A Family of Inequalities and the Sparsity of Imprimitive Matrices

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1. Introduction.

A square matrix A is *cogredient* to the matrix E, if for some permutation matrix P we have $PAP^t = E$. A matrix is *reducible* if it is cogredient to a matrix of the form $\binom{B\ O}{C\ D}$, where B and D are square matrices. Otherwise it is *irreducible* (see [1]). A nonnegative, irreducible matrix is *primitive* if some power of it is positive; otherwise it is termed *imprimitive*. The *index of imprimitivity* d of a nonnegative irreducible matrix A is the number of eigenvalues of A of maximum modulus. A positive d is ensured by the Perron-Frobenius Theorem [1], [4], and A is primitive if and only if $d = 1$, and imprimitive if $d > 1$.

Let A be an irreducible, imprimitive matrix with index of imprimitivity d. It is well known that A is cogredient to

$$\begin{pmatrix} 0 & A_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_d & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

where the zero blocks along the diagonal are square (see, for example, [1, p. 32]). We shall refer to such a matrix as being in *Frobenius Normal Form*.

Imprimitive matrices are widely discussed in [1], [4], [5], [7], [8], and others. In [6] it was shown, by using matrix inequalities, that an irreducible matrix having more positive than zero elements is necessarily primitive. Brualdi [2] noted that this result follows immediately from the Frobenius Normal Form of an imprimitive matrix.

Proceeding along this line of thought we wish to consider a nonnegative, irreducible matrix assuming the knowledge of its index of imprimitivity.

We shall introduce a family of inequalities that are interesting in themselves, from which the results on imprimitive matrices will easily follow.

2. Some inequalities for a specific rational function of positive real numbers.

Let t be a positive integer and let $x_1, x_2, \dots, x_t$ be a sequence of positive, real numbers. Then

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Lemma 1. *For $t \leq 4$, we have*

$$\left(\sum_{i=1}^{t} x_i\right)^2 \Big/ \sum_{i=1}^{t} x_i x_{i+1} \geq t$$

where i is taken modulo t.

Proof: For $t = 1$ the lemma is trivially true. For $t = 2$ we get, applying the Arithmetic-Geometric-Mean inequality,

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \geq 4x_1x_2.$$

Let $t = 3$. Then

$$\begin{aligned} 2x_1^2 + 2x_2^2 + 2x_3^2 &= \left(x_1^2 + x_2^2\right) + \left(x_2^2 + x_3^2\right) + \left(x_3^2 + x_1^2\right) \\ &\geq 2x_1x_2 + 2x_2x_3 + 2x_3x_1 \end{aligned}$$

so that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1.$$

Adding $2x_1 x_2 + 2x_2x_3 + 2x_3x_1$ to both sides of the inequality we obtain the desired result.

Now put $t = 4$. We have

$$\begin{aligned} &(x_1 + x_2 + x_3 + x_4)^2 \\ &\quad = (x_1 - x_2 + x_3 - x_4)^2 + 4\,(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) \\ &\quad \geq 4\,(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) \end{aligned}$$

and the result follows. Moreover, equality holds if and only if $x_1 + x_3 = x_2 + x_4$. Lemma 1 is thus proved. ∎

We now have

Lemma 2. *Let $t \geq 5$. Then*

$$\left(\sum_{i=1}^{t} x_i\right)^2 \Big/ \sum_{i=1}^{t} x_i x_{i+1} > 4.$$

Proof: Let $x_1, \dots, x_t$ be positive real numbers and let

$$G(x_1, \dots, x_t) = \left(\sum_{i=1}^{t} x_i\right)^2 - 4\sum_{i=1}^{t} x_i x_{i+1}.$$

Note that Lemma 1 implies that $G(x_1, \dots, x_t) \geq 0$ for $t = 4$. We now show by induction on t that $G(x_1, \dots, x_t) > 0$ for $t \geq 5$. If all the x_i's are equal, then $G(x_1, \dots, x_t) = t^2 x_1^2 - 4t x_1^2 > 0$. Hence, we may assume that there exist a j with $x_j < x_{j-1}$ (where the indices are read modulo t). But the expression $G(x_1, \dots, x_t)$ is invariant under cyclic rotation of the arguments and, hence, without loss of generality we may assume $j = 2$. Then

$$\begin{aligned} 0 &\leq G(x_1, x_2 + x_3, x_4, \dots, x_t) \\ &= G(x_1, \dots, x_t) - 4x_1x_3 - 4x_2x_4 + 4x_2x_3 \\ &= G(x_1, \dots, x_t) - 4x_3(x_1 - x_2) - 4x_2x_4 \\ &< G(x_1, \dots, x_t). \end{aligned}$$

The result now follows by induction. ∎

We may now combine Lemma 1 and Lemma 2 and state

Lemma 3. *Let t be a positive integer and let $x_1, x_2, \dots, x_t$ be a sequence of positive real numbers. Then*

$$\left(\sum_{i=1}^{t} x_i\right)^2 \Big/ \sum_{i=1}^{t} x_i x_{i+1} \geq \min(4, t).$$

For $t < 5$ we may obtain equality; for $t \geq 5$ strict inequality prevails.

For $t \geq 5$ the lemma is the best possible as the following example shows. Put $x_1 = x_t = m, x_i = 1$ for $1 < i < t$. Put $Z = (2m+t-2)^2, N = m^2+2m+t-3$. It is clear that $\lim_{m\to\infty}(Z/N) = 4$, so that for $t \geq 5$ and positive real ε we may find an integer $n_0(\varepsilon)$ such that for every $n > n_0$ we can produce a sequence $x_1, x_2, \dots, x_t$ of positive integers for which

$$X = \sum_{i=1}^{t} x_i = n$$

and

$$4 < X^2 \Big/ \sum_{i=1}^{t} x_i x_{i+1} < 4 + \varepsilon.$$

If in the above example we choose $t = 4$, we get $Z/N = 4$ for every positive integer m.

3. The Matrix Sparsity results.

Let A be a nonnegative matrix and let $\sigma(A)$ denote the number of positive entries in A.

Considering the Frobenius Normal Form of a nonnegative,irreducible matrix of order n and index of imprimitivity d, where the zero blocks of the diagonal are square of orders $k_1, k_2, \ldots, k_d$ and speculating on the possible number of positive entries in the given matrix, we immediately come to the conclusion that

$$\sigma(A) \leq k_1 k_2 + k_2 k_3 + \ldots + k_{d-1} k_d + k_d k_1 .$$

We may now state

Theorem 1. *Let A be an irreducible matrix of order n and index of imprimitivity $d \leq 4$. Then*

$$\sigma(A) \leq n^2/d. \tag{1}$$

Proof: The theorem follows from Lemma 1 and the fact that cogredient matrices have the same number of positive elements and the same index of imprimitivity.

Let $d = 3$. Put $n = 3m + \delta$ with $\delta = 0, 1, 2$. If $\delta = 0$, let all the diagonal zero blocks be $m \times m$, so that clearly $\sigma(A) = n^2/3$. We now assume δ to be nonzero. Let $a = m$, $b = m + \delta - 1$, $c = m + 1$. Then $n = a + b + c$. Since $\delta < 3$, we have $\delta^2/3 < \delta$ and so, the zero blocks being of orders a, b and c, we may get $\sigma(A) = ab + bc + ca = 3m^2 + 2m\delta + \delta - 1 > 3m^2 + 2m\delta + \delta^2/3 - 1 = (3m + \delta)^2/3 - 1 = n^2/3 - 1$. We thus get $\sigma(A) = \lfloor n^2/3 \rfloor$ where $\lfloor s \rfloor$ denotes the greatest integer not exceeding s.

Leaving similar considerations for the cases $d = 2$ and $d = 4$ to the reader we are now in the position to strengthen Theorem 1 by stating

Theorem 1'. *Let A be an irreducible matrix of order n and index of imprimitivity $d \leq 4$. Then*

$$\sigma(A) \leq \lfloor n^2/d \rfloor . \tag{2}$$

Equality in (2) may be obtained for every n and d, $1 \leq d \leq 4$.

From Theorem 1 follows immediately

Corollary. *[6, Theorem 1] A nonnegative, irreducible matrix having more positive than zero entries is necessarily primitive.*

Let $A = (a_{ij})$ with $a_{ij} \neq 0$ if and only if $j = i + 1$ modulo n, where n is the order of the matrix. We shall refer to such a matrix as a full cycle matrix. It is easily seen that a full cycle matrix A of order n is imprimitive with index of imprimitivity $d = n$, so that $\sigma(A) = n^2/d$; and yet inequality (1) no longer holds in the general case for $n \geq 5$. A counterexample of smallest order is the following matrix of order 7.

$$A_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix A_7 is irreducible with index of imprimitivity $d = 5$. But $\sigma(A_7) = 10$ and $n^2/d = 49/5 < 10 = \sigma(A_7)$.

Theorem 2. *Let the conditions for A be as stated in Theorem 1 except for d which will be assumed greater than 4. Then*

$$\sigma(A) < n^2/4. \tag{3}$$

Proof: Apply Lemma 2. ∎

As previously noted inequality (3) is the best possible as for every positive, real ε there exists a positive integer $n_0(\varepsilon)$ and an infinite sequence of matrices $A_i(\varepsilon)$ of order i and index of imprimitivity $d > 4$ such that for $n \geq n_0(\varepsilon)$ we get $n^2/(4+\varepsilon) < \sigma(A_n) < n^2/4$.

Remark.

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