The Basis Number of the Lexicographic Product of Graphs

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Abstract. We define the basis number, $b(G)$, of a graph G to be the least integer k such that G has a k-fold basis for its cycle space. We investigate the basis number of the lexicographic product of paths, cycles and wheels. It is proved that

$$b(P_n \otimes P_m) = b(p_n \otimes C_m) = 4 \text{ for } n, m \geq 7,$$
$$b(C_n \otimes P_m) = b(C_n \otimes C_m) = 4 \text{ for } n, m \geq 6,$$
$$b(P_n \otimes W_m) = 4 \text{ for } n, m \geq 9, \text{ and}$$
$$b(C_n \otimes W_m) = 4 \text{ for } n, m \geq 8.$$

It is also shown that $\max\{4, b(G)+2\}$ is an upper bound for $b(P_n \otimes G)$ and $b(C_n \otimes G)$ for every semi-hamiltonian graph G.

1. Introduction.

Throughout this paper we consider only finite, undirected, simple graphs. Our terminology and notations will be standard except as indicated. For undefined terms, see [5] and [7].

Let G be a connected graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of the edges. Then any subset S of edges corresponds to a q-dimensional vector $(a_1, a_2, \ldots, a_q)$ with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$, for $i = 1, 2, \ldots, q$. These vectors form a vector space over the field Z_2. These vectors which correspond to cycles of G generate a subspace called the cycle space of G, denoted by $C(G)$. Strictly speaking the vectors generate $C(G)$, but we usually think of the corresponding cycles as elements of the space. It is well-known that

$$\dim C(G) = \gamma(G) = q - p + 1,$$

where p is the number of vertices and $\gamma(G)$ is the cyclomatic number of G. A basis for $C(G)$ is called *k-fold* if each edge of G occurs in at most k of the cycles in the basis. The *basis number* of G (denoted by $b(G)$) is the smallest integer k such that $C(G)$ has a k-fold basis, and such a basis is called a *required basis* of G and denoted by $B_r(G)$. If B is a basis for $C(G)$ and e is an edge of G, then the *fold of e in B* (denoted by $f_B(e)$) is defined to be the number of cycles in B containing e. The *lexicographic product* [6] (also called composition) of two graphs $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$, denoted by $G_1 \otimes G_2$, is the graph with vertex set $X_1 \times X_2$ in which (x_1, x_2) is joined to (y_1, y_2) whenever x_1 is joined to y_1 in G_1 or $x_1 = y_1$ and x_2 is joined to y_2 in G_2. It is clear that the

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number of edges in $G_1 \otimes G_2$ is $p_1 q_2 + p_2^2 q_1$, where p_i and q_i are the number of vertices and edges in G_i, $i = 1, 2$. Note that the lexicographic products $G_1 \otimes G_2$ and $G_2 \otimes G_1$ are not isomorphic.

The path with n vertices is denoted by P_n, and the cycle with m edges is denoted by C_m.

The first important result concerning the basis number was given in 1937 by MacLane [8]. He proved that a graph G is planar if and only if $b(G) \leq 2$. In 1981, Schmeichel [11] proved that for $n \geq 5$, $b(K_n) = 3$; and for $m, n \geq 5$, $b(K_{m,n}) = 4$ except for $K_{6,10}$, $K_{5,n}$, and $K_{6,n}$, with $n = 5, 6, 7$, and 8.

Moreover, in 1984 Al-Sardary [10] established $b(K_{5,n}) = b(K_{6,n}) = 3$, for $n = 5, 6, 7$, and 8.

In 1982, Banks and Schmeichel [4] proved that for $n \geq 7$, $b(Q_n) = 4$, where Q_n is the n-cube. Furthermore, Ali [1], [2], and [3], considered the basis number of some complete multipartite graphs, the basis number of the join of graphs and the basis number of the direct products of paths and cycles, respectively. His main results are given in the following theorems.

1. $b(K_{m(n)}) \leq 9$, *for* $m, n \geq 3$, *where* $K_{m(n)}$ *is a complete* m*-partite graph.*

2. $b(K_{n,n,n}) = 3$, $n \geq 3$,

3. $b(K_{m,n,\ell}) \leq 4$, *for any positive integers* ℓ, m *and* n.

4. $b(C_m \cdot P_n) \leq 2$.

5. $b(C_m \cdot C_n) \leq 2$, *where* $C_m \cdot C_n$ *denotes the direct product [3] of the cycles* C_m *and* C_n.

6. *If* G_1 *and* G_2 *are vertex disjoint graphs, and each has a spanning forest of valency not more than* 4, *then*

$$b(G_1 + G_2) \leq \max\{4, b(G_1) + 1, b(G_2) + 1\}.$$

The purpose of this paper is to determine the basis number of the lexicographic products of paths, cycles, and wheels.

2. The basis number of $P_n \otimes P_m$.

Let the vertex sets of P_n and C_n be Z_n, where Z_n denotes the addition group of residus modulo n. Let the path P_n be $012\ldots(n-1)$, and the cycle C_n be $012\ldots(n-1)0$.

The following lemma is a simple and useful result.

Lemma 1. *If* G *and* H *are connected graphs, then* $G \otimes H$ *is also connected.*

Proof: From the definition of the lexicographic product of two graphs, one can deduce that $G \times H \subseteq G \otimes H$, where $G \times H$ dentoes the cartesian product [9] of the graphs G and H. In [9] it is proved that $G \times H$ is connected, therefore, $G \times H$ is also connected. ∎

Now we consider the complete bipartite graph $K_{m,m}$. Let $\{(0,0),(0,1),\dots,(0,m-1)\}$ and $\{(1,0),(1,1),\dots,(1,m-1)\}$ be the partition of the vertices of $K_{m,m}$ into independent sets. Schmeichel [11] proved that

$$B_r(K_{m,m}) = \{(0,i)(1,j)(0,i+1)(1,j+1)(0,i) \mid i,j = 0,1,\dots,m-2\}, \quad (1)$$

is a basis for $C(K_{m,m})$ of fold 4, and so deduced that $b(K_{m,m}) \leq 4$. Ali [2] showed that

$$A_1 = \{(0,0)(1,0),(0,0)(1,m-1),(0,m-1)(1,0),(0,m-1)(1,m-1)\}, \quad (2)$$

is the set of all 1-fold edges of $K_{m,m}$ in $B_r(K_{m,m})$, and

$$A_2 = \{(0,0)(1,j),(0,m-1)(1,j),(0,j)(1,0),(0,j)(1,m-1) \mid j = 1,2,\dots,m-2\}, \quad (3)$$

is the set of all 2-fold edges of $K_{m,m}$ in $B_r(K_{m,m})$; and all other edges are of fold 4.

Lemma 2. *If G is connected, and has a spanning tree of valency not more than 4, then*

$$b(P_2 \otimes G) \leq \max\{4, b(G) + 1\}.$$

Proof: It is clear that

$$P_2 \otimes G = G + G.$$

Therefore, from Theorem 1 in [2],

$$b(P_2 \otimes G) = b(G + G) \leq \max\{4, b(G) + 1\}.$$

∎

Corollary 1. $b(P_2 \otimes P_m) = 4$ *for* $m \geq 12$.

Proof: From Lemma 2, we have

$$b(P_2 \otimes P_m) \leq 4.$$

It is clear that for $m \geq 3$, $P_2 \otimes P_m$ is nonplanar, thus, $b(P_2 \otimes P_m) \geq 3$. On the other hand, suppose $b(P_2 \otimes P_m) = 3$ and let B_r be a required basis of $P_2 \otimes P_m$. Then $|B_r| = m^2 - 1$. As in Theorem 1 in [2], the number of 3-cycles in B_r is at most $3.2(m-1)$, and all other cycles in B_r are of length at least 4.

Hence,

$$4\left((m^2-1)-6(m-1)\right)+3.6(m-1)\le 3(m^2+2m-2),$$

that is,

$$m^2+8\le 12m.$$

The above inequality does not hold when $m\ge 12$. Hence,

$$b(P_2\otimes P_m)\ge 4,\ \text{if } m\ge 12.$$

∎

By finding a 3-fold basis, we show that $b(P_2\otimes P_m)=3$ for $m=3,4$, and 5. It seems likely that $b(P_2\otimes P_m)=3$ for $m=6,7,\dots,11$.

Corollary 2. $b(P_2\otimes C_m)=4$ *for* $m\ge 12$.

Proof: From Lemma 2, we have

$$b(P_2\otimes C_m)\le 4.$$

It is clear that $P_2\otimes C_m$ is nonplanar for $m\ge 3$. Thus, $b(P_2\otimes C_m)\ge 3$. On the other hand, suppose $b(P_2\otimes C_m)=3$. The number of 3-cycles in a required basis of $P_2\otimes C_m$ is at most $3.2(m)$, and all other cycles are of length at least 4. Since

$$\gamma(P_2\otimes C_m)=m^2+1,$$

and

$$|E(P_2\otimes C_m)|=m^2+2m,$$

then

$$4(m^2-6m+1)+3(6m)\le 3(m^2+2m),$$

which implies

$$m^2+4\le 12m.$$

It can be easily verified that the above inequality does not hold when $m\ge 12$. Hence,

$$b(P_2\otimes C_m)\ge 4 \text{ if } m\ge 12.$$

∎

By finding a 3-fold basis, it is shown that

$$b(P_2 \otimes C_m) = 3 \text{ for } m = 3, 4, \text{and } 5.$$

Theorem 1. *For* $n \geq 2$, $b(P_n \otimes P_m) \leq 4$.

Proof: Let $V_0, V_1, \ldots, V_{n-1}$, be a partition of the vertex set of $P_n \otimes P_m$ such that

$$V_i = \{(i,0), (i,1), \ldots, (i, m-1)\},$$

and let E_i, $i = 0, 1, \ldots, n-2$, be the set of all edges in $P_n \otimes P_m$ joining a vertex of V_i to a vertex of V_{i+1}. That is, for each i, the subgraph $H_i = (V_i \cup V_{i+1}, E_i)$ is isomorphic to $K_{m,m}$. Also let P_m^j be the j-copy of P_m, that is P_m with each vertex y replaced by (j, y), for $j = 0, 1, \ldots, n-1$. Then, it is a simple matter to verify that

$$E_0, E_1, \ldots, E_{n-2}, E(P_m^0), \ldots, E(P_m^{n-1})$$

is a partition of the edge-set of $P_n \otimes P_m$.

We shall prove that

$$B(P_n \otimes P_m) = \left(\bigcup_{k=0}^{n-2} B_r(H_k) \right) \cup S$$

is a basis for $\mathcal{C}(P_n \otimes P_m)$, in which

$$B_r(H_k) = \{(k,i)(k+1,j)(k,i+1)(k+1,j+1)(k,i) \mid i, j = 0, 1, \ldots, m-2\},$$

and

$$S = \{(i,j)(i,j+1)(i+1,m-1)(i,j), (i+1,j)(i+1,j+1)(i,m-1)(i+1,j) \mid i = 0, 1, \ldots, n-2 \text{ and } j = 0, 1, \ldots, m-2\}.$$

It is clear that

$$|(B(P_n \otimes P_m)| = (m-1)(m-1)(n-1) + 2(m-1)(n-1) = m^2(n-1) + n(m-1) - nm + 1 = \gamma(P_n \otimes P_m).$$

From the construction of $P_n \otimes P_m$, we notice that every chordless cycle C of $P_n \otimes P_m$ is either a 3-cycle or a 4-cycle of the graph H_k for some k. Therefore, C is generated by $B(P_n \otimes P_m)$. Thus, $B(P_n \otimes P_m)$ generates $\mathcal{C}(P_n \otimes P_m)$ and so it is a basis.

To consider the fold of the basis $B(P_n \otimes P_m)$, let

$$T = \bigcup_{k=0}^{n-2} B_r(H_k),$$

$$L = \{(i, m-1)(i+1, j), (i+1, m-1)(i, j) \mid i = 0, 1, \ldots, n-2, \text{ and } j = 0, 1, \ldots, m-1\},$$

and

$$M = \left(\bigcup_{k=0}^{n-2} E(H_k) \right) - L.$$

It is clear that $E(P_n \otimes P_m)$ is partitioned into $\bigcup_{i=0}^{n-1} E(P_m^i)$, L and M. Then using (1), (2), and (3), we have

$$f_T(e) = 0, \quad f_S(e) \leq 2, \quad \text{for } e \text{ in } \bigcup_{i=0}^{n-1} E(P_m^i),$$

$$f_T(e) \leq 2, \quad f_S(e) \leq 2, \quad \text{for } e \in L,$$

$$f_T(e) \leq 4, \quad f_S(e) = 0, \quad \text{for } e \in M.$$

Thus, $B(P_n \otimes P_m)$ is a 4-fold basis of $P_n \otimes P_m$. ∎

Theorem 2. *For every integers $m, n \geq 7$, we have $b(P_n \otimes P_m) = 4$.*

Proof: By Theorem 1, it suffices to exhibit $b(P_n \otimes P_m) \geq 4$, for $m, n \geq 7$. Suppose $b(P_n \otimes P_m) = 3$. The number of 3-cycles in a required basis of $P_n \otimes P_m$ is at most $3(n)(m-1)$, and all other cycles are of length at least 4. Thus,

$$4\left((m^2-1)(n-1) - 3n(m-1)\right) + 3\left(3n(m-1)\right) \leq 3\left(m^2(n-1) + n(m-1)\right),$$

that is

$$m^2(n-1) + 2n + 4 \leq 6mn.$$

It can be easily verified that the above inequality does not hold when $m, n \geq 7$. Hence,

$$b(P_n \otimes P_m) = 4 \text{ if } n, m \geq 7.$$

∎

Remark: Although the lexicographic product $P_n \otimes P_m$ is not isomorphic to $P_m \otimes P_n$, for $m \neq n$, but from Theorem 1 and Theorem 2, if $m, n \geq 7$

$$b(P_n \otimes P_m) = b(P_m \otimes P_n) = 4.$$

3. The basis number of $C_n \otimes C_m$.

In this section, we consider the basis number of the lexicographic product of a cycle with a path or a cycle.

Theorem 3. *For all integers* $m, n \geq 7$,

$$b(P_n \otimes C_m) = 4.$$

Proof: Let $B(P_n \otimes P_m)$ be the required basis of $P_n \otimes P_m$ defined in Theorem 1. We shall prove that

$$B(P_n \otimes C_m) = B(P_n \otimes P_m) \cup W$$

is a basis for $C(P_n \otimes C_m)$, in which

$$W = \{(i,0)(i,1)\ldots(i,m-1)(i,0) | i = 0,1,\ldots,n-1\}.$$

It is clear that

$$\begin{aligned} |B(P_n \otimes C_m)| &= \gamma(P_n \otimes P_m) + n \\ &= (n-1)(m^2-1) + n \\ &= (n-1)m^2 + nm - nm + 1 \\ &= \gamma(P_n \otimes C_m). \end{aligned}$$

Since $B(P_n \otimes P_m)$ is an independent set of cycles, and each cycle $(i,0)(i,1)\ldots(i,m-1)(i,0)$ contains an edge $(i,0)(i,m-1)$ not in $E(P_n \otimes P_m)$, then $B(P_n \otimes C_m)$ is an independent set of cycles in $C(P_n \otimes C_m)$. One can easily observe from the proof of Theorem 1, that for all $i = 0,1,\ldots,n-1$ and $j = 0,1,\ldots,m-1$, the fold of the edge $(i,j)(i,j+1)$ is 1 in W and it is not more than 2 in $B(P_n \otimes P_m)$, in which $j+1$ is taken with respect to modulo m. The fold of each other edge of $P_n \otimes C_m$ is at most 4. Thus, $b(P_n \otimes C_m) \leq 4$.

On the other hand, suppose $b(P_n \otimes C_m) = 3$. The number of 3-cycles in a required basis of $P_n \otimes C_m$ is at most $3nm$, and all other cycles are of length at least 4. Hence,

$$4\left(m^2(n-1) - 3mn + 1\right) + 9nm \leq 3\left(m^2(n-1) + mn\right),$$

that is

$$m^2(n-1) + 4 \leq 6mn.$$

It can be easily verified that the above inequality does not hold when $m, n \geq 7$. Hence,

$$b(P_n \otimes C_m) \geq 4 \text{ if } m, n \geq 7.$$

The proof of Theorem 3 is complete. ∎

Theorem 4. *For $m, n \geq 3$, we have $b(C_n \otimes C_m) \leq 4$; equality holds when $m, n \geq 6$.*

Proof: Let

$$B(C_n \otimes C_m) = B(P_n \otimes C_m) \cup B(P_2' \otimes P_m) \cup \{C_0\},$$

in which $B(P_n \otimes C_m)$ is the basis for $C(P_n \otimes C_m)$ defined in the proof of Theorem 3, $B(P_2' \otimes P_m)$ is the basis for $C(P_2' \otimes P_m)$ defined in Theorem 1, for $n = 2$, where P_2' is the edge joining vertices 0 and $n-1$, and

$$C_0 = (0,0)(1,0)(2,0)\ldots(n-1,0)(0,0).$$

To obtain $B(P_2' \otimes P_m)$ from $B(P_2 \otimes P_m)$ replace 1's in the first positions of the ordered pairs by $(n-1)$. Now C_0 is independent from the cycles in $B(P_2' \otimes P_m)$, because C_0 contains the edge $(0,0)(1,0)$, which is not present in any linear combination of cycles in $B(P_2' \otimes P_m)$. Therefore, $B(P_2' \otimes P_m) \cup \{C_0\}$, is an independent set of cycles. Also $B(P_n \otimes C_m) \cup B(P_2' \otimes P_m) \cup \{C_0\}$ is an independent set of cycles, because if C is any cycle generated from cycles in $B(P_2' \otimes P_m) \cup \{C_0\}$, then C contains at least one edge of E_{n-1} (E_{n-1} is the set of all edges joining a vertex in V_{n-1} to a vertex in V_0). On the other hand, no linear combination of cycles in $B(P_n \otimes C_m)$ contains an edge of E_{n-1}. Therefore, $B(C_n \otimes C_m)$ is an independent set of cycles. Since

$$\begin{aligned}|B(C_n \otimes C_m)| &= (n-1)m^2 + 1 + (m^2 - 1) + 1 \\ &= nm^2 + 1 = \gamma(C_n \otimes C_m),\end{aligned}$$

then $B(C_n \otimes C_m)$ is a basis for $C_n \otimes C_m$.

To find the fold of $B(C_n \otimes C_m)$, let

$$N = B(P_2' \otimes P_m) \cup B(P_n \otimes C_m).$$

From the proofs of Theorem 1 and Theorem 3, one can easily check that

$$f_N(e) \leq 4, \quad \text{for each } e \in E(C_n \otimes C_m) - E(C_0),$$

and

$$f_N(e) = 1, \quad \text{for each } e \in E(C_0).$$

Thus, $B(C_n \otimes C_m)$ is a 4-fold basis of $C_n \otimes C_m$.

To complete the theorem, suppose $b(C_n \otimes C_m) = 3$. The number of 3-cycles in a required basis of $C_n \otimes C_m$ is at most $3nm$, and all other cycles are of length at least 4.

Hence,

$$4(nm^2 - 3nm + 1) + 9nm \leq 3(m^2n + mn),$$

that is

$$m^2n + 4 \leq 6mn.$$

It is clear that the above inequality does not hold when $m, n \geq 6$. Hence,

$$b(C_n \otimes C_m) \geq 4 \text{ if } m, n \geq 6.$$

The proof of Theorem 4 is complete. ∎

Theorem 5. *For* $n \geq 3$ *and* $m \geq 2$,

$$b(C_n \otimes P_m) \leq 4;$$

equality holds for $m, n \geq 6$.

Proof: Consider the set of cycles

$$B(C_n \otimes P_m) = B(P_n \otimes P_m) \cup B(P_2' \otimes P_m) \cup \{C_0\},$$

in which $B(P_n \otimes P_m)$ is that defined in the proof of Theorem 1, and $B(P_2' \otimes P_m) \cup \{C_0\}$ is defined in the proof of Theorem 4. It is clear that

$$B(C_n \otimes P_m) \subseteq B(C_n \otimes C_m),$$

and

$$|B(C_n \otimes P_m)| = m^2n - n + 1 = \gamma(C_n \otimes P_m).$$

Thus, $B(C_n \otimes P_m)$ is a basis for $C_n \otimes P_m$, and

$$b(C_n \otimes P_m) \leq 4.$$

Now, suppose

$$b(C_n \otimes P_m) = 3,$$

and $B_r(C_n \otimes P_m)$ is a required 3-fold basis. Then the number of 3-cycles in $B_r(C_n \otimes P_m)$ is at most $3n(m-1)$ when $n \geq 4$ and $m \geq 3$.

Since

$$|E(C_n \otimes P_m)| = m^2n + mn - n,$$

then

$$4(m^2n - 3mn + 2n + 1) + 3.3n(m-1) \leq 3(m^2n + mn - n),$$

that is

$$m^2n + 2n + 4 \leq 6mn.$$

The above inequality does not hold when $n, m \geq 6$. Hence, for $m, n \geq 6$

$$b(C_n \otimes P_m) = 4.$$

∎

4. An upper bound for $b(P_n \otimes G)$ and $b(C_n \otimes G)$.

The basis number of the lexicographic product of a semi-hamiltonian graph with a path or a cycle is studied in this section.

Theorem 6. *For every semi-hamiltonian graph G,*

$$b(P_n \otimes G) \leq \max\{4, b(G) + 2\}.$$

Proof: Let m be the order of G and let P_m be a hamiltonian path of G. Consider the set of cycles

$$B(P_n \otimes G) = B(P_n \otimes P_m) \cup \left(\bigcup_{i=0}^{n-1} B_r(G^i)\right)$$

where $B_r(G^i)$ is a required basis of the ith copy G^i of G, and $B(P_n \otimes P_m)$ is the basis of $P_n \otimes P_m$ defined in the proof of Theorem 1. It is clear that any linear combination of cycles in $B(P_n \otimes P_m)$ contains an edge joining a vertex of $V(G^i)$ with a vertex of $V(G^{i+1})$ for some i, $0 \leq i \leq n-2$. Moreover, any linear combination of cycles in $\cup_{i=0}^{n-1} B_r(G^i)$ contains edges of G^i for some i, $0 \leq i \leq n-1$. Therefore, $B(P_n \otimes G)$ is independent. Since

$$\begin{aligned} |B(P_n \otimes G)| &= \gamma(P_n \otimes P_m) + n(q - m + 1) \\ &= m^2(n-1) + nq - nm + 1 \\ &= \gamma(P_n \otimes G), \end{aligned}$$

in which q is the size of G, then $B(P_n \otimes G)$ is a basis for $P_n \otimes G$.

From Theorem 1, the fold in $B(P_n \otimes G)$ of each edge xy, $x \in V(G^i)$ and $y \in V(G^{i+1})$, for $i = 0, 1, \ldots, n-2$, does not exceed 4. And the fold in $B(P_n \otimes P_m)$ of each edge of P_m^i, for $i = 0, 1, \ldots, n-1$, does not exceed 2. Therefore, the fold in $B(P_n \otimes G)$ of each edge of G^i, for $i = 0, 1, \ldots, n-1$, does not exceed $b(G) + 2$. Thus,

$$b(P_n \otimes G) \leq \max\{4, b(G) + 2\}.$$

∎

Theorem 7. *For every semi-hamiltonian graph G,*

$$b(C_n \otimes G) \leq \max\{4, b(G) + 2\}.$$

Proof: Using Theorem 5, one may show, as in the proof of Theorem 6, that

$$B(C_n \otimes P_m) \cup \left(\bigcup_{i=0}^{n-1} B_r(G^i)\right)$$

is a basis for $C_n \otimes G$ of fold at most

$$\max\{4, b(G) + 2\},$$

in which m is the order of G. ∎

Corollary 1. *For $m,n \geq 9$, $b(P_n \otimes W_m) = 4$, where W_m is a wheel of order m.*

Proof: From Theorem 6, $b(P_n \otimes W_m) \leq 4$. Suppose $b(P_n \otimes W_n) = 3$ and $B_r(P_n \otimes W_m)$ is a 3-fold basis. It is clear that

$$|E(P_n \otimes W_m)| = m^2 n + 2mn - m^2 - 2n,$$
$$\gamma(P_n \otimes W_m) = m^2 n + mn - m^2 - 2n + 1.$$

The number of 3-cycles in $B_r(P_n \otimes W_m)$ is at most $3n(2m-2)$. Therefore,

$$4(m^2 n - m^2 - 5mn + 4n + 1) + 3.3n(2m-2) \leq 3(m^2 n + 2mn - m^2 - 2n),$$

that is

$$m^2 n + 4n + 4 \leq m^2 + 8mn.$$

The above inequality does not hold for $m,n \geq 9$. Hence, for $m,n \geq 9$, $b(P_n \otimes W_m) = 4$. ∎

Corollary 2. *For $m,n \geq 8$, $b(C_n \otimes W_m) = 4$.*

The proof is similar to that of Corollary 1.

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