On Harmonious Labelings of Graphs

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Abstract. In this paper, partial answers to some open problems on harmonious labelings of graphs listed in [2] are given.

1. Introduction

Let $G = (V, E)$ be a finite simple graph with p vertices and q edges. We write (a, b) for the edge whose endpoints are a and b. A harmonious labeling of a graph with $q \geq p$ edges is a function

$$h: V(G) \to Zq$$

(the group of integers modulo q) such that the induced edge labeling given by $h(a, b) = h(a) + h(b) \pmod{q}$ for every edge (a, b) is $1-1$. If $q = p - 1$, exactly one label may be used on two vertices and the resulting edge labels are distinct.

Since general results about harmonious graphs seem to be exceptionally difficult to find, research has focused on specific classes of graphs (see [2]).

In the next section we investigate some general characterizations of harmonious graphs. Section 3 presents partial answers to some of the open problems listed in [2].

2. General results on harmonious graphs

Theorem 2.1. *The total number of possible harmonious labelings of a graph with $q \geq p$ is*

$$\left(\frac{q-1}{2}\right)^q \qquad \text{if } q \text{ is odd,}$$

$$\left(\frac{q(q-2)}{4}\right)^{q/2} \qquad \text{if } q \text{ is even.}$$

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Proof: When q is odd, the set of labelings for all edges is $\{0, 1, \ldots, q-1\}$ as follows.

$h(a,b)$	edge (a,b)	the number of edges
1	$(0,1)\ (q-1,2), \ldots, \left(\frac{q+3}{2}, \frac{q-1}{2}\right)$	$\frac{q-1}{2}$
...	...	...
$q-2$	$(0,q-2)\ (1,q-3) \ldots \left(\frac{q-3}{2}, \frac{q-1}{2}\right)$	$\frac{q-1}{2}$
$q-1$	$(0,q-1)\ (1,q-2) \ldots \left(\frac{q-3}{2}, \frac{q+1}{2}\right)$	$\frac{q-1}{2}$
0	$(1,q-1)\ (2,q-2) \ldots \left(\frac{q-1}{2}, \frac{q+1}{2}\right)$	$\frac{q-1}{2}$

Picking an (a, b) from each row in the above table, we can construct a harmonious graph with q edges. Conversely, each edge (a, b) of a harmonious labeling comes from exactly one row in the table. So, the number of possible harmonious labelings is $\left(\frac{q-1}{2}\right)^q$, if q is odd.

Similarly, q is even, the number of possible harmonious labelings is $\left(\frac{q(q-2)}{4}\right)^{q/2}$. ∎

Remark. Not all the harmonious graphs produced in the proof of Theorem 2.1 are connected.

Example. $G = (V, E)$ with $E = \{(0,1), (3,7), (0,3), (1,3), (0,5), (2,4), (1,6), (1,7)\}$ is not connected.

Theorem 2.2. *Every graph can be embedded in a harmonious graph.*

Proof: For a given graph $G = (V, E)$ with p vertices and q edges, we construct a graph $G^* = \bar{K}_n + K_1 + G$, where n is defined as follows.

$$\text{Let } V(G) = \{x_1, x_2, \ldots, x_p\},$$
$$V(K_1) = \{x_0\},$$
$$V(\bar{K}_n) = \{y_1, y_2, \ldots, y_n\},$$

and define h by

$$h(x_i) = \begin{cases} i & \text{for } i = 0, 1, 2, \\ h(x_{i-1}) + h(x_{i-2}) + 1 & \text{for } i = 3, 4, \ldots, p. \end{cases}$$

Then h is an injection from $V(\bar{K}_n)$ to the set

$$\{1, 2, \ldots, \max_{(x_i,x_j)\in E(K_1+G)} h(x_i, x_j))\}\backslash\{h(x_i, x_j) \mid x_i, x_j \in E(K_1 + G)\},$$

where

$$n = \max_{(x_i,x_j)\in E(K_1+G)} (h(x_i, x_j)) - p - q \geq 0.$$

It is easy to verify that h is a harmonious labeling and G is a subgraph of G^*. ∎

The following theorem provides a necessary condition for harmonious graphs.

Theorem 2.3. *Let* $(d_1, d_2, \ldots, d_p)$ *be the degree sequence of* G.
If G *is a harmonious graph, then*

$$\sum_{t=1}^{p} d_t y_t \equiv \binom{q}{2} \pmod{q} \qquad (*)$$

has a solution $(y_1, y_2, \ldots, y_p)$ *of nonnegative integers, where* $0 \le y_i \le q-1$, *and*

$$y_i \neq y_j,\ i \neq j,$$

if G *is not a tree.*

If G is a tree, there is exactly one pair i, j such that $y_i = y_j$.

Proof: Let $h: V(G) \to \{0, 1, 2, \ldots, q-1\}$ be a harmonious labeling of G. Let $h(x_i) = y_i$, $x_i \in V(G)$ and $\deg x_i = d_i$, $i = 1, 2, \ldots, p$, then $y_i \in \{0, 1, 2, \ldots, q-1\}$, and $y_i \neq y_j$ if $i \neq j$. Hence

$$\begin{aligned}\sum_{i=1}^{p} d_i y_i &= \sum_{\substack{(x_i, x_j) \in E(G) \\ i \neq j}} h(x_i, x_j) + qt \text{ (t is a nonnegative integer)} \\ &= 0 + 1 + 2 + \cdots + q - 1 + qt \\ &= \binom{q}{2} + qt = \binom{q}{2} \pmod{q}.\end{aligned}$$

∎

Lemma 2.4. ([5,Ch.2, §6, Th.2]) *Equation (*) has a solution of integer if and only if*

$$g.c.d.(d_1, d_2., \ldots, d_p, q) \mid \binom{q}{2}.$$

As a corollary to Theorem 2.3 we have the Graham-Sloane necessity condition [4, Th. 11].

Corollary 2.5. [4]) *If a harmonious graph* G *with degree sequence* $(d_1, d_2, \ldots, d_p)$ *has an even number* q *of edges and* $2^k \mid d_i$ *(*d_i *is divisible by* 2^k*) for* $i = 1, 2, \ldots, p$, *then* $2^{k+1} \mid q$.

Proof: Suppose that $2^{k+1} \nmid q$. Consider

$$\sum_{i=1}^{p} d_i x_i \equiv \binom{q}{2} \pmod{q}.$$

Since $2^k \mid d_i$ $2 \mid q$, and $q = \frac{1}{2}\sum_{i=1}^{p} d_i$, then $2^{k-1} \mid$ g.c.d.$(d_1, d_2, \ldots, d_p, q)$, $k \geq 2$.

(1) If $2^k \nmid$ g.c.d.$(d_1, \ldots, d_p, q)$, then $2^{k-1} \nmid \frac{1}{2}q(q-1) = \binom{q}{2}$. Thus g.c.d.$(d_1, \ldots, d_p, q) \nmid \binom{q}{2}$.

(2) If $2^k \mid$ g.c.d.$(d_1, \ldots, d_p, q)$. Since $2^{k+1} \nmid q$, $2^k \nmid \frac{1}{2}q(q-1)$. Thus g.c.d.$(d_1, \ldots, d_p, q) \nmid \frac{1}{2}q(q-1)$, i.e., g.c.d.$(d_1, \ldots, d_p, q) \nmid \binom{q}{2}$.

By Lemma 2.4 equation (*) has no solution in the integers. Hence G is not harmonious. ∎

Remark. Theorem 2.3 is not a consequence of Graham-Sloane Theorem (Corollary 2.5), as will be shown in Theorem 3.11.

Corollary 2.6. *If a graph G is k-regular, then G is not harmonious, if*

(1) $p \equiv 1 \pmod 2$ *and* $k \equiv 0 \pmod 4$, *or*
(2) $p \equiv 2 \pmod 4$ *and* $k \equiv 0 \pmod 2$.

Proof: Observe that $q = \frac{1}{2}pk$. The result now follows from Corollary 2.5. ∎

3. Answers to some open problems in harmonious graphs

In [2] Gallian summerized much of the work done on harmonious labelings of graphs and provided a plethora of open problems and conjectures. In this section we'll give partial answers to some of them.

First, we want to point out that Grace's conjecture that an even cycle with one additional endpoint for each vertex is harmonious has been proved by us (see [6]).

Now we investigate specific classes of graphs.

(1) mK_n (the union of m disjoint copies of the complete graph K_n).

According to [2], up to now no results have been obtained pertaining to the harmoniousness of mK_n.

Theorem 3.1. *If $n \equiv 1 \pmod 2$ and $m \equiv 2 \pmod 1$, then mK_n is not harmonious.*

Proof: The graph mK_n is a $(n-1)$-regular graph with $m\binom{n}{2}$ edges.

By Corollary 2.6, mK_n is not harmonious if

$$\begin{cases} m\binom{n}{2} \equiv 1 & \pmod 2 \\ n-1 \equiv 0 & \pmod 4 \end{cases} \quad \text{or} \quad \begin{cases} m\binom{n}{2} \equiv 2 & \pmod 4 \\ n-1 \equiv 0 & \pmod 2 \end{cases}$$

It is not difficult to get $\begin{cases} m \equiv 2 & \pmod 4 \\ n-1 \equiv 0 & \pmod 2 \end{cases}$ from the above congruence expressions. ∎

We next consider mK_3. According to Theorem 3.1, a necessary condition for mK_3 to be harmonious is that m is odd or $m \equiv 0 \pmod 4$. We have the following.

Theorem 3.2. *mK_3 is harmonious when m is odd.*

Proof: Arrange the m complete graphs K_3 in their natural sequence. Let x_i, y_i, z_i, be three vertices in the ith copy of K_3, $i = 1, 2, \ldots, m$.

$$h: V(mK_3) \to \{0, 1, 2 \ldots, 3m-1\}, (m \geq 3), \text{ is given by}$$

$$h(x_i) = i - 1, \qquad i = 1, 2, \ldots, m,$$

$$h(y_i) = \begin{cases} 2(m-i) \;, & 1 \leq i \leq (m-1)/2, \\ 3m - 2i \;, & (m+1)/2 \leq i \leq m, \end{cases}$$

$$h(z_i) = \begin{cases} (5m-1)/2 + i \;, & 1 \leq i \leq (m-1)/2, \\ (3m-1)/2 + i \;, & (m+1)/2 \leq i \leq m. \end{cases}$$

Remark: When $m = 3$, $h: \{x_1, x_2, x_3\} \to \{0, 1, 2\}$.

One can easily verify that h is one-to-one. In the following, we give the value of $h(x_i, y_i)$.

(a) The set of $h(x_1, y_i)$, $i = 1, 2, \ldots, m$, is

$$\left\{\frac{3m-1}{2}, \frac{3m+1}{2}, \ldots, \frac{5m-3}{2}\right\}.$$

(b) The set of $h(x_i, z_i)$, $i = 1, 2, \ldots, m$, is

$$\left\{\frac{5m-1}{2}, \frac{5m+1}{2}, \ldots, \frac{7m-3}{2},\right\}.$$

(c) The set of $h(y_i, z_i)$, $i = 1, 2, \ldots, m$, is

$$\left\{\frac{7m-1}{2}, \frac{7m+1}{2}, \ldots, \frac{9m-3}{2}\right\}.$$

Thus the labels of the edges are the integers from $\frac{3m-1}{2}$ to $\frac{9m-3}{2}$ and therefore h is a harmonious labeling. ∎

It is known (see [2]) that mK_n is graceful if and only if $m = 1$ and $n \leq 4$. We conjecture that mK_3 is not harmonious when $m \equiv 0 \pmod 4$. We have verified that the conjecture is true for $m = 4$.

(2) Helms H_n.

Helms H_n (wheels with a pendent edge at each cycle vertex) have been shown to be graceful (see [2]), but their harmoniousness is an open problem. In fact, the proof of the harmoniousness of wheels W_{2n+2} in [3] has given us a hint as to how to solve this problem.

In [3], the path of length $2n$ was drawn as a bipartite graph in the plane with $n+1$ vertices on the left and n vertices on the right, and with the first vertex of the path on the left. Label the vertices on the left by $0,1,2,\ldots,n$ and the vertices on the right by $n+1, n+2,\ldots,2n$, Then the vertex labeled n was connected to the vertex labeled 0 to give a harmonious cycle C_{2n+1}. Labeling C_{2n+1} in the preceding way and using $3n+1$ as the center label produces a harmonious labeling for $W_{2n+2} = K_1 \odot C_{2n+1}$. Now, add a pendent edge at each cycle vertex of the above labeling for W_{2n+2} and label the endpoints adjacent to the cycle vertices $0,1,2,\ldots,n$ with $5n+2, 5n+3,\ldots,6n+2$, and the endpoints adjacent to the cycle vertices labeled $n+1, n+2,\ldots,2n$ with $4n+2, 4n+3,\ldots,5n+1$.

This yields a harmonious labeling of helms with an odd cycle. Hence we have the following.

Theorem 3.3. *Helms with an odd cycle are harmonious.*

We conjecture that helms with an even cycle also are harmonious. We have checked that the conjecture is true for helms with a 4-cycle or 6-cycle.

(3) Windmills $K_n^m (n > 3)$ (the family of graphs consisting of m copies of K_n with a vertex in common).

In 1982, K_4^m was proved to be harmonious for all m (see [2]). Graham and Sloane conjectured that K_n^2 is harmonious if and only if $n = 4$. They verified this conjecture for the cases that n is odd or $n = 6$ (see [4]).

We'll show that Graham and Sloane's conjecture is true for infinitely many even n.

Lemma 3.4. *(see [1, §18, Th.1]) Let $n = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where each p_i is prime. Then n is not a sum of two squares if and only if there is a*

$$p_i \equiv 3 \pmod 4 \quad \textit{and } \alpha_i \textit{ is odd.} \tag{*}$$

Theorem 3.5. *If n is even and $n/2$ satisfies condition $(*)$ in Lemma 3.4, then is not harmonious.*

Proof: $|E(K_n^2)| = 2\binom{n}{2}$ is even. Suppose that K_n^2 is harmonious, and $x, a_1, a_2, \ldots, a_{n-1}$ are labels of the vertices of one K_n and $x, b_1, b_2, \ldots, b_{n-1}$ are labels of the vertices of the other K_n, where x is the label of the vertex common to the two K_n's.

We introduce the following function

$$H(Z) = (Z^x + Z^{a_1} + \cdots + Z^{a_{n-1}})^2 + (Z^x + Z^{b_1} + \cdots + Z^{b_{n-1}})^2.$$

Let A_e (A_o) be the number of even (odd) numbers in set $\{x, a_1, \ldots, a_{n-1}\}$, and B_e (B_o) be the corresponding number in the set $\{x, b_1, \ldots, b_{n-1}\}$. Clearly, $A_e + A_o = B_e + B_o = n$ also,

$$H(-1) = (A_e - A_o)^2 + (B_e - B_o)^2 \tag{1}$$

and

$$H(Z) = Z^{2x} + Z^{2a_1} + \cdots + Z^{2a_{n-1}}$$
$$+ Z^{2x} + Z^{2b_1} + \cdots + Z^{2b_{n-1}}$$
$$+ 2(Z^{x+a_1} + Z^{x+a_2} + \cdots + Z^{x+a_{n-1}} + Z^{a_1+a_2} + \ldots$$
$$+ Z^{x+b_{n-1}} + Z^{b_1+b_2} + \cdots + Z^{b_{n-2}+b_{n-1}})$$

Since K_n^2 is harmonious, and $|E(K_n^2)|$ is even, we have

$$H(-1) = 2n \tag{2}$$

By (1) and (2), we have

$$(A_e - A_o)^2 + (B_e - B_o)^2 = 2n \tag{3}$$

where $A_e - A_o$ and $B_e - B_o$ are even because $A_e + A_o$ and $B_e + B_o$ are even.
Since n is even, (3) is equivalent to

$$X^2 + Y^2 = n/2 \tag{4}$$

where X, Y are integers (even numbers).
By Lemma 3.4, if $n/2$ doesn't satisfy condition $(*)$, then equation (4) doesn't hold. Thus K_n^2 is not harmonious. ∎

According to Theorem 3.5 the values of n up to 100 for which K_n^2 is not harmonious are 6, 12, 14, 22, 24, 28, 30, 38, 42, 44, 46, 48, 54, 56, 60, 62, 66, 70, 76, 78, 84, 86, 88, 92, 94, 96.

Lemma 3.6. (see [1, §18, Ex.13]) *A positive integer n is not a sum of three squares if and only if $n = 4^e(8k+7)$, where e, k are nonnegative integers.*

Theorem 3.7. *If*

(1) $n \equiv 0 \pmod 4$ *and* $\frac{3n}{4} = 4^e(8k+7)$, *where e and k are nonnegative integers or*
(2) $n \equiv 5 \pmod 8$, *then K_n^3 is not harmonious.*

Proof: The argument of this proof is similar to that of Theorem 3.5.
We introduce the following function:

$$H(Z) = (Z^x + Z^{a_1} + \cdots + Z^{a_{n-1}})^2$$
$$+ (Z^x + Z^{b_1} + \cdots + Z^{b_{n-1}})^2$$
$$+ (Z^x + Z^{c_1} + \cdots + Z^{c_{n-1}})^2.$$

Counting $H(-1)$ in two ways, we easily see that a necessary condition for K_n^3 to be harmonious is

$$\alpha^2 + \beta^2 + \gamma^2 = 3n \tag{5}$$

where α, β, γ are even when $n \equiv 0 \pmod 4$ and α, β, γ are odd when $n \equiv 5 \pmod 8$.

But, by Lemma 3.6, when $n \equiv 0 \pmod 4$, $\frac{3n}{4} = 4^e(8k+7)$; or when $n \equiv 5 \pmod 8$ equation (5) doesn't hold. ∎

According to Theorem 3.7, $K_5^3, K_{13}^3, K_{20}^3$ are not harmonious.

(4) $B(n, r, m,), r > 1$, (the graph consisting of m copies of K_n with a K_r in common).

It is known that $B(n, r, m)$ is gracefui in the following cases: $n = 3, r = 2, m \geq 1$; $n = 4, r = 3, m \geq 1$ (see [4]). But up to now, no results have been obtained regarding the harmoniousness of $B(n, r, m)$ for $r > 1$. Analogous to the result for the gracefulness of $B(n, r, m)$ we have the following.

Theorem 3.8. *$B(3, 2, m)$ $m \geq 1$ is harmonious.*

Proof: Label the two vertices in common to the K_3's with $\{0, 1\}$. Label the rest with $\{2, 4, 6, \ldots, 2m\}$.

An easy computation shows that the labeling is of the desired type. ∎

Theorem 3.9. *$B(4, 3, m)$ $m \geq 1$ is harmonious.*

Proof: Label the three vertices in common to the K_4's with $\{0, 1, 2\}$. Label the rest with $\{4, 7, 10, \ldots, 3i+1, \ldots, 3m+1\}$.

This labeling is clearly harmonious. ∎

We believe that the graphs $B(4, 2, m)$ also are harmonious, since we have verified the cases when $m = 2, 3, 4$.

(5) $C^{n^{(t)}}$ (the one point union of t cycles of length n). Graham and Sloane proved $C^{3^{(t)}}$ is harmonious if and only if $t \not\equiv 2 \pmod 4$. S.C. Shee (see [2]) proved $C^{4^{(t)}}, t > 1$, is harmonious.

We have the following negative result.

Theorem 3.10. *If*

(1) $n \equiv 1 \pmod 2, t \equiv 2 \pmod 4$; *or*
(2) $n \equiv 2 \pmod 4, t \equiv 1 \pmod 2$, *then $C^{n^{(t)}}$ is not harmonious.*

Proof: If n is odd and $t \equiv 2 \pmod 4$ or $n \equiv 2 \pmod 4$ and t is odd, then $|E(C^{n^{(t)}})| = tn$ is even, and the degree of every vertex is divisible by 2 but $4 \nmid tn$ so, by Corollary 2.5, $C^{n^{(t)}}$ is not harmonious. ∎

(6) C_n^{+t} (the class of graphs formed by adding a single pendent edge to t vertices of a cycle of length n).

Grace has shown that C_n^{+t} is harmonious when n is odd. But not results have been obtained regarding the harmoniousness of C_n^{+t} when n is even. The following result shows that the graphs C_n^{+t} are not harmonious for even n.

First we prove a more general result.

Theorem 3.11. *Let C_n' be the class of graphs formed by adding a path to a vertex of a cycle of length n, and $q = |E(C_n')|$. Then a necessary condition for C_n' to be harmonious is that q is even.*

Proof: Suppose that C_n' is harmonious. Label the vertex of degree 3 and the end-point with x_1 and x_2, and label the other vertices of degree 2 with $x_3, x_4, \ldots, x_q$.

According to the equation $(*)$ in Theorem 2.3

$$3x_1 + x_2 + 2(x_3 + \cdots + x_q) \equiv \binom{q}{2} \pmod{q}, \text{ so}$$

$$x_1 - x_2 + 2\binom{q}{2} \equiv \binom{q}{2} \pmod{q} \tag{6}$$

$$x_1 - x_2 + \frac{q}{2}(q-1) \equiv 0 \pmod{q}$$

If $q \equiv 1 \pmod{2}$, from (6) we obtain $x_1 - x_2 \equiv 0 \pmod{q}$, which is not possible because $x_1 \neq x_2$ and $x_1, x_2 \in Z_q$. ∎

Remark: Theorem 3.11 can't be derived from Corollary 2.5.

As an immediate consequence we have the following.

Corollary 3.12. *When n is even, C_n^{+1} is not harmonious.*

(7) Triangular snakes (a triangular cactus block-cutpoint-graph is a path of length n).

David Moulton [7] has proved that triangular snakes are graceful, but no results have been obtaired as to their harmoniousness. Our next result provides a partial solution.

Theorem 3.13. *When n is odd, Δ_n is harmonious.*

Proof: Arrange the triangles in their natural sequence. Let x_{i-1}, x_i, y_i be the three vertices in ith triangle for $i = 1, 2, \ldots, n$.

Case 1: $n \equiv 1 \pmod{4}$.

If $n = 1$, label the three vertices with $\{0, 1, 2\}$. We consider $n \geq 3$ as follows.

The function h is defined by

$$h(x_i) = \begin{cases} \frac{i}{2} & \text{for even } i, \\ \frac{n+i}{2} & \text{for odd } i, \end{cases}$$

$$h(y_i) = \begin{cases} 2n - i & \text{for even } i \neq 2, \\ 2n + \frac{i-3}{2} & \text{for odd } i, \text{ and } i \geq \frac{n+1}{2}, \\ \frac{5n+i}{2} - 1 & \text{for odd } i, \text{ and } 3 \leq i < \frac{n+1}{2}, \\ 2n - 1 & \text{if } i = 1, \\ 3n - 1 & \text{if } i = 2. \end{cases}$$

One can easily verify that $h(x_i) \neq h(x_j)$, $h(y_i) \neq h(y_j)$ and $h(x_i) \neq h(y_j)$. In the following we give the set of induced edge labels.

(a) The set of edge labels on (x_{i-1}, x_i) $i = 1, 2, \ldots, n$, is

$$\left\{\frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, \frac{3n-1}{2}\right\}.$$

(b) The set of edge labels on (y_i, x_{i-1}), (y_i, x_i) even and $i \geq 4$, and (y_1, x_0), (y_1, x_1), $(y_{\frac{n+1}{2}}, x_{\frac{n-1}{2}})$ is

$$\left\{\frac{3n-1}{2}+1, \frac{3n-1}{2}+2, \ldots, \frac{5n-1}{2}\right\}.$$

(c) The set of edge labels on (y_i, x_{i-1}), (y_i, x_i), where i is odd and $i \neq 1$, $i \neq \frac{n+1}{2}$ and (y_2, x_1), (y_2, x_2) $(y_{\frac{n+1}{2}}, x_{\frac{n+1}{2}})$ is

$$\left\{\frac{5n-1}{2}+1, \frac{5n-1}{2}+2, \ldots, \frac{7n-1}{2}\right\}.$$

Thus the set of induced edge labels of Δ_n is

$$\left\{\frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, \frac{7n-1}{2}\right\},$$

and the function h is a harmonious labeling.

Case 2: $n \equiv 3 \pmod 4$.

If $n = 3$, label x_0, x_1, x_2, x_3 with $0, 2, 1, 3$; label y_1, y_2, y_3 with $8, 4, 6$. Clearly this is a harmonious labeling. Now we consider $n > 3$ as follows.

The function h is given by

$$h(x_i) = \begin{cases} \frac{i}{2} & \text{for even } i, \\ \frac{n+1}{2} & \text{for odd } i, \end{cases}$$

$$h(y_i) = \begin{cases} 2n-i & \text{for even } i \neq 2, \\ 2n+\frac{i-3}{2} & \text{for odd } i, \text{ and } i \geq \frac{n+7}{2}, \\ \frac{5n+i}{2}-2 & \text{for odd } i, \text{ and } 3 \leq i \leq \frac{n-1}{2}, \\ \frac{9n-7}{4} & \text{if } i = \frac{n+3}{2}, \\ 2n-1 & \text{if } i = 1, \\ 3n-1 & \text{if } i = 2. \end{cases}$$

Analogously, we can show that h is a harmonious labeling. ∎

Theorem 3.14. *When* $n \equiv 2 \pmod 4$, Δ_n *is not harmonious.*

Proof: Note that $q = |\Delta_n| = 3n$, and Δ_n has $n-1$ vertices of degree 4 and $n+2$ vertices of degree 2. Since $n \equiv 2 \pmod 4$ and $4 \nmid 3n$, according to Corollary 2.5, is not harmonious. ∎

The remaining problem is whether Δ_n is harmonious when $n \equiv 0 \pmod 4$. We have checked that Δ_4 is harmonious.

We remark that K_4-snakes analogous to triangular snakes are harmonious (see [4]). Now we introduce the ladder $+nK_2$ defined as follows.

Let $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$ be the consecutive vertices of two disjoint paths on n vertices. A ladder is the graph obtained by connecting each x_i to y_i, y_{i+1} $(0 \le i \le n-2)$ and each y_i to x_{i+1} $(0 \le i \le n-2)$.

We have

Theorem 3.15. *The ladders* $+nK_2$ *are harmonious.*

Proof: Denote the n vertices on one side of ladder by $x_0, x_1, x_2, \ldots, x_{n-1}$ natural order and the n vertices on other side by $y_0, y_1, y_2, \ldots, y_{n-1}$.

The function h: $V(+nK_2) \to \{0, 1, 2, \ldots, q\}$ is given by

$$h(x_i) = \begin{cases} 1 & \text{if } i = 0, \\ 3 + h(x_{i-1}), & \text{for odd } i, \\ 2 + h(x_{i-1}), & \text{for even } i \neq 0. \end{cases}$$

$$h(y_i) = \begin{cases} 0 & \text{if } i = 0, \\ 2 + h(y_{i-1}) & \text{for odd } i, \\ 3 + h(y_{i-1}) & \text{for even } i \neq 0. \end{cases}$$

It is not difficult to verify that h is a harmonious labeling. ∎

Further, we define subladders nK_2, as the class of graphs formed by deleting edges (x_{i-1}, y_i) or (y_{i-1}, x_i) for each i $(i = 1, 2, \ldots, n-1)$ from the ladders above.

Theorem 3.16. *The subladders* $\oplus nK_2$ *are harmonious.*

Proof: The function h is given by

$$h(x_i) = \begin{cases} 1 & \text{if } i = 0, \\ 2 + h(x_{i-1}), & \text{if } i \geq 1. \end{cases}$$

$$h(y_i) = \begin{cases} 0 & \text{if } i = 0, \\ 2 + h(y_{i-1}) & \text{if } i \geq 1. \end{cases}$$

This labeling is clearly harmonious. ∎

(8) Double cone $C_n + \bar{K}_2$.

According to [2], when n is odd $C_n + \bar{K}_2$ has been shown to be harmonious. But when n is even the harmoniousness of $C_n + \bar{K}_2$ still is open. The following theorem give a partial answer to this question.

Theorem 3.17. *If* $n = 2, 4, 6 \pmod{8}$, $C_n + \bar{K}_2$ *is not harmonious.*

Proof: Note that the double cone $C_n + \bar{K}_2$ has two vertices of degree n, n vertices of degree 4 and $q = 3n$.

Case 1. $n \equiv 2 \pmod{4}$. Note that $4 \nmid 3n$. *Case 2.* $n \equiv 4 \pmod{8}$. We have $8 \nmid 3n$.

By Corollary 2.5, $C_n + \bar{K}_2$ are not harmonious. ∎

We don't know if $C_n + \bar{K}_2$ is harmonious when $n \equiv 0 \pmod{8}$. Once this is answered, this problem will be solved completely.

Finally, we want to point out that one of open problems in the table of [2], namely the harmoniousness of $S_m + K_1$, was settled by Chang, Hsu and Rogers in 1981 (see [2]) ($K_{mn} + K_1$ are harmonious).

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