Interval Orders and Linear Extension Cycles

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Abstract. Let $p(x > y)$ be the probability that a random linear extension of a finite poset has x above y. Such a poset has a LEM (linear extension majority) cycle if there are distinct points $x_1, x_2, \ldots, x_m$ in the poset such that $p(x_1 > x_2) > 1/2, p(x_2 > x_3) > 1/2, \ldots, p(x_m > x_1) > 1/2$. We settle an open question by showing that interval orders can have LEM cycles.

1. Introduction.

For all distinct x and y in a finite poset $P = (X, >)$, let $p(x > y)$ be the proportion of linear extensions of P in which x is greater than y. We define the linear extension majority relation $>_*$ on X for P by

$$x >_* y \text{ if } p(x > y) > 1/2$$

and say that P has a LEM *cycle* if there are distinct $x_1, x_2, \ldots, x_m$ in X for which

$$x_1 >_* x_2, x_2 >_* x_3, \ldots, x_m >_* x_1.$$

We have learned recently from Ivan Rival (see [14] for reference) that $>_*$ appeared in the work of S.S. Kislitsyn as early as 1967. Kislitsyn conjectured that $>_*$ is transitive, in which case LEM cycles could not arise and $>_*$ would join other methods of constructing representative linear or weak order extensions of finite posets [8]. The fact that $>_*$ can be intransitive seems to have been demonstrated first by Fishburn [5] in 1974. Since then, LEM cycles have been shown to occur in many types of posets. Our purpose here is to show that these include interval orders, thus settling an open question in Gehrlein and Fishburn [12]. We prove this shortly, but first give some background to place it in perspective.

We recall that the width W of a finite poset P is the maximum cardinality of an antichain in P; the height H of P is the maximum cardinality of a chain in P minus 1; and the dimension D of P is the minimum cardinality of a set of linear extensions of P whose intersection equals P. P is an *interval order* if it has no pair of order-disjoint 2-point chains, so $x > b$ or $a > y$ whenever $x > y$ and $a > b$, and is a *semiorder* if it is an interval order that has no 3-point chain that is order-disjoint from a fourth point. Standard representation theorems [3, 6, 15] say that:

T1. P is an interval order if and only if there is a map f from X into closed real intervals such that $x > y \Leftrightarrow \min f(x) > \max f(y)$;

T2. P is a semiorder if and only if there is a map f from X into *unit-length* closed real intervals such that $x > y \Leftrightarrow \min f(x) > \max f(y)$;

T3. P has $D \leq 2$ if and only if there is a map g from X into closed real intervals such that $x > y \Leftrightarrow g(x)$ properly includes $g(y)$.

Because of T3, $D \leq 2$ posets are also referred to as interval inclusion orders. Semiorders always have $D \leq 3$ [13], but interval orders can have arbitrarily large dimensions [2].

2. LEM cycles.

Since the discovery of the existence of LEM cycles in finite posets, efforts have been made to determine the simplest types of posets that exhibit the phenomenon. Gehrlein and Fishburn [11] used computer search to establish that LEM cycles cannot occur when $n \leq 8$, where $n = |X|$. They showed also that exactly five 9-point posets have LEM cycles, each on three points $(1 >_* 2 >_* 3 >_* 1)$. These are posets (a), (b), and (c), of Figure 1 and the inverses of (b) and (c). Poset (a), noted earlier in [7, 9], is a 3-dimensional width-3 height-2 poset with $p(1 > 2) = p(2 > 3) = p(3 > 1) = 80/159$. Posets (b) and (c) are 2-dimensional with (W, H) equal to $(3,4)$ and $(4,3)$ respectively. Gehrlein and Fishburn [12] and Gehrlein [10] identify other posets for $n \in \{10, 11, 12\}$ with LEM cycles. All of these have $H \geq 2$ and none is an interval order. Examples are shown in (d) - (f) of Figure 1.

The question of whether height-1 posets can have LEM cycles was settled affirmatively by Ewacha, Fishburn, and Gehrlein [4]. Their smallest example has $n = 15$: see Figure 1 (g). The analysis of [4] suggests that no smaller height-1 poset has a LEM cycle.

Along with the $n \leq 8$ posets, we know of two nontrivial types of posets that never have LEM cycles. They are semiorders [12] and width-2 posets. Results in [10, 12] invite the conjecture that interval orders also never have LEM cycles, but we now know that this is false.

Theorem. *There are 2-dimensional interval orders with as few as 25 points that have LEM cycles.*

In other words, there are interval orders that are also interval inclusion orders that have LEM cycles. We do not know whether $n = 25$ is best possible for this conclusion, but even if it were there might be interval orders with $D > 2$ and $n < 25$ that have LEM cycles.

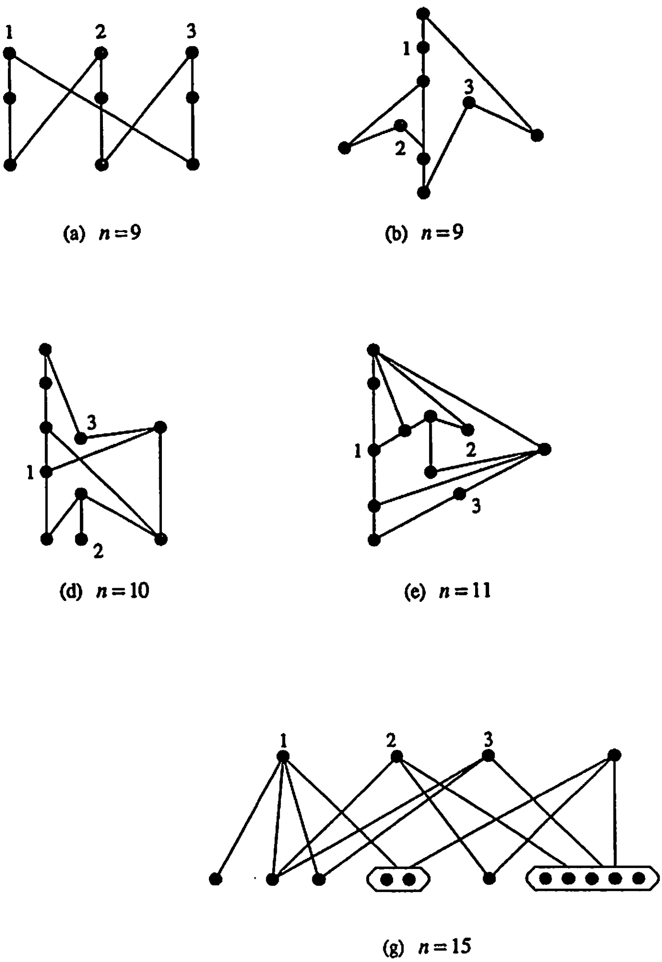

Figure 1. Posets with $1 >_* 2 >_* 3 >_* 1$.

3. Proof of Theorem.

Figure 2(i) pictures a 25-point poset P with isolated point 3 alongside a 24-point component. Antichains A, B, and C have 9, 5 and 6 points respectively, every point in A covers 1 and x, 1 covers every point in B, ... Figure 2(ii) and Figure 2(iii) show representations of P as an interval order and an interval inclusion order respectively. In these representations, all intervals for A (or B, or C) are identical.

The large component of the diagram was constructed to have 1 above 2 in a majority of its linear extensions, so $1 >_* 2$. However, when $1 > 2$, they are separated by at most x and y; when $2 > 1$, they can be separated by as many as seven points. Define the *height of a point in a linear extension* to be its bottom-up position, that is, 1 or 2 or $\cdots$. The separation differential for 1 and 2 allows 1's average height to be less than 2's. In our large component, $|A|$, and $|B|$ and $|C|$ where chosen to minimize $|A| + |B| + |C|$ subject to $1 >_* 2$, average height of 2 greater than the global average (12.5 for the 24-point component), and average height of 1 less than the global average. The latter two constraints imply that when point 3 is merged with the linear extensions of the large component, we get $2 >_* 3$ and $3 >_* 1$.

To compute p values we first replace A, B, and C by same-sized chains with no loss of generality. For $1 > 2$, 1 can fit between 2 and A in 2 ways, y can then fit into the 12-point chain above $B \cup C$ in 13 ways, B and C merge in $\binom{11}{6}$ ways, and 3 fits into a 24-point linear extension of the large component in 25 ways. Therefore, given the initial linearizations of A, B, and C, $1 > 2$ for $2(13)\binom{11}{6}25$ linear extensions. The number for $2 > 1$ is $\left[12\binom{12}{6} + \binom{11}{6}\right]25$. Similar computations for $2 > 3$ and $3 > 1$ lead to

$$p(1 > 2) = 26/51 = 0.509803 \ldots$$
$$p(2 > 3) = 214/425 = 0.503529 \ldots$$
$$p(3 > 1) = 298/595 = 0.500840 \ldots .$$

Hence, $1 >_* 2 >_* 3 >_* 1$.

4. Discussion.

A general conclusion of the research summarized here is that only certain very restrictive classes of posets, including those of semiorders and width-2 posets, are devoid of LEM cycles. Interval orders, height-1 posets, width-3 posets, and 2-dimensional posets can have LEM cycles. Two minor questions left open are the minimum $|X|$ for which a height-1 poset has a LEM cycle (≤ 15), and for which an interval order has a LEM cycle (≤ 25).

Other open problems abound. We conclude with two that are suggested by work on proportional transitivity [7] and on random posets [**1, 8, 10, 16, 17**].

(1) Determine sup min $\{p(1 > 2), p(2 > 3), p(3 > 1)\}$, where the sup is taken over all finite posets that contain points 1, 2, and 3. The largest

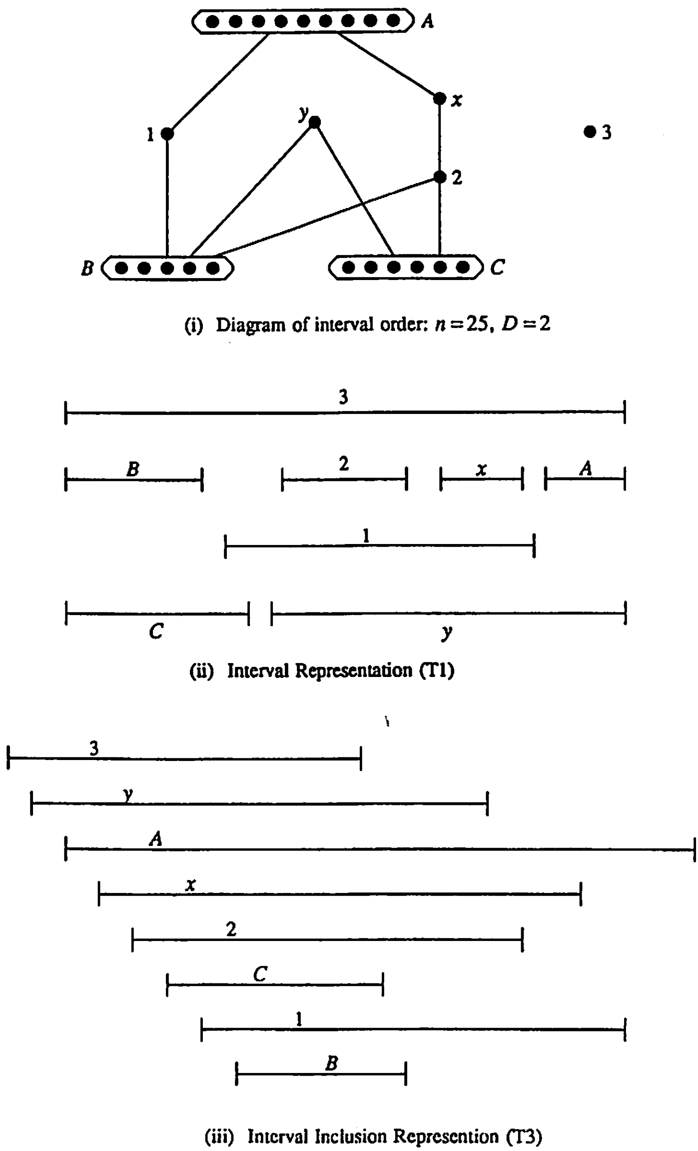

Figure 2. Interval order with $1 >_\star 2 >_\star 3 >_\star 1$.

lower bound on the sup that is presently known, about 0.54, follows from Theorem 3 in [7].

(2) Determine the limit as $n \to \infty$ of the probability that an n-point poset has a LEM cycle, or show that the limit does not exist. The probability model intended here assigns equal probability to each n-point poset (unlabeled). The question can also be posed for random posets on n points defined in other ways.

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