A PARTIAL SOLUTION TO A QUESTION RAISED BY R. L. GRAHAM

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Abstract. A group $(G, \cdot)$ with the property that, for a particular integer $r > 0$, every r-set S of G possesses an ordering, $s_1, s_2, \cdots, s_r$, such that the partial products $s_1, s_1s_2, \cdots, s_1s_2\cdots s_r$ are all different, is called an r-set-sequenceable group. We solve the question as to which abelian groups are r-set-sequenceable for all r except that, for $r = n-1$, the question is reduced to that of determining which groups are R-sequenceable.

The following question was raised by R. L. Graham [4].

"Given a group $(G, \cdot)$, of order n, for which integers $0 < r \leq n$, does every r-set of G have an ordering such that all partial products are different?"

For our discussion of this question, we shall find it convenient to make the following two definitions.

Definition 1. An ordering of the elements of a subset S of G, in which all the partial products are different, is called a *sequencing* of S in $(G, \cdot)$.

Definition 2. A group $(G, \cdot)$ with the property that, for a particular positive integer r, every r-set of G possesses a sequencing, is called an *r-set-sequenceable group.*

In terms of these definitions Graham's question is equivalent to asking: "For which values of r is a given group r-set-sequenceable?"

In this paper we solve this question for abelian groups, for all r, except that for $r = n - 1$ the question remains partly unsolved for some groups, namely those groups for which it is unknown whether the group is R-sequenceable. In particular, all groups are both 1-set and 2-set-sequenceable. The only abelian groups which are 3-set-sequenceable are the generalised Klein groups. For $3 < r < n-2$, no abelian group is r-set sequenceable. The only abelian groups of order $n > 4$ which are $(n-2)$-set-sequenceable are the generalised Klein groups. It is well known that a group is n-set-sequenceable if and only if it contains a unique element of order two. We will also show that a group is $(n-1)$-set-sequenceable if and only if it is R-sequenceable. Regarding non-abelian groups we observe that, if $(G, \cdot)$ is a non-abelian group which contains a subgroup which is isomorphic to an abelian group $(H, \cdot)$, then for those values of r for which $(H, \cdot)$ is not r-set-sequenceable neither is $(G, \cdot)$.

We shall need the following further definitions and lemmas.

Definition 3. A finite group $(G, \cdot)$ of order n is said to be *sequenceable* if its elements can be ordered in a sequence, $a_1, a_2, a_3, \cdots, a_n$, such that all of the partial products $b_1 = a_1$, $b_2 = a_1a_2$, $b_3 = a_1a_2a_3, \cdots, b_n = a_1a_2a_3\cdots a_n$ are different.

Definition 4. A finite group $(G, \cdot)$ of order n is said to be *R-sequenceable* if its elements can be ordered in a sequence, $a_1, a_2, a_3, \cdots, a_n$, such that all of the partial products $b_1 = a_1$, $b_2 = a_1a_2$, $b_3 = a_1a_2a_3, \cdots, b_{n-1} = a_1a_2a_3\cdots a_{n-1}$ are different and so that the product $b_n = a_1a_2a_3\cdots a_n = e$.

Lemma 1 (Miller [5]). If $(G, \cdot)$ is an abelian group of order n, then $\prod_{i=1}^{n} g_i = e$, unless $(G, \cdot)$ possesses a unique element of order two, say g', in which case $\prod_{i=1}^{n} g_i = g'$.

[Lemma 1 has been rediscovered on several occasions. For an historical note, see [1].]

Lemma 2. If $e \in S$ and S possesses a sequencing $s_1, s_2, \cdots, s_r$, then $s_1 = e$.

Lemma 3. If S is an r-set of G such that $e \in S$ and $\prod_{i=1}^{r} s_i = e$, then, in any ordering of S, there must be a repeated partial product.

Lemma 4. An abelian group $(G, \cdot)$ of order n is n-set-sequenceable if and only if it possesses a unique element of order two.

Proof. By definition, $(G, \cdot)$ is n-set-sequenceable if and only if $(G, \cdot)$ is sequenceable. By Gordon [3], an abelian group is sequenceable if and only if it possesses a unique element of order two. □

Theorem 1. An abelian group $(G, \cdot)$ of order n is $(n-1)$-set-sequenceable if and only if $(G, \cdot)$ is R-sequenceable.

Proof. If $(G, \cdot)$ possesses a unique element g' of order two, then, by Lemmas 1 and 3, the $(n-1)$-set $G\backslash\{g'\}$ possesses no sequencing. Thus, using Lemma 4, if $(G, \cdot)$ is sequenceable, then it is not $(n-1)$-set-sequenceable. Hence, if $(G, \cdot)$ is $(n-1)$-set-sequenceable, then $(G, \cdot)$ does not possess a unique element of order two and so $\prod_{i=1}^{n} g_i = e$, by Lemma 1.

Now suppose $(G, \cdot)$ is $(n-1)$-set-sequenceable, then $\forall$ $(n-1)$-sets S $\exists$ an ordering, say $s_1, s_2, \cdots, s_{n-1}$, which forms a sequencing of S. If the omitted element is the identity, then the ordering $e, s_1, s_2, \cdots, s_{n-1}$ is an R-sequencing of $(G, \cdot)$. If, however, the omitted element s_n is not the identity, then the ordering $s_1, s_2, \cdots, s_{n-1}, s_n$ is an R-sequencing of $(G, \cdot)$.

Conversely, suppose $(G, \cdot)$ is R-sequenceable and let $a_1, a_2, a_3, \cdots, a_n$ be one such R-sequencing. Then $a_1, a_{i+1}, a_{i+2}, \cdots, a_n, a_2, \cdots, a_{i-1}, a_i$ is also an R-sequencing of $(G, \cdot)$ for all i and so $a_1, a_{i+1}, a_{i+2}, \cdots, a_n, a_2, \cdots, a_{i-1}$ is a sequencing for the $(n-1)$-set $G\backslash\{a_i\}$. To complete the proof we observe that $a_2, a_3, \cdots, a_n$ is a sequencing for $G\backslash\{e\}$. □

Theorem 1 raises the question: which finite abelian groups are R-sequenceable? We know that a necessary condition for R-sequenceability is that the group does not possess a unique element of order two. It has been conjectured that this condition is also sufficient. In [2], Friedlander, Gordon and Miller showed that the following types of abelian group are R-sequenceable:

(i) Cyclic groups of odd order;

(ii) Abelian groups of odd order whose Sylow 3-subgroup is cyclic;

(iii) Abelian groups of orders which are relatively prime to six;

(iv) Elementary abelian p-groups, except C_2;

(v) Abelian groups of type $C_2 \times C_{4k}$;

(vi) Abelian groups whose Sylow 2-subgroup S is one of the following kinds:

(a) $S = (C_2)^m$, $m > 1$ but $m \neq 3$.

(b) $S = C_2 \times C_h$, where $h = 2^k$ and either k is odd or else $k \geq 2$ is even and G/S has a direct cyclic factor of order congruent to 2 modulo 3.

Also Ringel [6] has claimed that abelian groups of type $C_2 \times C_{6k+2}$ are R-sequenceable.

It is easy to see that all groups are both 1-set and 2-set-sequenceable. To consider r-set-sequenceability of abelian groups for $2 < r < n - 1$, we examine several cases.

Theorem 2. If $(G, \cdot)$ is an abelian group of order n, where either n is odd or $(G, \cdot)$ has a unique element η of order two, then it is not r-set-sequenceable for $2 < r < n - 1$.

Proof. For r odd, $2 < r < n - 1$, the r-set $S = \{e, g_1, g_1^{-1}, g_2, g_2^{-1}, \cdots, g_{(r-1)/2}, g_{(r-1)/2}^{-1}\}$ (where if n is even, then $\eta \notin S$) possesses no sequencing, by Lemma 3.

For r even, $2 < r < n - 1$, the r-set $S = \{e, g_1, g_2, (g_1g_2)^{-1}, g_3, g_3^{-1}, \cdots, g_{r/2}, g_{r/2}^{-1}\}$, where if n is odd, then $S \cap \{g_1^{-1}, g_2^{-1}, g_1g_2\} = \emptyset$ and if n is even, then $g_1 = \eta$ and $S \cap \{g_2^{-1}, \eta g_2\} = \emptyset$, possesses no sequencing, by Lemma 3. [Note that for this construction when n is odd we require $r \leq n - 3$ but, since $n - 2$ is odd, this is sufficient for all even values of $r < n - 1$.] □

Theorem 3. If $(G, \cdot)$ is an abelian group of order n which possesses more than one element of order two and at least one non-identity element not of order two, then $(G, \cdot)$ is not r-set-sequenceable for $2 < r < n - 1$.

Proof. $(G, \cdot)$ must possess at least $n/2$ non-identity elements not of order two since the elements of order two, together with the identity, form a subgroup of index at least two in $(G, \cdot)$.

For r odd, $2 < r \leq (n/2) + 1$, the r-set $S = \{e, g_1, g_1^{-1}, g_2, g_2^{-1}, \cdots, g_{(r-1)/2}, g_{(r-1)/2}^{-1}\}$ possesses no sequencing, by Lemma 3, where none of the elements of S has order two.

For r even, $2 < r \leq (n/2) + 1$, the r-set $S = \{e, g_1, g_2, g_1g_2, g_3, g_3^{-1}, \cdots, g_{r/2}, g_{r/2}^{-1}\}$, possesses no sequencing, by Lemma 3, where g_1, g_2 and hence g_1g_2 are of order two whilst the remaining elements of S are not.

For r, when $n/2 < r < n-1$, the r-set $S = (G \backslash S') \cup \{e\}$, where S' is an $(n-r+1)$-set of the appropriate one of the two types constructed above, possesses no sequencing, by Lemmas 1 and 3. □

Theorem 4. If $(G, \cdot)$ is an abelian group of order $n \geq 4$ in which every non-identity element has order two, then $(G, \cdot)$ is 3-set-sequenceable.

Proof. Any 3-set S must possess a sequencing since, if we consider the ordering s_1, s_2, s_3, we obtain the partial products $s_1, s_1s_2, s_1s_2s_3$. These partial products must all be distinct since $s_1 = s_1s_2 \Leftrightarrow s_2 = e$ and $s_1s_2 = s_1s_2s_3 \Leftrightarrow s_3 = e$, but, by Lemma 2, $s_1 = e$ when $e \in S$. The remaining possibility is $s_1 = s_1s_2s_3$, but $s_1 = s_1s_2s_3 \Leftrightarrow s_2s_3 = e$, contrary to the hypothesis that every element has order two. □

Theorem 5. If $(G, \cdot)$ is an abelian group of order $n \geq 8$ in which every non-identity element has order two, then $(G, \cdot)$ is not r-set-sequenceable for $3 < r < n-2$ and for $r = n$.

Proof. The proof is by induction on the size of a minimal generating set of $(G, \cdot)$. We first note that, since every element has order 2, $(G, \cdot)$ must be the generalised Klein group of order $n = 2^m$, for some m. We denote a minimal set of generators of $(G, \cdot)$ by $\{c_1, c_2, \cdots, c_m\}$.

As our starting point for the induction argument we take the generalised Klein group of order eight. Now, for $r = 4$, the set $S = \{e, c_1, c_2, c_1c_2\}$ has no sequencing and, for $r = 5$, the set $S = \{e, c_1, c_2, c_3, c_1c_2c_3\}$ has no sequencing, also, for $r = 8$, the group has no sequencing, by Lemma 4. Thus for $3 < r < 6$ and $r = 8$, $\exists$ r-sets S which possess no sequencing.

We now consider $(G, \cdot)$ to be an arbitrary generalised Klein group of order 2^m, $m > 3$, and assume that, for the generalised Klein group of order 2^{m-1}, the theorem is true. That is, $\exists$ r-sets S for which no sequencing is possible when $3 < r < 2^{m-1} - 2$ and $r = 2^{m-1}$. Since the set $\{c_1, c_2, \cdots, c_{m-1}\}$ generates a subgroup $(H, \cdot)$ of $(G, \cdot)$ isomorphic to the generalised Klein group of order 2^{m-1}, $\exists$, by assumption, r-sets S of G for which no sequencing exists $\forall$ r when $3 < r < 2^{m-1} - 2$ and $r = 2^{m-1}$. Furthermore, the product of all the elements in H is the identity element, by Lemma 1. It follows that:

for $r = 2^{m-1} - 1$, the set $S = H \backslash \{c_2, c_3, c_1c_2c_3\} \cup \{c_2c_m, c_1c_2c_m\}$ possesses no sequencing, by Lemma 3;

for $r = 2^{m-1} - 2$, the set $S = H \backslash \{c_1, c_1c_2, c_1c_3, c_2c_3\} \cup \{c_2c_m, c_1c_2c_m\}$ possesses no sequencing, by Lemma 3.

Thus $\exists$ r-sets S of G which possess no sequencing for $3 < r \leq 2^{m-1}$. For $2^{m-1} < r < 2^m-2$, the set $S = (G \backslash S') \cup \{e\}$ possesses no sequencing, where S' is an $(n-r+1)$-set of the appropriate one of the two types constructed above.

We have shown that the generalised Klein group $(G, \cdot)$ of order $n = 2^m$, $m \geq 3$, is not r-set-sequenceable for r, when $3 < r < 2^m - 2$. Furthermore, since $(G, \cdot)$ is not sequenceable, by Lemma 1, it is not n-set-sequenceable. □

In [2], Friedlander, Gordon and Miller give the following construction for an R-sequencing of the additive group of $GF(p^m) \neq Z_2$, where $\ell = p^m - 2$ and α is a generator of the (cyclic) multiplicative group.

0		$1-\alpha$		$\alpha-\alpha^2$		$\cdots$		$\alpha^{\ell-1}-\alpha^{\ell}$		$\alpha^{\ell}-1$
	0		$1-\alpha$		$1-\alpha^2$		$\cdots$		$1-\alpha^{\ell}$	0

Here the top row contains the sequencing whilst the bottom row contains the partial "products". Since we are dealing with the additive group of a field the "products" are in fact sums in this case. We make use of this result in the following theorem.

Theorem 6. If $(G, \cdot)$ is an abelian group of order $n \geq 8$ in which every non-identity element has order two, then $(G, \cdot)$ is $(n-2)$-set-sequenceable.

Proof. For any given pair of non-identity elements $\{x, y\}$ $\exists$ a minimal set of generators C such that $\{x, y\} \subset C$. Hence, given any R-sequencing $a_1, a_2, \cdots, a_{n-2}, a_{n-1}, a_n$ of $(G, \cdot)$, we can always find an isomorphism θ of $(G, \cdot)$ such that $\theta(a_{n-1}) = x$ and $\theta(a_n) = y$. Then, since the ordering $\theta(a_1), \theta(a_2), \cdots, \theta(a_{n-2}), \theta(a_{n-1}), \theta(a_n)$ forms an R-sequencing of $(G, \cdot)$, the ordering $\theta(a_1), \theta(a_2), \cdots, \theta(a_{n-2})$ forms a sequencing of the set $G \backslash \{x, y\}$. To complete the proof we note that $\theta(a_2), \theta(a_3), \cdots, \theta(a_{n-2}), \theta(a_{n-1})$ is a sequencing of $G \backslash \{e, y\}$. □

The question of r-set-sequenceability of the Klein group itself is trivial, since, for $r = 1, 2, 3$, we have that all r-sets possess a sequencing whilst for $r = 4$ no such sequencing is possible, by Lemma 1.

References

[1] J. Dénes and A. D. Keedwell, *A new conjecture concerning the admissibility of groups*, European J. Combin. 10 (1989), 171-174.

[2] R. J. Friedlander, B. Gordon and M. D. Miller, *On a group sequencing problem of Ringel,* Proc. Ninth S.E. Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium 21 (1978), 307-321.

[3] B. Gordon, *Sequences in groups with distinct partial products,* Pacific J. Math. 11 (1961), 1309-1313.

[4] R. L. Graham, personal communication to A. D. Keedwell.

[5] G. A. Miller, *A new proof of the generalized Wilson's theorem,* Ann. Math., 4 (1903), 188-190.

[6] G. Ringel, *Cyclic arrangements of the elements of a group,* AMS Notices 21 (1974), A95-96.

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