Theorem 3. *For all $n > 0$, $a_{3,n} = \lceil 2n/3 \rceil$.*

Proof. Figure 5 shows that $a_{3,n} \geq \lceil 2n/3 \rceil$. Lemma 3 shows the result holds for $n \leq 3$. For $n > 3$, assume the result holds for $n - 3$. Then by Lemma 2,

$$a_{3,n} \leq a_{3,n-3} + a_{3,3} = \lceil 2(n-3)/3 \rceil + 2 = \lceil 2n/3 \rceil.$$

The result follows by induction. □

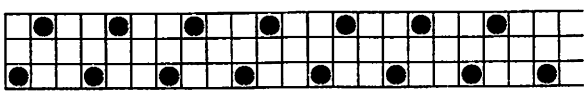

Figure 5 – Maximal 2-packings of $3 \times n$ Board. The first n columns (the pattern repeats every 3 columns) show that $a_{3,n} \geq \lceil 2n/3 \rceil$. Theorem 3 shows this is maximal.

Theorem 4. *For all $n > 0$,*

$$a_{4,n} = \begin{cases} \lceil 6n/7 \rceil + 1 & \text{if } n \bmod 7 = 1 \\ \lceil 6n/7 \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $f(n)$ be the right side of the equation in Theorem 4. Figure 6 shows that $a_{4,n} \geq f(n)$. Lemma 3 gives the result when $n \leq 7$. For $n > 7$, assume the result holds for $n - 7$. Then by Lemma 2,

$$a_{4,n} \leq a_{4,n-7} + a_{4,7} = f(n-7) + 6 = f(n).$$

The result follows by induction. □

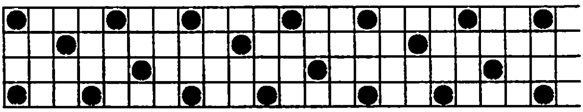

Figure 6 – Maximal 2-packings of $4 \times n$ Board. The first n columns (the pattern repeats every 7 columns) show the right side of the equation in Theorem 4 is a lower bound for $a_{4,n}$. Theorem 4 shows this is maximal.

3. The Cases where $m = 5, 6, 7$

The only values of m where Lemma 2 does not give $a_{m,n}$ for all but a finite number of n is when $m = 5, 6, 7$. For these, proving (1) becomes more difficult.

Lemma 4. *Let S be a 2-packing of $m \times n$ board.*

(a) If for some $0 < r < n$, $|S| = a_{m,r} + a_{m,n-r}$, then S has exactly $a_{m,r}$ checkers in columns 1 to r.

(b) If for some $q < r$, $|S| = a_{m,q} + a_{m,n-q} = a_{m,r} + a_{m,n-r}$, then S has exactly $a_{m,r} - a_{m,q}$ checkers in columns $q+1$ to r.

Proof. For (a), S can at most $a_{m,r}$ checkers in columns 1 to r, and S has at most $a_{m,n-r}$ checkers in columns $r+1$ to n. From these, (a) follows. Statement (b) follows immediately from (a). □

Theorem 5. *For all $n > 0$, $a_{5,n} = n + 1$.*

Proof. Figure 7 shows that $a_{5,n} \geq n + 1$. Lemma 3 gives the result when $n \leq 3$. For $n > 3$, assume the result holds for all $k < n$. Suppose S 2-packs a $5 \times n$ board with $n + 2$ checkers. Then for all $0 < k < n$, $a_{m,k} + a_{m,n-k} = k + 1 + n - k + 1 = n + 2$. Let s_i be the number of checkers of S in column i. Then by Lemma 4, S has column count 2, 1, 1, 1, ..., 1, 2 (*i.e.*, $s_1 = s_n = 2$ and $s_k = 1$ for all k with $1 < k < n$). The only way for $s_1 = 2$ and $s_2 = 1$ is to have checkers at $(1, 1)$, $(1, 5)$ and $(2, 3)$. But this allows no checkers in column 3. The result follows by induction. □

Lemma 5. *A 2-packing of a 6×5 board cannot have column count 2, 1, 1, 2, 1.*

Proof. Suppose S 2-packs a 6×5 board with column count 2, 1, 1, 2, 1. In order for $s_3 = s_5 = 1$, the checkers in column 4 must be $(1, 4)$ and $(6, 4)$, and at least one of the checkers in columns 3 or 5 must be in row 4. Without loss of generality, assume $(3, 4) \in S$. Then $(5, 2) \in S$. But then 2 checkers cannot be placed in column 1. □

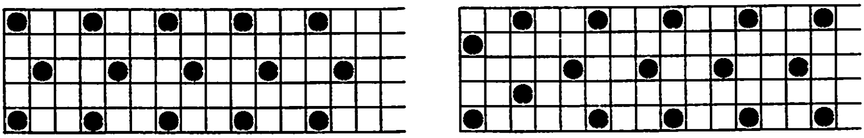

Figure 7 – Maximal 2-packings of $5 \times n$ board. The first n columns of the left 2-packing (the pattern repeats every 3 columns) shows $a_{5,n} \geq n+1$ for n mod $3 \neq 0$. The first n columns of the right 2-packing (after the first 4 columns, the pattern repeats every 3 columns) shows $a_{5,n} \geq n+1$ for n mod $3 = 0$. These are maximal by Theorem 5.

Lemma 6. *A 2-packing of a 6×7 board cannot have column count 1, 2, 1, 1, 1, 2, 1.*

Proof. Suppose a 2-packing of a 6×7 board had column count 1, 2, 1, 1, 1, 2, 1. In order for columns 1, 3, 5, and 7 to each have a checker, the checkers in columns 2 and 6 must be at $(1,2)$, $(6,2)$, $(1,6)$ and $(6,6)$. Then the checkers in columns 3 and 5 must be either $(3,3)$ and $(4,5)$, or $(4,3)$ and $(3,5)$. Either way prohibits a checker in column 4. □

Theorem 6. *For all $n > 0$, $a_{6,n} = \lceil (6n+2)/5 \rceil$.*

Proof. For n mod $5 = 1, 2, 3$, Lemma 1 shows that $a_{6,n} \geq \lceil 6n/5 \rceil = \lceil (6n+2)/5 \rceil$. Figure 8 shows that $a_{6,n} \geq \lceil (6n+2)/5 \rceil$ for n mod $5 = 0, 4$. Lemma 3 gives the result for $n \leq 3$. For $n > 3$, assume the result holds for all $k < n$. Then there are three cases.

If n mod $5 = 0, 1, 4$, then from Lemma 2,

$$a_{6,n} \leq a_{6,n-3} + a_{6,3} = \lceil (6(n-3)+2)/5 \rceil + 3 = \lceil (6n+2)/5 \rceil.$$

For $n = 5j+2$, suppose S 2-packs a $6 \times n$ board with $\lceil (6n+2)/5 \rceil + 1 = 6j+4$ checkers. For all k where $0 < k < n$,

$$a_{6,n-k} + a_{6,k} = \lceil (6(n-k)+2)/5 \rceil + \lceil (6k+2)/5 \rceil$$

$$= 6j + 2 + \lceil (4 - (k \bmod 5))/5 \rceil + \lceil ((k \bmod 5)+2)/5 \rceil = 6j+4.$$

So by Lemma 4, S has column count 2, 1, 1, 2, 1, ... This violates Lemma 5. So the result is true in this case.

For $n = 5j+3$, suppose S 2-packs a $6 \times n$ board with $\lceil (6n+2)/5 \rceil + 1 = 6j+5$ checkers. For all k where $0 < k < n$,

$$a_{6,n-k} + a_{6,k} = \lceil (6(n-k)+2)/5 \rceil + \lceil (6k+2)/5 \rceil$$

$$= 6j+4+\lceil -(k \bmod 5)/5 \rceil + \lceil ((k \bmod 5)+2)/5 \rceil = \begin{cases} 6j+6 & \text{if } k \bmod 5 = 4 \\ 6j+5 & \text{otherwise.} \end{cases}$$

So by Lemma 4, $s_1 = s_n = 2$, $s_k = 1$ for $1 < k < n$ and $k \bmod 5 = 1, 2, 3$, and $s_k + s_{k+1} = 3$ for $1 < k < n$ with $k \bmod 5 = 4$. Since columns 1, 2 and 3 have 2, 1 and 1 checkers, respectively, $s_4 = 1$ and $s_5 = 2$ for the reverse violates Lemma 5. Now since columns 4 to 8 have column count 1, 2, 1, 1, 1, we must have $s_9 = 1$ and $s_{10} = 2$ for the reverse violates violates Lemma 6. Continuing, $s_k = 1$ for $1 < k < n$ and $k \bmod 5 = 4$ and $s_k = 2$ for $1 < k < n$ and $k \bmod 5 = 0$, for the first instance where this does not happen violates Lemma 6. Then, columns $n-4$ to n have column count 1, 2, 1, 1, 2 which violates Lemma 5. So the result is true in this final case. □

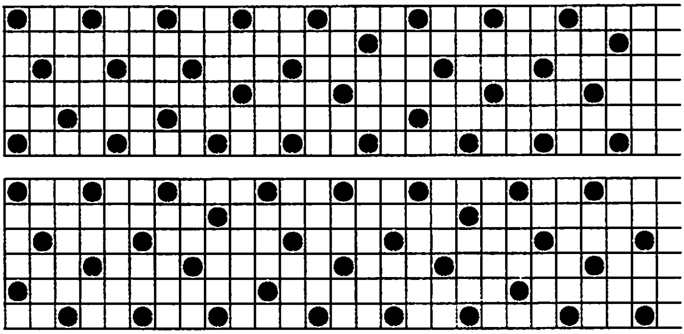

Figure 8 – Maximal 2-packings of $6 \times n$ board. The first n columns of the upper 2-packing (after the first 6 columns, the pattern repeats every 10 columns) show that $a_{6,n} \geq \lceil (6n+2)/5 \rceil$ for $n \bmod 5 = 0$. The first n columns of the lower 2-packing (the pattern repeats every 10 columns) show that $a_{6,n} \geq \lceil (6n+2)/5 \rceil$ for $n \bmod 5 = 4$. Figure 2 shows that $a_{6,n} \geq \lceil (6n+2)/5 \rceil$ for $n \bmod 5 \neq 0, 4$. Theorem 6 shows these are maximal.

Lemma 7. *A 2-packing of* 7×5 *board cannot have column count 2, 1, 2, 1, 2.*

Proof. Suppose S 2-packs a 6×7 board with column count 2, 1, 2, 1, 2. In order for $s_2 = s_4 = 1$, there must be a checker at either $(1,3)$ or $(7,3)$. Without loss of generality, assume $(1,3) \in S$. The other checker in column 3 must be either $(4,3)$, $(5,3)$, $(6,3)$, or $(7,3)$.

If there is a checker at $(4,3)$, at least one checker in columns 2 and 4 must be in row 6. Without loss of generality, assume $(6,2)$ has a checker. Then column 1 cannot have 2 checkers.

If there is a checker at $(5,3)$ or $(6,3)$, at least one checker in columns 2 and 4 must be in row 2. Without loss of generality, assume $(2,2)$ has a checker. Then column 1 cannot have 2 checkers.

If there is a checker at $(7,3)$, at least one checker in columns 2 and 4 must be in row 3 or 5. Without loss of generality, assume there is a checker at $(3,2)$. Then column 1 cannot have 2 checkers. □

Theorem 7. *For all $n > 0$,*

$$a_{7,n} = \begin{cases} 3 & \text{if } n = 1 \\ 10 & \text{if } n = 7 \\ \lceil (7n+2)/5 \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $f(n)$ be the right side of the above equation. Figure 9 shows that $a_{7,n} \geq \lceil (7n+2)/5 \rceil$ for $n \bmod 5 = 3, 5$ with $n > 7$. Otherwise, Lemma 1 shows that $a_{7,n} \geq \lceil 7n/5 \rceil = f(n)$. Lemma 3 shows the result holds for $n \leq 18$. For $n > 18$, assume the result holds for all $k < n$. Then there are two cases.

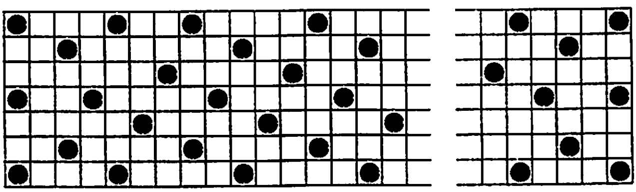

Figure 9 – Maximal 2-packings of a $7 \times n$ Board. The above 2-packing (after the first 6 columns, the pattern repeats every 5 columns until the last 6 columns) show that $a_{7,n} \geq \lceil (7n+2)/5 \rceil$ for $n \bmod 5 = 2$ and $n > 7$. Removing the last two columns shows that $a_{7,n} \geq \lceil (7n+2)/5 \rceil$ for $n \bmod 5 = 0$ and $n > 7$. Figure 2 shows that $a_{7,n} \geq \lceil (7n+2)/5 \rceil$ for $n \bmod 5 \neq 0, 2$. Theorem 7 shows these are maximal.

If $n \bmod 5 \neq 4$, then from Lemma 2,

$$a_{7,n} \leq a_{7,n-7} + a_{7,7} = \lceil (6(n-7)+2)/5 \rceil + 10 = \lceil (6n+2)/5 \rceil.$$

For $n = 5j+4$, suppose S 2-packs a $7 \times n$ board with $\lceil (7n+2)/5 \rceil + 1 = 7j+7$ checkers. For all k where $7 < k < n-7$,

$$a_{7,n-k} + a_{7,k} = \lceil (7(n-k)+2)/5 \rceil + \lceil (7k+2)/5 \rceil$$

$$= 7j + 6 + \lceil -2(k \bmod 5)/5 \rceil + \lceil (2(k \bmod 5)+2)/5 \rceil$$

$$= \begin{cases} 7j+8 & \text{if } k \bmod 5 = 2 \\ 7j+7 & \text{otherwise.} \end{cases}$$

Further, $a_{7,n-7} + a_{7,7} = \lceil (7(n-7)+2)/5 \rceil + 10 = 7j - 3 + 10 = 7j + 7$. So by Lemma 4, $s_8 = s_{n-8} = 2$, $s_k = 1$ for $7 < k < n-7$ and $k \bmod 5 = 1, 4$, $s_k = 2$ for $7 < k < n-7$ and $k \bmod 5 = 0$, and $s_k + s_{k+1} = 3$ for $7 < k < n-7$ with $k \bmod 5 = 2$. Since columns 8 to 11 have column count 2, 1, 2, 1, we must have $s_{12} = 1$ and $s_{13} = 2$ in order not to violate Lemma 7. Now since columns 13 to 16 have column count 2, 1, 2, 1, we must have

$s_{17} = 1$ and $s_{18} = 2$ in order not to violate Lemma 7. Continuing, $s_k = 1$ with $7 < k < n - 7$ and $k \bmod 5 = 2$ and $s_k = 2$ with $7 < k < n - 7$ and $j \bmod 5 = 3$, for the first instance where this does not happen violates Lemma 7. Then, columns $n - 12$ to $n - 8$ have column count 2, 1, 2, 1, 2 which violates Lemma 7. So the result is true in this final case. □

4. The Case where $m \geq 8$

Theorem 8. *For all $m, n \geq 8$,*

$$a_{m,n} = \begin{cases} 17 & \textit{if } (m,n) = (10,8) \textit{ or } (m,n) = (8,10) \\ \lceil mn/5 \rceil & \textit{otherwise.} \end{cases}$$

Proof. Lemma 3 shows the result holds when $m, n \leq 18$, and for $m = 8$ with $n \leq 25$ (Figure 10 shows the exceptionally dense 2-packing of an 8×10 board). Lemma 1 shows the result is a lower bound for all m and n.

Without loss of generality, assume $m \leq n$. Assume the result holds for all $j < m$ and $k < n$. If $m = 8$, then $n \geq 26$. Then Lemma 2 and induction give

$$a_{8,n} \leq a_{8,n-15} + a_{8,15} = \lceil 8(n-15)/5 \rceil + 24 = \lceil 8n/5 \rceil.$$

If $m > 8$, then $n \geq 19$. Then Lemma 2 and induction give

$$a_{m,n} \leq a_{m,n-10} + a_{m,10} = \lceil m(n-10)/5 \rceil + 2m = \lceil mn/5 \rceil. \quad \square$$

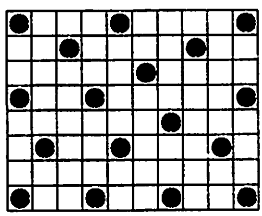

Figure 10 – Maximal 2-packing of 8×10 Board. This is the unique (up to symmetries) maximal 2-packing of an 8×10 board with 17 checkers. Figure 2 gives a maximal 2-packing for all $m \times n$ board with $m, n \geq 8$ except for 10×8 and 8×10 boards.

5. Suggestions for Further Work

The regularities found here suggest similar problems might have similar regularities. Three directions for explorations are listed below.

k-packings. Checkers form a *k-packing* if each pair of checkers are separated by at least k squares. It is easy to show that the maximal number of checkers in a 1-packing on an $m \times n$ board is $\lceil mn/2 \rceil$. For $k > 2$, what is the maximal number of checkers in a k-packing on an $m \times n$ board?

Domination and Fractional Domination. A set of checkers *dominate* a board if each square has a checker on it or is next to (either vertically or horizontally) a square with a checker on it. Let $A_{m,n}$ be the minimum number of checkers needed to dominate an $m \times n$ board. A set of "fractional checkers" (checkers weighted with a nonnegative number) are a *fractional domination* of a board if, for each square, the weight on the square plus the weights on its neighbors is at least one. Let $B_{m,n}$ be the minimum sum of weights needed to fractionally dominate an $m \times n$ board. A set of fractional checkers are a *fractional 2-packing* of a board if, for each square, the weight on the square plus the weights on its neighbors is at most one. Let $b_{m,n}$ be the maximum sum of weights needed to fractionally 2-pack an $m \times n$ board. Chandrasekharan, Hedetniemi, Laskar and Majumdar [1] give

$$a_{m,n} \leq b_{m,n} = B_{m,n} \leq A_{m,n}. \tag{2}$$

Considerable work has been directed toward finding $A_{m,n}$ and $B_{m,n}$. Jacobson and Kinch [6] showed that

$$A_{1,n} = \lceil n/3 \rceil, \; A_{2,n} = \lceil (n+1)/2 \rceil, \; A_{3,n} = \lceil (3n+1)/4 \rceil$$

$$\text{and } A_{4,n} = \begin{cases} n+1 & \text{if } n = 1,2,3,5,6,9 \\ n & \text{otherwise.} \end{cases} \tag{3}$$

Hare, Hedetniemi and Hare [5] devised a dynamic programming algorithm to find $A_{m,n}$. It found that $A_{5,n} = \lceil (6n+4)/5 \rceil$ for $8 \leq n \leq 500$. Cockayne, Hare, Hedetniemi and Wimer [2] showed that for all $n \geq 8$,

$$\lceil (n^2+n-3)/5 \rceil \leq A_{n,n} \leq \lceil (n^2+4n-20)/5 \rceil.$$

Since it is a linear programming problem, computation of $B_{m,n}$ is straightforward. Equations (1), (2) and (3) give $B_{1,n} = \lceil n/3 \rceil$. Hare [3] showed that

$$B_{2,n} = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd} \\ (n^2+2n)/2(n+1) & \text{if } n \text{ is even.} \end{cases}$$

2-Packing of Cartesian Products. A checker board can be thought of as the Cartesian product of two path graphs. It should be possible to find the 2-packing number for the Cartesian product of other infinite families of graphs.

References

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