A Note on Constructing Magic Rectangles of Even Order

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In this note we consider the problem of constructing magic rectangles of size m by n where m and n are both multiples of two. What seems to be a new and relatively simple method for constructing many such rectangles is presented.

Introduction. An $m \times n$ magic rectangle is an $m \times n$ array containing the integers $1, 2, \cdots, mn$ such that if the (i,j)th entry of the array Z is denoted by $z_{ij}, i = 1, \cdots, m, j = 1, \cdots, n$, then

$$\sum_{j=1}^{n} z_{ij} = n(mn+1)/2 = R \text{ for } i = 1, \cdots, m.$$

and

$$\sum_{i=1}^{m} z_{ij} = m(mn+1)/2 = C \text{ for } j = 1, \cdots, n.$$

Magic rectangles are useful in the statistical design of experiments, e.g., see Phillips [8]. For example, in agricultural field trial experiments, it is often necessary to apply treatments sequentially to plots of ground (experimental units) occurring in long narrow rows. If a linear fertility trend exists parallel to the rows of plots to which treatments are applied, then observations obtained occur not only as a result of the treatments applied but also as a result of the fertility trend. The analysis of the data obtained from such an experiment can be confusing since it may be difficult or impossible to separate the actual effects of the treatments from the effects of the fertility trend. Magic rectangles can be used to apply the treatments of a two factor factorial experiment sequentially to plots occurring in a row so that the usual least squares estimates of the corresponding factorial effects are not contaminated by the linear trend. Similarly, magic rectangles can be used in many other experimental situations to apply treatments sequentially to experimental units over space or time so that the usual least squares estimates of the treatment effects are free of any unknown linear trend effects.

A number of methods for constructing magic rectangles are known. In fact, Harmuth [4, 5] proved the existence of all magic rectangles of size $m \times n$ where m and n are both even or odd by giving some not very precise rules for their construction. Andrews [1] gives an excellent discussion of magic rectangles as well as a description of another method of constructing magic rectangles called the method of complementary differences which Andrews [2] attributes to Planck. Phillips [7] gives a simple method of construction when either m or n is a multiple of four. In this paper we give what seems to be a new and fairly simple method of constructing a large number of $m \times n$ magic rectangles which can be applied whenever m and n are both even integers.

Construction Method. We now describe a general method for constructing an $m \times n$ magic rectangle where m and n are both even integers. There are five steps to the procedure. To make the procedure clear, we shall illustrate each step by constructing a 6×8 magic rectangle when describing the particular steps of the process.

Step 1.

Example 1. To construct a 6×8 magic rectangle, we begin by finding a 16×1 vector $p' = (p_1, \cdots, p_{16})$ whose entries are all ± 1 with $p_i = -p_{16-i+1}$ for $i = 1, \cdots, 8$ and which is orthogonal to both the 16×1 vector of 1's and the 16×1 vector $(1, 2, 3, \cdots, 16)'$. One such vector is given by

$$p' = (-1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1).$$

More generally, for a given value n, in step 1 any $2n \times 1$ vector $p' = (p_1, \cdots, p_{2n})$ satisfying (i) $p_i = \pm 1$ for $i = 1, \cdots, 2n$, (ii) $p_i - p_{2n-i+1}$ for $i = 1, \cdots, n$ and (iii) $\sum_{i=1}^{2n} i p_i = 0$ will suffice. One convenient method for constructing such a vector is to begin by letting $s' = (-1, 1, 1, -1)$ and $t' = (-1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1, 1)$. Now if $n = 2l$ for some $l = 2, 4, 6, \cdots$, let $p' = (s_1', \cdots, s_l', -s_1', \cdots, -s_l')$ where $s_i = s$ for $i = 1, \cdots, l$ or if $n = 6$, let $p' = t'$ or finally if $n = 2l$ for some $l = 5, 7, 9 \cdots$, let $p' = (s_1', \cdots, s'_{(l-3)/2}, t', -s_1', \cdots,$ where again $s_i = s$ for $i = 1, \cdots, (l-3)/2$. It is easy to verify that the construction methods just described yield vectors p satisfying conditions (i), (ii) and (iii) given above.

Step 2.

Example 1 (continued). Now write down the 3×16 array W by writing the integers $1, 2 \cdots, 48$ sequentially in columns. For this example,

$$W = \begin{pmatrix} 1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 & 25 & 28 & 31 & 34 & 37 & 40 & 43 & 46 \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32 & 35 & 38 & 41 & 44 & 47 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & 33 & 36 & 39 & 42 & 45 & 48 \end{pmatrix}.$$

For arbitrary values of m and n, construct the $(m/2) \times (2n)$ array W by sequentially writing down the numbers $1, 2, \cdots, mn$ in columns, i.e. if $W = (\omega_{ij})$, then for $i = 1, \cdots, m/2, j = 1, \cdots, 2n, \omega_{ij} = (m/2)(j-1) + i$.

Step 3.

Example 1 (continued). At this step, use the vector $p' = (p_1, \cdots, p_{16})$ from Step 1 and the 3×16 matrix W from Step 2 to create another 6×16 array X. In particular, if the ith entry in p is -1, let the first three entries in the ith column of X be the same as the three entries in the ith column of W followed by three zeros. If the ith entry of p is 1, let the ith column of X consist of three zeros followed by the three entries in the ith column of W listed in reverse order. For this example,

$$p' = (-1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1)$$

$$\downarrow$$

$$X = \begin{pmatrix} 1 & 0 & 0 & 10 & 13 & 0 & 0 & 22 & 0 & 28 & 31 & 0 & 0 & 40 & 43 & 0 \\ 2 & 0 & 0 & 11 & 14 & 0 & 0 & 23 & 0 & 29 & 32 & 0 & 0 & 41 & 44 & 0 \\ 3 & 0 & 0 & 12 & 15 & 0 & 0 & 24 & 0 & 30 & 33 & 0 & 0 & 42 & 45 & 0 \\ 0 & 6 & 9 & 0 & 0 & 18 & 21 & 0 & 27 & 0 & 0 & 36 & 39 & 0 & 0 & 48 \\ 0 & 5 & 8 & 0 & 0 & 17 & 20 & 0 & 26 & 0 & 0 & 35 & 38 & 0 & 0 & 47 \\ 0 & 4 & 7 & 0 & 0 & 16 & 19 & 0 & 25 & 0 & 0 & 34 & 37 & 0 & 0 & 46 \end{pmatrix}.$$

For the general case, $p' = (p_1, \cdots, p_{2n})$ and if we denote the ith column of W by W_i and the ith column of X by X_i, then $W = (W_1, \cdots, W_{2n})$ with $W_i = \binom{X_i}{0}$ if $p_i = -1$, $\binom{0}{\bar{X}_i}$ if $p_i = 1$ where 0 is the $n \times 1$ vector of zeros and $\bar{X}_i$ is the $n \times 1$ vector whose entries are obtained by listing the entries of X_i in reverse order.

Step 4.

Example 1 (continued). From the 6×16 array X, we construct another 6×8 array Y. The first column of Y is obtained by adding columns 1 and 16 of X together, the second column of Y is obtained by adding columns 2 and 15 of X together, etc. Thus

$$Y = \begin{pmatrix} 1 & 43 & 40 & 10 & 13 & 31 & 28 & 22 \\ 2 & 44 & 41 & 11 & 14 & 32 & 29 & 23 \\ 3 & 45 & 42 & 12 & 15 & 33 & 30 & 24 \\ 48 & 6 & 9 & 39 & 36 & 18 & 21 & 27 \\ 47 & 5 & 8 & 38 & 35 & 17 & 20 & 26 \\ 46 & 4 & 7 & 37 & 34 & 16 & 19 & 25 \end{pmatrix}.$$

For the general case, we construct the $m \times n$ array Y from the $m \times 2n$ array X. If we let X_i denote the ith column of X for $i = 1, \cdots, 2n$ and Y_i denote the ith column of Y for $i = 1, \cdots, n$, then $Y_i = X_i + X_{2n+1-i}$ for $i = 1, \cdots, n$.

Comment. After step 4 of the construction process, all column sums of the array Y are equal to $n(mn \times 1)/2$ whereas the entries in rows i and $n+1-i$ of Y both sum to $ni + mn(2n-1)/4$ for $i = 1, \cdots, m/2$. Also, if $Y = (y_{ij})$, then for $i = 1, \cdots, (m/2) - 1$, $y_{i+1,j} = y_{i,j} + 1$ for $j = 1, \cdots, n$ and for $i = 1, \cdots, (m/2) - 1$, $y_{m-i,j} = y_{m-i+1,j} + 1$ for $j = 1, \cdots, n$.

Step 5.

Example 1 (continued). We now create the desired 6×8 magic rectangle $Z = (z_{ij})$ from the 6×8 array $Y = (y_{ij})$ obtained in step 4. In particular, select any 4 columns from Y and interchange the elements in the first and third rows of the columns selected. Now select again any 4 columns of Y and interchange the elements in the fourth and sixth rows of the columns selected. The resulting array Z obtained by making these interchanges is a 6×8 magic rectangle. For instance, if columns 1,4,5 and 7 are selected first and then columns 2,4,5 and 6 are selected and the appropriate row interchanges are made within these columns, the resulting 6×8 magic rectangle obtained from Y is

$$Z = \begin{pmatrix} 3 & 43 & 40 & 12 & 15 & 31 & 30 & 22 \\ 2 & 44 & 41 & 11 & 14 & 32 & 29 & 23 \\ 1 & 45 & 42 & 10 & 13 & 33 & 28 & 24 \\ 48 & 4 & 9 & 37 & 34 & 16 & 21 & 27 \\ 47 & 5 & 8 & 38 & 35 & 17 & 20 & 26 \\ 46 & 6 & 7 & 39 & 36 & 18 & 19 & 25 \end{pmatrix}.$$

We note that the row sums of Z are all equal to 196 and the column sums of Z are all equal to 147.

For the general case, we obtain the desired $m \times n$ magic rectangle $Z = (z_{ij})$ from the $m \times n$ array $Y = (y_{ij})$ by appropriately interchanging column elements of Y. In particular, if we let $[y]$ denote the integer part of the decimal expansion for $y > 0$, then for $i = 1, \cdots, [m/4]$, we create sets $\mathcal{A}_i$ and β_i of size $n/2$ where the elements of each of the sets $\mathcal{A}_i$ and β_i consist of any $n/2$ distinct integers selected from $1, 2, \cdots, n$. Now, for each element j in $\mathcal{A}_i, i = 1, \cdots, [m/4]$, we interchange the entries in rows i and $(m/2) - i$ of column j of Y. Similarly, for each element j in $\beta_i, i = 1, \cdots, [m/4]$, interchange the entries in rows $(m/2) + i$ and $m - i + 1$ of column j of Y. Using the comment made following step 4, it is easy to verify that the array Z obtained from Y by making these interchanges is the desired magic rectangle.

Comment. With regard to the construction process outlined above, there is a good deal of flexibility. For example, any vector p satisfying the conditions given in step 1 and any choices for the sets $\mathcal{A}_i$ and β_i of integers described in step 5 will be satisfactory. Thus the construction process given here will yield a great many magic rectangles for specified values of m and n.

We now give another example to further illustrate the construction process outlined above.

Example 2. In this example, we construct a 6×10 magic rectangle.

Step 1. Let $p' = (-1,1,1,-1,-1,1,-1,1,1,1,-1,-1,-1,1,-1,1,1,-1,-1,1)$

Step 2.

$$W = \begin{pmatrix} 1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 & 25 & 28 & 31 & 34 & 37 & 40 & 43 & 46 & 49 & 52 & 55 & 58 \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32 & 35 & 38 & 41 & 44 & 47 & 50 & 53 & 56 & 59 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & 33 & 36 & 39 & 42 & 45 & 48 & 51 & 54 & 57 & 60 \end{pmatrix}.$$

Step 3. Using p' and W, we get

$$p' = (-1,1,1,-1,-1,1,-1,1,1,1,-1,-1,-1,1,-1,1,1,-1,-1,1)$$

$$\downarrow$$

$$X = \begin{pmatrix} 1 & 0 & 0 & 10 & 13 & 0 & 19 & 0 & 0 & 0 & 31 & 34 & 37 & 0 & 43 & 0 & 0 & 52 & 55 & 0 \\ 2 & 0 & 0 & 11 & 14 & 0 & 20 & 0 & 0 & 0 & 32 & 35 & 38 & 0 & 44 & 0 & 0 & 53 & 56 & 0 \\ 3 & 0 & 0 & 12 & 15 & 0 & 21 & 0 & 0 & 0 & 33 & 36 & 39 & 0 & 45 & 0 & 0 & 54 & 57 & 0 \\ 0 & 6 & 9 & 0 & 0 & 18 & 0 & 24 & 27 & 30 & 0 & 0 & 0 & 42 & 0 & 48 & 51 & 0 & 0 & 60 \\ 0 & 5 & 8 & 0 & 0 & 17 & 0 & 23 & 26 & 29 & 0 & 0 & 0 & 41 & 0 & 47 & 50 & 0 & 0 & 59 \\ 0 & 4 & 7 & 0 & 0 & 16 & 0 & 22 & 25 & 28 & 0 & 0 & 0 & 40 & 0 & 46 & 49 & 0 & 0 & 58 \end{pmatrix}.$$

Step 4. From X, we obtain

$$Y = \begin{pmatrix} 1 & 55 & 52 & 10 & 13 & 43 & 19 & 37 & 34 & 31 \\ 2 & 56 & 53 & 11 & 14 & 44 & 20 & 38 & 35 & 32 \\ 3 & 57 & 54 & 12 & 15 & 45 & 21 & 39 & 36 & 33 \\ 60 & 6 & 9 & 51 & 48 & 18 & 42 & 24 & 27 & 30 \\ 59 & 5 & 8 & 50 & 47 & 17 & 41 & 23 & 26 & 29 \\ 58 & 4 & 7 & 49 & 46 & 16 & 40 & 22 & 25 & 28 \end{pmatrix}.$$

Step 5. To obtain the desired 6×10 magic rectangle Z, for $i = 1$, suppose $\mathcal{A}_1 = \{1, 2, 3, 7, 9\}$ and $\beta_1 = \{2, 5, 6, 7, 8\}$. Upon interchanging the elements in rows 1 and 3 of Y corresponding to the columns given in $\mathcal{A}_1$ and interchanging the elements in rows 4 and 6 of Y corresponding to the columns given in β_1 we obtain

$$Z = \begin{pmatrix} 3 & 57 & 54 & 10 & 13 & 43 & 21 & 37 & 36 & 31 \\ 2 & 56 & 53 & 11 & 14 & 44 & 20 & 38 & 35 & 32 \\ 1 & 55 & 52 & 12 & 15 & 45 & 19 & 39 & 34 & 33 \\ 60 & 4 & 9 & 51 & 46 & 16 & 40 & 22 & 27 & 30 \\ 59 & 5 & 8 & 50 & 47 & 17 & 41 & 23 & 26 & 29 \\ 58 & 6 & 7 & 49 & 48 & 18 & 42 & 24 & 25 & 28 \end{pmatrix}.$$

The row sums of Z are all equal to 305 and the column sums of Z are all equal to 183.

References

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