Partitions for Quadruples

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Abstract. Partitions of all quadruples of an n-set into pairwise disjoint packings with no common triples, have applications in the design of constant weight codes with minimum Hamming distance 4. Let $\theta(n)$ denote the minimal number of pairwise disjoint packings, for which the union is the set of all quadruples of the n-set. It is well known that $\theta(n) \geq n-3$ if $n \equiv 2$ or $4 \pmod 6$, $\theta(n) \geq n-2$ if $n \equiv 0, 1$ or $3 \pmod 6$, and $\theta(n) \geq n-1$ for $n \equiv 5 \pmod 6$. $\theta(n) = n-3$ implies the existence of a large set of Steiner quadruple systems of order n. We prove that $\theta(2^k) \leq 2^k - 2$, $k \geq 3$, and if $\theta(2n) \leq 2n-2$, $n \equiv 2$ or $4 \pmod 6$, then $\theta(4n) \leq 4n-2$. Let $D(n)$ denote the maximum number of pairwise disjoint Steiner quadruple systems of order n. We prove that $D(4n) \geq 2n + \min\{D(2n), n-2\}$ for $n \equiv 1$ or $5 \pmod 6$, $n > 7$, and $D(28) \geq 18$.

1. Introduction.

A *packing quadruple system* (PQ) of order n ($PQ(n)$) is a pair (Q, q) where $Q = (0, 1, \ldots, n-1\}$ is a set of n *points* and q is a collection of 4-element subsets of Q called *blocks* such that every 3-element subset of Q is a subset of at most one block of q. A PQ is *optimal* if there is no PQ of the same order with a larger size. A *Steiner quadruple system* (SQS) of order n ($SQS(n)$) is a ($PQ(n)$) such that every 3-element subset of Q is a subset of exactly one block of q. It is well known that an $SQS(n)$ exists if and only if $n \equiv 2$ or $4 \pmod 6$. It is clear that an $SQS(n)$ is an optimal PQ and it is well known that an $SQS(v)$, has $b_v = \frac{1}{4}\binom{v}{3}$ blocks. Hanani [7] proved that Steiner quadruple systems of order v exist if and only if $v \equiv 2$ or $4 \pmod 6$. Two SQSs (Q, q_1) and (Q, q_2) are *disjoint* if $q_1 \cap q_2 = \emptyset$. Let $D(v)$ denote the maximum number of *pairwise disjoint* SQSs (PDQSs) of order v. It is clear that $D(v) \leq v-3$ and a set of $v-3$ PDQSs of order v is called a *large set*. The main constructions of PDQSs are the $2v$ and the $3v$ constructions of Lindner [10], [11], the constructions of n mutually 2-chromatic PDQSs of order $2n$, n odd, of Phelps and Rosa [13], and the constructions of Etzion and Hartman [5]. Pairwise disjoint $PQ(n)$s can be represented in a graph whose vertex set is the set of all quadruples of the n-set. Two vertices are connected with an edge if the corresponding quadruples have a common triple. A coloring of the vertices partitions the quadruples into pairwise disjoint PQs of order n. Let $\theta(n)$ denote the chromatic number of this graph. It is clear that $\theta(n) = n-3$ for some n if and only if a large set of order n exists. Hence, $\theta(n) \geq n-3$ if $n \equiv 2$ or $4 \pmod 6$. It is well known (for example,

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[1]) that an optimal $PQ(n)$ for $n \equiv 1$ or $3 \pmod 6$ has size $n(n-1)(n-3)$ and, hence, $\theta(n) \geq n-2$. For $n \equiv 0 \pmod 6$ an optimal $PQ(n)$ has size $\frac{n(n^2-3n-6}{24}$ and, hence, $\theta(n) \geq n-2$. For $n \equiv 5 \pmod 6$ an optimal $PQ(n)$ has by the Johnson bound [7] at most size $\frac{(n-1)(n^2-3n-4)}{24} + \lfloor \frac{n-5}{12} \rceil$ and, hence, $\theta(n) \geq n-1$. Graham and Sloane [6] proved that $\theta(n) \leq n$. It is also well known that $\theta(7) = 6$. van Pul and Etzion [14] proved that $\theta(n) \leq n-1$ for $n = 2^i$, or $n = 3 \cdot 2^i$, $i \geq 1$, and if $\theta(2n) \leq 2n-1$ then $\theta(4n) \leq 4n-1$. Brouwer *et al* [2] proved that $\theta(n) \leq n-1$ for $n = 5 \cdot 2^i$, and $n = 7 \cdot 2^i, i \geq 1$.

Partitions of quadruples have applications in the construction of constant weight codes with minimum Hamming distance 4. Let $A(n, d, w)$ denote the maximum number of codewords in a binary code of length n, minimum Hamming distance d, and constant weight code w. A (n, d, w) is a fundamental combinatorial quantity, which is also used in the construction of codes for asymmetric channels, DC-free codes, and spherical codes [2]. It seems that the best known method to design constant weight codes with distance 4 is the partitioning method [2]. To apply this method we have to partition sets of n-tuples into disjoint constant weight codes of weight w and minimum Hamming distance 4. Partition of quadruples is, of course, partition of n-tuples with weight 4.

In Section 2 we present the $2v$ construction of Lindner and a variant of the partitioning method which results in a PQ. In Section 3 we show that the set of PDQSs of Lindner is not maximal, that is, it can be extended, and prove that $D(4n) \geq 2n + \min\{D(2n), n-2\}$ for $n \equiv 1$ or $5 \pmod 6$, $n > 7$, and $D(28) \geq 18$. In Section 4 we show that $\theta(2^k) \leq 2^k - 2$, $k \geq 3$, and if $\theta(2n) \leq 2n-2, n \equiv 2$ or $4 \pmod 6$, then $\theta(4n) \leq 4n-2$.

2. The Lindner construction.

Our constructions for PDQSs and partitions of quadruples use many kinds of latin squares and latin rectangles. For a $k \times n$, $k \leq n$, latin rectangle A, $A(i,j)$, $1 \leq i \leq k$, $1 \leq j \leq n$, denote cell (i,j) of A. Two of these latin squares are based on the table of the cyclic group [4]. These two $n \times n$ latin squares are denoted by A_n and B_n and defined by

$$A_n(i,j) \equiv i + j - 2 \pmod n$$
$$B_n(i,j) \equiv i - j \pmod n.$$

Figure 1 presents A_7 and B_7.

0	1	2	3	4	5	6		0	6	5	4	3	2	1
1	2	3	4	5	6	0		1	0	6	5	4	3	2
2	3	4	5	6	0	1		2	1	0	6	5	4	3
3	4	5	6	0	1	2		3	2	1	0	6	5	4
4	5	6	0	1	2	3		4	3	2	1	0	6	5
5	6	0	1	2	3	4		5	4	3	2	1	0	6
6	0	1	2	3	4	5		6	5	4	3	2	1	0

Figure 1: A_7 and B_7

Now, we present the Lindner construction [10] for constructing v SQSs of order $2v$ from an $SQS(v)$ and a $v \times v$ latin square. Let (Q, B) be an $SQS(v)$ with $Q = \{1, 2, \ldots, v\}$ and let V be a latin square of order v. Denote by α_i the permutation on $Q = \{1, 2, \ldots, v\}$ defined by $x\alpha_i = y$ if and only if $V(i, x) = y$. Set $S = Q \times \{1, 2\}$ and for each $i = 1, 2, \ldots, v$ define a collection of quadruples B_i on S as follows

(1) For each quadruple $[x, y, z, w] \in B$, the following 8 quadruples belong to B_i:

$$\begin{array}{ll}
[(x,1),(y,1),(z,1),(w\alpha_i,2)], & [(x,2),(y,2),(z,2),(w\alpha_i^{-1},1)], \\
[(x,1),(y,1),(z\alpha_i,2),(w,1)], & [(x,2),(y,2),(z\alpha_i^{-1},1),(w,2)], \\
[(x,1),(y\alpha_i,2),(z,1),(w,1)], & [(x,2),(y\alpha_i^{-1},1),(z,2),(w,2)], \\
[(x\alpha_i,2),(y,1),(z,1),(w,1)], & [(x\alpha_i^{-1},1),(y,2),(z,2),(w,2)],
\end{array}$$

(2) For each 2-element subset $[x, y]$ of Q, $[(x,1),(y,1),(x\alpha_i,2),(y\alpha_i,2)] \in B_i$.

If the latin square V contains no 2×2 subsquare then the v SQSs are v PDQSs of order $2v$. If the latin square has 2×2 subsquares then to obtain v pairwise disjoint PQs of order v we made the following change in (2). For each 2-element subset $[x, y]$ of Q, such that there is no j, $j < i$, with $V(i, x) = V(j, y)$ and $V(i, y) = V(j, x)$, $[(x,1),(y,1),(x\alpha_i,2),(y\alpha_i,2)] \in B_i$.

We will use the latin squares with no subsquares of order 2 which were defined by Kotzig, Lindner, and Rosa, [9]. Let $n \equiv 2 \pmod 4$, $n = 2k$, so that k is odd. Let C, D be $k \times k$ latin squares defined by

$C(i, j) = A_k(i, j)$ reduced modulo k to the range $\{k, k+1, k+2, \ldots, 2k-1\}$, and

$D(i, j) = A_k(i, j) - 1$ reduced modulo k to the range $\{k, k+1, k+2, \ldots, 2k-1\}$.

The square

$$M_k = \begin{bmatrix} B_k & C \\ D & B_k \end{bmatrix}$$

is a latin square of order n with no subsquare of order 2.

The following properties of M_k are significant for extending the v SQSs obtained by the $2v$ construction to a larger set (but not a large set) of PDQSs. The proof of the following lemma can be easily verified from the definition of M_k, A_k, and B_k.

Lemma 1. *M_k has the following properties.*

(1) $M_k(i, j) - M_k(i, s) \equiv s - j \pmod k$, $1 \leq i \leq 2k$, $1 \leq j < s \leq k$ *or* $k + 1 \leq j < s \leq 2k$.

(2) $M_k(i,j) - M_k(i,s) \equiv j + s - 2 \pmod{k}, 1 \leq i \leq k, 1 \leq s \leq k, k+1 \leq j \leq 2k.$

(3) $M_k(i,s) - M_k(i,j) \equiv s + j - 3 \pmod{k}, k+1 \leq i \leq 2k, 1 \leq s \leq k, k+1 \leq j \leq 2k.$

The set of PDQSs from Lindner construction is not maximal. To extend it we use a well known variant of the partitioning construction (for example, see Lindner and Rosa [12]). This variant results in an SQS. Let $(\mathcal{Q}_1, q_1)$ and $(\mathcal{Q}_2, q_2)$ be any two SQSs of order $2v$ where $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$. Let $F = \{F_1, F_2, \ldots, F_{2v-1}\}$ and $G = \{G_1, G_2, \ldots, G_{2v-1}\}$ be two one-factorizations of K_{2v} based on $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively, and let α be any permutation on the set $\{1, 2, \ldots, 2v-1\}$. Define a collection of blocks A on $\mathcal{Q}_1 \cup \mathcal{Q}_2$ as follows:

(B.1) Any block belonging to q_1 or $q2$ belongs to A; and

(B.2) If $x_1, x_2 \in \mathcal{Q}_1$ and $y_1, y_2 \in \mathcal{Q}_2$ then $[x_1, x_2, y_1, y_2] \in A$ if and only if $[x_1, x_2] \in F_i$, $[y_1, y_2] \in G_j$ and $i\alpha = j$.

We call this method *partitioning quadruple system* (PQS) construction. This method results in PQ which is not an SQS if either we use a PQ instead of an SQS, or that instead of a permutation α we use a partial permutation, where a *partial permutation* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2v-1})$, is defined by $\alpha_i \in \{0, 1, \ldots, 2v-1\}$ and all the nonzero α_i's are distinct. If $i\alpha = 0$ we ignore F_i in (B.2).

3. Extending the non-maximal set of PDQSs.

In order to extend the set of v PDQSs constructed in Section 2, we first describe the near-one-factorization F of K_n, $n \equiv 1$ or $5 \pmod{6}$, $n = 2k+1$. The following sets are the first disjoint near-one-factors of F.

$$F_1 = \{[i, i+k] : 0 \leq i \leq k-1\}$$
$$F_2 = \{[i, i+k+1] : 0 \leq i \leq k-1\}$$
$$F_3 = \{[2i-1, 2i] : 1 \leq i \leq k\}$$
$$F_4 = \{[2i, 2i+1] : 1 \leq i \leq k-1\} \cup \{[0, 2k]\}.$$

It is clear that $[a,b] \in F_1 \cup F_2$ implies that $a - b$ equal k or $n - k$ modulo n and $[a,b] \in F_3 \cup F_4$ implies that $a - b$ equal 1 or $n-1$ modulo n. The only other two pairs for which $a - b$ equal either k, $n-k$, 1, or $n-1$ modulo n are $[0,1]$ and $[k, 2k]$. Using the results of Chetwynd and Hilton [3] we have that these near-one-factors can be extended to near-one-factorizations for $n \geq 29$. If we extend these four near-one-factors by adding a near-one-factor F_{n-1} which contains $[k, 2k]$ but not $[0,1]$ (this can be easily done for $n \geq 11$, and for $n = 7$ let F_7 contains $[k, 2k]$ and $[0,1]$) then the theorem of Chetwynd and Hilton implies that these near-one-factors can be extended to a near-one-factorization F for $n \geq 33$ (In F let $[0,1]$ be in F_n.) We found out that this is true also for

$n = 19, 23, 29$, and 31, which are the only 4 parameters (except 7) below 33 that we need, since for all other n, $n < 33$, $n \equiv 1$ or $5 \pmod 6$ we have $D(4n) \geq 3n$ [5]. Given the near-one-factorization $F = \{F_1, F_2, \ldots, F_n\}$ of order $n \equiv 1$ or $5 \pmod 6$, where in F_i vertex f_i is isolated, we construct the following two one-factorizations G and H of K_{2n}.

Construction of G.

$$G_i = \{[a,b] : [a,b] \in F_i\} \cup \{[2n-1-a, 2n-1-b] : [a,b] \in F_i\} \cup \{[f_i, 2n-1-f_i]\}, \quad 1 \leq i \leq n.$$

$$G_{n+i} = \{[a,b] : 0 \leq a \leq n-1, n \leq b \leq 2n-1, a+b \equiv i-1 \pmod n\}, \quad 1 \leq i \leq n-1.$$

Construction of H.

$$H_i = \{[a,b] : [a,b] \in F_i\} \cup \{[n+a, n+b] : [a,b] \in F_i\} \cup \{[f_i, n+f_i]\}, \quad 1 \leq i \leq n.$$

$$H_{n+i} = \{[a,b] : 0 \leq a \leq n-1, n \leq b \leq 2n-1, b-a \equiv i \pmod n\}, \quad 1 \leq i \leq n-1.$$

Now, we apply the Lindner construction with the $(2n) \times (2n)$ latin square M_n and an SQS$(2n)$. By the structure of the one-factorizations G and H, and by Lemma 1, we can observe that the quadruples $[(a,1),(b,1),(c,2),(d,2)]$, $[a,b] \in G_i$, $[c,d] \in H_j$, which might have been used in the Lindner construction are for the following pairs of i and j.

(P.1) $1 \leq i \leq n$ with $j = 2n-2$ or $j = 2n-1$.
(P.2) $i = n+1$ or $i = n+2$ with $1 \leq j \leq n$.
(P.3) $i = n+1$ with $j = 2n-1$.
(P.4) $i = n+2$ with $j = n+1$.
(P.5) $n+3 \leq i \leq 2n-1$ with $j = i-1$ or $j = i-2$.

If $[0,1]$ and $[k,2k]$ are in the same near-one-factor (when $n = 7$) we have

(Q.1) $i = 1$ or $i = 2$ with $j = 1$ or $j = 2$ or $j = n$.
(Q.2) $i = 3$ or $i = 4$ with $j = 3$ or $j = 4$ or $j = n$.
(Q.3) $5 \leq i \leq n-1$ with $5 \leq j \leq n$.
(Q.4) $i = n$ with $1 \leq j \leq n$.

If $[0,1]$ and $[k,2k]$ are in two different near-one-factors we have

(R.1) $i = 1$ or $i = 2$ with $j = 1$ or $j = 2$ or $j = n-1$.
(R.2) $i = 3$ or $i = 4$ with $j = 3$ or $j = 4$ or $j = n$.
(R.3) $5 \leq i \leq n-2$ with $5 \leq j \leq n$.
(R.4) $i = n-1$ with $j = 1$ or $j = 2$ or $5 \leq j \leq n$.
(R.5) $i = n$ with $3 \leq j \leq n$.

We have to construct an $r \times (2n-1)$ latin rectangle β such that if $\beta(s,i) = j$ then the quadruple $[(a,1),(b,1),(c,2),(d,2)]$ for $[a,b] \in G_i$ and $[c,d] \in H_j$ does not appear in any quadruple of the first $2n$ PQs, that is, $\beta(s,i) \neq j$ for pairs i,j defined above by (P.1) through (P.5), (Q.1) through (Q.4), and (R.1) through (R.5). If $[0,1]$ and $[k,2k]$ are in the same near-one-factors we have $r = n-3$ while if $[0,1]$ and $[k,2k]$ are in two different near-one-factors we have $r = n-2$. In the Appendix we present the near-one-factorization, latin squares with no 2×2 subsquares and the 4×13 latin rectangle which are used to obtain $D(28) \geq 18$. Now we give the construction for the case where $[0,1]$ and $[k,2k]$ are in two different near-one-factors. To make the construction of this latin rectangle clearer we will build it in two steps. In the first step there will be some entries which are inconsistent with the requirements defined by (P.1) through (P.5) and (R.1) through (R.5). In the second step we will fix those entries to obtain our rectangle β.

Step 1: For $1 \leq i \leq n-2$

(1) For $j = 1,2,3$, $\alpha(i,j) = i+j+1$ reduced modulo n to the range $\{1,2,\ldots,n\}$.
(2) For $4 \leq j \leq n+2$, $\alpha(i,j) = j-i+n-2$ reduced modulo $n-1$ to the range $\{n+1, n+2, \ldots, 2n-1\}$.
(3) For $n+3 \leq j \leq 2n-1$, $\alpha(i,j) = i+j+2$ reduced modulo n to the range $\{1,2,\ldots,n\}$.

The only violations that we have are of the requirements defined by (P.1) for columns 4 through n and by (R.1) and (R.2) for columns 1, 2, and 3, and rows $n-4, n-3$, and $n-2$. We correct these in the next step.

Step 2: $\beta(i,j) = \alpha(i,j)$ except for the following cases.

(1) $\beta(2,4) = 2$ and $\beta(2,2n-2) = 2n-1$.
(2) For $3 \leq i \leq n-5$ if $\alpha(i,j) = 2n-2$ then $\beta(i,j) = 1$. If $\alpha(i,j) = 2n-1$ then $\beta(i,j) = 2$. If $\alpha(i,j) = 1$ then $\beta(i,j) = 2n-2$. If $\alpha(i,j) = 2$ then $\beta(i,j) = 2n-1$.
(3) $\beta(n-4,2) = n+2$, $\beta(n-4,3) = n+5$, $\beta(n-4,4) = n-1$, $\beta(n-4,n-3) = 3$, $\beta(n-4,n-2) = 1$, $\beta(n-4,n) = 2$, $\beta(n-4,n+3) = 2n-2$, $\beta(n-4,2n-2) = n$, $\beta(n-4,2n-1) = 2n-1$, and for $n+4 \leq j \leq 2n-3$, $\beta(n-4,j) = j$ reduced modulo n to the range $\{1,2,\ldots,n\}$.
(4) $\beta(n-3,1) = n+4$, $\beta(n-3,4) = 5$, $\beta(n-3,n-2) = 4$, $\beta(n-3,n-1) = 3$, $\beta(n-3,n+3) = 2$, $\beta(n-3,n+4) = 2n-1$, $\beta(n-3,2n-2) = 2n-2$, $\beta(n-3,2n-1) = n-1$, and for $n+5 \leq j \leq 2n-3$, $\beta(n-3,j) = j+1$ reduced modulo n to the range $\{1,2,\ldots,n\}$.
(5) $\beta(n-2,1) = n+3$, $\beta(n-2,2) = 3$, $\beta(n-2,4) = 6$, $\beta(n-2,n-1) = 4$, $\beta(n-2,n) = 1$, $\beta(n-2,n+3) = 2n-1$, $\beta(n-2,n+4) = 5$, $\beta(n-2,2n-3) = 2n-2$, $\beta(n-2,2n-2) = n-1$, $\beta(n-2,2n-1) =$

n, and for $n+5 \leq j \leq 2n-4$, $\beta(n-2,j) = j+2$ reduced modulo n to the range $\{1,2,\dots,n\}$.

We leave to the reader to check that the β is a latin rectangle which fulfill all the requirements. We use the r rows of the $r \times (2n-1)$ latin rectangle β as the r permutations for the PQS construction (each point a of G is changed to $(a,1)$ and each point b of H is changed to $(b,2)$). Therefore, we have the following theorem.

Theorem 1. *$D(4n) \geq 2n + \min\{D(2n), n-2\}$, $n \equiv 1$ or 5 (mod 6) and $D(28) \geq 18$.*

Figure 2 and Figure 3 present the latin rectangles α and β, for $n = 11$.

3	4	5	12	13	14	15	16	17	18	19	20	21	6	7	8	9	10	11	1	2
4	5	6	21	12	13	14	15	16	17	18	19	20	7	8	9	10	11	1	2	3
5	6	7	20	21	12	13	14	15	16	17	18	19	8	9	10	11	1	2	3	4
6	7	8	19	20	21	12	13	14	15	16	17	18	9	10	11	1	2	3	4	5
7	8	9	18	19	20	21	12	13	14	15	16	17	10	11	1	2	3	4	5	6
8	9	10	17	18	19	20	21	12	13	14	15	16	11	1	2	3	4	5	6	7
9	10	11	16	17	18	19	20	21	12	13	14	15	1	2	3	4	5	6	7	8
10	11	1	15	16	17	18	19	20	21	12	13	14	2	3	4	5	6	7	8	9
11	1	2	14	15	16	17	18	19	20	21	12	13	3	4	5	6	7	8	9	10

Figure 2: The latin rectangle α for $n = 11$

3	4	5	12	13	14	15	16	17	18	19	20	21	6	7	8	9	10	11	1	2
4	5	6	2	12	13	14	15	16	17	18	19	20	7	8	9	10	11	1	21	3
5	6	7	1	2	12	13	14	15	16	17	18	19	8	9	10	11	20	21	3	4
6	7	8	19	1	2	12	13	14	15	16	17	18	9	10	11	20	21	3	4	5
7	8	9	18	19	1	2	12	13	14	15	16	17	10	11	20	21	3	4	5	6
8	9	10	17	18	19	1	2	12	13	14	15	16	11	20	21	3	4	5	6	7
9	13	16	10	17	18	19	3	1	12	2	14	15	20	4	5	6	7	8	11	21
15	11	1	5	16	17	18	19	4	3	12	13	14	2	21	6	7	8	9	20	10
14	3	2	6	15	16	17	18	19	4	1	12	13	21	5	7	8	9	20	10	11

Figure 3: The latin rectangle β for $n = 11$

4. Upper bounds on $\theta(n)$.

Two one-factors $[H_1, H_2]$, of K_{2k}, are called consistent if there exists an integer r such that for every edge $[a,b] \in H_i$, $i = 1,2$, either $a - b \equiv r \pmod{2k}$ or $b - a \equiv r \pmod{2k}$. One-factorization $F = \{F_1, F_2, \dots, F_{2k-1}\}$, is called consistent if $[F_{2i-1}, F_{2i}]$, $1 \leq i \leq 2k-1$, are consistent, and in F_{2k-1}, for each edge $[a,b]$, $a - b \equiv k \pmod{2k}$.

Lemma 2. *The graph G with the set of vertices Z_n and edges $\{[a,b] : a - b \equiv i \pmod{n}\}$ is a union of at most three sets of edges such that no two edges in a set have a vertex in common.*

Proof: Follows immediately from the fact that G is a union of cycles. ∎

Let (i,n) denote the greatest common divisor of i and n.

Lemma 3. *The graph G with the set of vertices Z_n and edges $\{[a,b] : a - b \equiv i \pmod{n}\}$ is a union of two one-factors if and only if $\frac{n}{(i,n)}$ is even and greater than 2.*

Proof: Since G is a union of cycles, G is a union of two one-factors if and only if it consists of cycles with even lengths. The length of the cycles of G is $\frac{n}{(i,n)}$ and, hence, G is a union of two one-factors if and only if $\frac{n}{(i,n)}$ is even and greater than 2. ∎

Lemma 4. *K_{2k} has a consistent one-factorization iff $2k = 2^r$.*

Proof: By Lemma 3, K_{2^r} is the only graph that can have a consistent one-factorization. By Lemma 3 each i, $1 \le i \le 2^{r-1} - 1$, the graph with the set of vertices Z_{2^r} and edges $\{[a,b]: a - b \equiv i \pmod{2^r}\}$ is a union of two one-factors. These two one-factors are taken as F_{2i-1} and F_{2i}. Now, let $F_{2^r-1} = \{[j, j + 2^{r-1}]: 0 \le j \le 2^{r-1} - 1\}$. It is easy to verify that $F = \{F_1, F_2, \dots, F_{2^r-1}\}$ is a consistent one-factorization. ∎

Theorem 2. *$\theta(2^k) \le 2^k - 2$ for $k \ge 3$.*

Proof: Let $n = 2^k$. We use the Lindner construction with the latin square A_n to obtain the first n sets of quadruples. Let $F = \{F_1, F_2, \dots, F_{n-1}\}$ be a consistent one-factorization of order 2^k. Let $\alpha(i,j)$ be the $2 \times (n-1)$ rectangle defined by $\alpha(1, 2j-1) = 2j$, $\alpha(1,2j) = 2j - 1$ for $j = 1, 2, \dots, 2^{k-1} - 1$, $\alpha(1, n-1) = n - 1$, $\alpha(2,j) = j$, $j = 1, 2, \dots, n-2$, and $\alpha(2, n-1) = 0$. It is clear that if $\alpha(i,j) = m$, $m \ne 0$, then all the quadruples of the form $[(a,1),(b,1),(c,2),(d,2)]$ where $(a,b) \in F_j$ and $(c,d) F_m$, appear in the quadruples of the Lindner construction. Let $\beta(i,j)$ be the $2 \times (n-1)$ latin rectangle defined by $\beta(2, n-2) = n-1$, $\beta(2, n-1) = n-2$, and $\beta(i,j) = \alpha(i,j)$ otherwise. This latin rectangle can be completed to an $(n-1) \times (n-1)$ latin square [4] γ. We can assume that $\gamma(n-1, n-2) = n-2$. Let $\delta(i,j)$ be the $(n-2) \times (n-1)$ rectangle defined by

$$\delta(i,j) = \gamma(i+2, j) \text{ for } 1 \le i \le n-3, 1 \le j \le n-1, \quad \text{except for } i = n-3,\ j = n-2 \tag{1}$$

$$\delta(n-3, n-2) = 0 \tag{2}$$

$$\delta(n-2, j) = 0 \text{ for } 1 \le j \le n-3 \tag{3}$$

$$\delta(n-2, n-2) = n-1 \tag{4}$$

$$\delta(n-2, n-1) = n-2 \tag{5}$$

Now we apply the PQS construction to obtain $n-2$ set of quadruples by using the rows of the rectangle δ as the $n-2$ permutations and partial permutations in the PQS construction. ∎

Corollary 1. $\theta(2^k - 1) \le 2^k - 2$ *for* $k \ge 3$.

Since there exists a partition for quadruples of order 8 with 6 PQs of sizes, 14, 14, 12, 12, 10, and 8 [2], [14], our construction implies that there exists a partition for quadruples of order 2^k with $2^{k-1} - 2$ PQs of size b_{2^k} $2^{k-1} - 2$ PQs of size $b_{2^k} - 2^{k-2}$, one PQ of size $b_{2^k} - 2^{2k-3} + 2^{k-1}$, and one PQ of size $2^{2k-2} - 2^k$.

For values $n \equiv 2$ or $4 \pmod 6$ which are not powers of two there is no consistent one-factorization. Hence, we use a different technique. In the rest of this section let $n = 2^k \cdot r$ for odd $r > 1, r \not\equiv 0 \pmod 3$.

Lemma 5. *Let* $i = 2^k \cdot s$ *and* $j = 2^{k-1} \cdot s$ *for odd* $s, s < r$. *Then there exist four one-factors* H_1, H_2, H_3, H_4, *of* K_n *such that in* H_4 *for each edge* $[a, b]$, $a - b \equiv j \pmod{2^k \cdot r}$ *and in* H_1, H_2, H_3, *for each edge* $[a, b]$, *either* $a - b \equiv j \pmod{2^k \cdot r}$ *or* $a - b \equiv i \pmod{2^k \cdot r}$.

Proof: Let $g = (j, n)$ and $d = \frac{n}{g}$. The graph G_1 with set of vertices Z_n and set of edges $\{[m, m+i]: 0 \le m \le n-1\}$ consists of $(i, n) = 2g$ cycles of length $\frac{n}{(i,n)} = \frac{d}{2}$. Vertices a and b are on the same cycle iff $a \equiv b \pmod{2g}$. Given $c \in Z_n$, each two vertices of the set $\{a: c \le a < c+g$ or $c+j \le a < c+g+j\}$ are on two different cycles of G_1. Now let $V = \{m: c \le m < c+g\}$ and

$$H_1 = \{[m, m+j]: m \in V\}$$
$$\cup \left\{[m+qi, m+qi+i]: m \in V,\ q \text{ odd and } 1 \le q \le \frac{d}{2} - 1\right\}$$
$$\cup \left\{[m+j+qi, m+j+qi+i]: m \in V,\ q \text{ odd and } 1 \le q \le \frac{d}{2} - 1\right\}$$
$$H_2 = \{[m+2j, m+3j]: m \in V\}$$
$$\cup \left\{[m+qi, m+qi+i]: m \in V,\ q \text{ even and } 2 \le q \le \frac{d}{2} - 1\right\}$$
$$\cup \left\{[m+j+qi, m+j+qi+i]: m \in V,\ q \text{ even and } 2 \le q \le \frac{d}{2} - 1\right\}$$
$$H_3 = \{[m, m+i]: m \in V\}$$
$$\cup \{[m+j, m+j+i]: m \in V\}$$
$$\cup \{[m+qj, m+qj+j]: m \in V,\ q \text{ even and } 4 \le q \le d-2\}$$
$$H_4 = \{[qj, qj+j]: q \text{ odd and } 1 \le q \le d-1\}.$$

We leave to the reader to verify that H_m, $m = 1, 2, 3, 4$ have the required properties. ∎

Lemma 6. *The set* $S = \{i: 1 \le i \le 2^{k-1} \cdot\}$ *can be partitioned into*

(1) *pairs* $[a, b]$ *such that* $a = 2^{k-1} \cdot s$ *for odd* s, *and either* $b = 2a$ *or* $n - b = 2a$,

(2) *singletons* $[a]$ *such that* $a \not\equiv 0 \pmod{2^k}$,

such that the number of singletons is at least 4, the intersection between two sets (two pairs, two singletons, one singleton and one pair) is empty, and the union of all the sets is S.

Proof: For $1 \leq b \leq 2^{k-1} \cdot r, b \equiv 0 \pmod{2^k}$ we distinguish between two cases. If $b \not\equiv 0 \pmod{2^{k+1}}$ we form the pair $[\frac{b}{2}, b]$. If $b \equiv 0 \pmod{2^{k+1}}$ we form the pair $[\frac{n-b}{2}, b]$. Note that $n - b \equiv 0 \pmod{2^k}$ but $n - b \not\equiv 0 \pmod{2^{k+1}}$. All the other integers in S are singletons. Since the number of pairs is $\frac{r-1}{2}$ then the number of singletons is $2^{k-1} \cdot r - (r-1) \geq 4$. It is easy to verify that the pairs and singletons satisfy the requirements of the lemma. ∎

One-factorization $F = \{F_1, F_2, \dots, F_{2k-1}\}$ of K_{2k} is called *quasi-consistent* if it can be partitioned into pairs of one-factors which are consistent, sets of four one-factors, H_1, H_2, H_3, H_4, for which in H_4 for each edge $[a, b]$, $a - b \equiv j \pmod{2k}$, and in H_1, H_2, H_3, for each edge $[a, b]$ either $a - b \equiv j \pmod{2k}$ or $a - b \equiv 2j \pmod{2k}$, and in F_{2k-1}, for each edge $[a, b]$, $a - b \equiv k \pmod{2k}$. The following lemma is an immediate consequence of Lemmas 3, 5, and 6.

Lemma 7. K_n *has a quasi-consistent one-factorization.*

For the next lemma let $\{H_1, H_2, H_3\}$ be the three sets of pairs such that if $[a, b] \in H_i$, $[c, d] \in H_j, 1 \leq i, j \leq 3$ then either $a - b \equiv c - d \pmod{n}$ or $a - b \equiv d - c \pmod{n}$. Let $[G_1, G_2]$, $[G_3, G_4]$, two pairs of consistent one-factors. The following lemma has a trivial proof.

Lemma 8. *All the quadruples of the form* $[(a,1), (b,1), (c,2), (d,2)]$, *where* $[a, b] \in H_1 \cup H_2 \cup H_3$, $[c, d] \in G_1 \cup G_2$ *can be partitioned into three PQs. All the quadruples of the form* $[(a,1), (b,1), (c,2), (d,2)]$, *where* $[a, b] \in G_1 \cup G_2$, $[c, d] \in G_3 \cup G_4$ *can be partitioned into two PQs.*

Theorem 3. $\theta(n) \leq n - 2$ *implies* $\theta(2n) \leq 2n - 2$.

Proof: We use the Lindner construction with the latin square A_n to obtain the first n sets of quadruples. We can order the singletons defined in Lemma 6 in pairs $[a, b]$ and one quadrple $[c, d, e, f]$, where $e = \frac{n-2}{2}$ and $f = \frac{n}{2}$ since both $\frac{n-2}{2}$ and $\frac{n}{2}$ are singletons. Let $F = \{F_1, F_2, \dots, F_{n-1}\}$ be a quasi-consistent one-factorization of order n. In F we first have all sets of four one-factors which satisfy Lemma 5. Then all sets of four one-factors, which are related to the pairs ordered from the singletons. These four one-factors consists of two pairs of consistent one-factors. Finally, we have the seven one-factors related to $[c, d, e, f]$ in this order. Let $\alpha(i, j)$ be the $5 \times (n-1)$ rectangle defined by $\alpha(1, 2j-1) = 2j$, $\alpha(1, 2j) = 2j - 1$ for $j = 1, 2, \dots, 2^{k-1} \cdot r - 1$, $\alpha(1, n-1) = n - 1$, $\alpha(2, j) = j, j = 1, 2, \dots, n-2$, $\alpha(2, n-1) = 0$, $\alpha(3, 2a-1) = 2b - 1$, $\alpha(3, 2a) = 2b$, $\alpha(4, 2a-1) = 2b$, $\alpha(4, 2a) = 2b - 1$, for pairs $[a, b]$ defined by Lemma 6 or by the ordering of the singletons.

$$\alpha(3,2c-1)=n-3,\ \alpha(3,2c)=n-2,\ \alpha(3,2d-1)=2c-1,$$
$$\alpha(3,2d)=n-1,\ \alpha(3,n-3)=2d-1,\ \alpha(3,n-2)=2d,$$
$$\alpha(3,n-1)=2c,\ \alpha(4,2c-1)=n-2,\ \alpha(4,2c)=n-3,$$
$$\alpha(4,2d-1)=n-1,\ \alpha(4,2d)=2c,\ \alpha(4,n-3)=2d,$$
$$\alpha(4,n-2)=2d-1,\ \alpha(4,n-1)=2c-1,\ \alpha(5,n-2)=n-1,$$

$\alpha(5,n-1)=n-2$, and $\alpha(5,j)=0$ for $j=1,2,\ldots,n-3$. It is clear that if $\alpha(i,j)=m$, $m\neq 0$, then all the quadruples of the form $[(a,1),(b,1),(c,2),(d,2)]$ where $(a,b)\in F_j$ and $(c,d)\in F_m$, either appear in the quadruples of the Lindner construction or can be partitioned into three PQs by using Lemmas 2, 3, and 8, and the PQS construction. Let $\beta(i,j)$ be the $4\times(n-1)$ latin rectangle defined by $\beta(2,n-2)=n-1$, $\beta(2,n-1)=n-2$, and $\beta(i,j)=\alpha(i,j)$, $1\leq j\leq n-1$, otherwise. This latin rectangle can be completed to an $(n-1)\times(n-1)$ latin square [4] γ. We can assume that $\gamma(n-1,n-2)=n-2$. Let $\delta(i,j)$ be the $(n-5)\times(n-1)$ rectangle defined by

$$\delta(i,j)=\gamma(i+4,j)\text{ for }1\leq i\leq n-5,\ 1\leq j\leq n-1,$$
$$\text{except for } i=n-5,\ j=n-2, \tag{1}$$

$$\delta(n-5,n-2)=0. \tag{2}$$

Now we apply the PQS construction to obtain $n-5$ sets of quadruples by using the rows of the rectangle δ as the $n-5$ permutations and partial permutations in the PQS construction. Hence, all the quadruples where partitioned into $n+3+n-5=2n-2$ PQs. ∎

Unfortunately, except for $n=2^k$, $k\geq 3$, there is no known n for which $\theta(n)\leq n-2$.

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Appendix

The near-one-factorization F of K_7.

F_1	0,3	1,4	2,5
F_2	0,4	1,5	2,6
F_3	1,2	3,4	5,6
F_4	0,6	2,3	4,5
F_5	0,5	1,3	4,6
F_6	0,2	1,6	3,5
F_7	0,1	2,4	3,6

The latin square M_7.

0	6	5	4	3	2	1	7	8	9	10	11	12	13
1	0	6	5	4	3	2	8	9	10	11	12	13	7
2	1	0	6	5	4	3	9	10	11	12	13	7	8
3	2	1	0	6	5	4	10	11	12	13	7	8	9
4	3	2	1	0	6	5	11	12	13	7	8	9	10
5	4	3	2	1	0	6	12	13	7	8	9	10	11
6	5	4	3	2	1	0	13	7	8	9	10	11	12
13	7	8	9	10	11	12	0	6	5	4	3	2	1
7	8	9	10	11	12	13	1	0	6	5	4	3	2
8	9	10	11	12	13	7	2	1	0	6	5	4	3
9	10	11	12	13	7	8	3	2	1	0	6	5	4
10	11	12	13	7	8	9	4	3	2	1	0	6	5
11	12	13	7	8	9	10	5	4	3	2	1	0	6
12	13	7	8	9	10	11	6.	5	4	3	2	1	0

The latin rectangle for the PQS construction.

3	6	5	2	11	9	10	8	12	7	1	13	4
4	3	6	5	8	10	11	9	13	12	7	2	1
5	4	1	6	3	11	8	10	9	13	12	7	2
6	5	2	1	4	8	9	11	10	3	13	12	7