Homogeneous Traceability In Claw-Free Graphs[1]

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Abstract. A graph G is homogeneously traceable if for each vertex v of G there exists a hamiltonian path in G with initial vertex v. A graph is called claw-free if it has no induced $K_{1,3}$ as a subgraph.

In this paper,we prove that if G is a k-connected ($k > 1$) claw-free graph of order n such that the sum of degrees of any $k+2$ independent vertices is at least $n-k$, then G is homogeneously traceable. For $k = 2$, the bound $n-k$ is best possible.

As a corollary we obtain that if G is a 2-connected claw-free graph of order n such that $NC(G) \geq (n-3)/2$, where $NC(G) = \min\{|N(u) \cup N(v)| : uv \notin E(G)\}$, then G is homogeneously traceable. Moreover, the bound $(n-3)/2$ is best possible.

Introduction

We use [2] for terminology and notation not defined here and consider simple finite graphs only.

Throughout, let G be a graph of order n. We say G is claw-free if no induced subgraph of G is isomorphic to $K_{1,3}$. If G has a hamiltonian cycle (a cycle containing every vertex of G), then G is called hamiltonian. A graph G is homogeneously traceable if for each vertex v of G there exists a hamiltonian path (a path containing every vertex of G) with initial vertex v. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$ and the set of vertices adjacent to a vertex v by $N(v)$. For a subset S of $V(G)$, let $N(S) = (\cup_{v \in S} N(v)) \setminus S$. We denote by $\sigma_t(G)$ the minimum value of the degree-sum of any t pairwise non-adjacent vertices if $t \leq \alpha(G)$. If $t > \alpha(G)$, we set $\sigma_t(G) = t(n-1)$. If G is non-complete, then $NC(G)$ denotes the $\min\{|N(u) \cup N(v)| : uv \notin E(G)\}$. If G is complete, we set $NC(G) = n-1$. If no ambiguity arises, we sometimes write α for $\alpha(G)$ and σ_t for $\sigma_t(G)$.

In 1979, Chartrand, Gould and Kapoor confirmed the existence of homogeneously traceable non-hamiltonian graphs:

Theorem 1[3]. *There exists a homogeneously traceable non-hamiltonian graph of order n for all positive integers n except $3 \leq n \leq 8$.*

In 1981, Gould obtained a result about the degree-set for homogeneously traceable non-hamiltonian graphs:

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Theorem 2[5]. *Suppose $S = \{n_0, n_1, \ldots, n_k\}$ is a set of $k + 1 (\geq 1)$ positive integers and $n_i \geq 2$ for all i $(0 \leq i \leq k)$. Then S is the degree set of a homogeneously traceable non-hamiltonian graph unless $S = \{2\}$, where the degree set of a graph G is defined to be the set of degrees of the vertices of G.*

In [4], Faudree, Gould and Lindquester gave a sufficient condition in terms of neighborhood unions for a claw-free graph to be homogeneously traceable.

Theorem 3[4]. *If G is a 3-connected claw-free graph of order n such that $NC(G) > (2n-5)/3$, then G is homogeneously traceable.*

Furthermore, they made the following

Conjecture 4[4]. *If G is a 3-connected claw-free graph of order n such that $NC(G) \geq (2n-5)/3$, then G is hamiltonian.*

The following result was obtained by Zhang.

Theorem 5[6]. *If G is a k-connected $(k \geq 2)$ claw-free graph of order n with $\sigma_{k+1} \geq n - k$, then G is hamiltonian.*

As Bauer, Fan and Veldman pointed out, the following consequence of Theorem 5 improves Theorem 3 and Conjecture 4.

Theorem 6[1]. *If G is a 2-connected claw-free graph of order n with $NC(G) \geq (2n-5)/3$, then G is hamiltonian.*

In this paper, we obtain an analogue of Theorem 5 for homogeneously traceable graphs.

Theorem 7. *If G is a k-connected $(k \geq 2)$ claw-free graph of order n with $\sigma_{k+2} \geq n - k$, then G is homogeneously traceable.*

Obviously, when $k = 2$, Theorem 7 gives an improvement of Theorem 3.

Corollary 8. *If G is a 2-connected claw-free graph of order n with $NC(G) \geq (n-3)/2$, then G is homogeneously traceable.*

Proof of Theorem 7

To prove Theorem 7, we first give some convenient terminology and notation. For any path $Q = u_1 u_2 \ldots u_q$ of G, let $Q[u_i, u_j]$ represent both the subpath of Q from u_i to u_j and its vertex-set. For convenience, we sometimes use u_i^+ for u_{i+1} and u_i^- for u_{i-1}. For any vertex v of G, we call v *insertible* on Q if there exists an integer i such that $u_i, u_{i+1} \in N(v)$ and $\{u_i, u_{i+1}\}$ the insertion-pair of v on Q. We also denote by $g(v, Q)$, all insertion-pairs of v on Q. If no ambiguity arises, we sometimes write $g(v)$ instead of $g(v, Q)$. If P is a path with initial vertex u, then we call P u-path.

Now, let G be a k-connected ($k \geq 2$) claw-free graph of order n with $\sigma_{k+2} \geq n-k$. If G is homogeneously traceable, we are done. Otherwise, for some vertex of G, say v_1, there exists no hamiltonian v_1-path in G. Let $P = v_1 v_2 \ldots v_t$ be a v_1-path, and v_m be the first vertex adjacent to v_t along $P[v_1, v_t]$. We choose P such that

(A) The path P is as long as possible,

(B) Under (A), the subscript m is as small as possible.

Let v_s be the first vertex non-adjacent to v_t along $P[v_m, v_t]$, and set $A=\{v_i: v_{i-1}v_t \in E(G)\}$. By the choice of P, we get

$$N(A) \cap (P[v_1, v_{m-1}] \cup (V(G)\backslash V(P))) = \emptyset.$$

Thus, we have $s < t-1$, for otherwise v_m is a cut-vertex of G, which contradicts the 2-connectedness of G.

Because $V(G)\backslash V(P) \neq \emptyset$, let H be a component of $G\backslash V(P)$. By the k-connectedness of G, there exist h edges joining H and P ($h \geq k \geq 2$). Note that $h \geq |N(x) \cap V(P)|$ for any $x \in H$. Let these edges be $\{x_i v_{j(i)}: i = 1, 2, \ldots, h\}$, where $x_i \in V(H)$, $v_{j(i)} \in V(P)$, for $i = 1, 2, \ldots, h$ and $1 \leq j(1) < j(2) < \cdots < j(h) < t$. Set $B = \{v_{j(1)}, v_{j(2)}, \ldots, v_{j(h)}\}$.

We may choose H such that

(C) Under (B), the subscript $j(h)$ is as large as possible.

Let $x_0 \in H$ and $x_i H x_j$ denote a path of H joining x_i and x_j. For any $i (0 < i < h)$, along $P[v_{j(i)+1}, v_{j(i+1)-1}]$ we choose $\mu(i)$ such that

(D) Under (C), $v_{j(i)+\mu(i)}$ is the first non-insertible vertex on $Q(i)$, where $Q(i) = v_1 v_2 \ldots v_{j(i)} x_i H x_{i+1} v_{j(i+1)} v_{j(i+1)+1} \ldots v_t$ i.e., for any β ($0 < \beta < \mu(i)$), $g(v_{j(i)+\beta}, Q(i)) \neq \emptyset$.

For any i ($0 < i < h$) and β ($0 < \beta \leq \mu(i)$), we denote by $f^{\beta}_{j(i)}(Q)$ an operation of inserting $\{v_{j(i)+1}, v_{j(i)+2}, \ldots, v_{j(i)+\beta-1}\}$ into Q, where Q is a path of G. We have

Lemma 1. *The operation $f^{\beta}_{j(i)}(Q)$ is well-defined for all i ($0 < i < h$) and β ($0 < \beta \leq \mu(i)$).*

Proof: To prove this assertion, choose the largest index q ($0 < q < \beta$) such that there exists an integer r with $\{v_r, v_{r+1}\} \in g(v_{j(i)+1}) \cap g(v_{j(i)+q})$. Replacing edge $v_r v_{r+1}$ of $Q(i)$ by subpath $v_r v_{j(i)+1} \ldots v_{j(i)+q} v_{r+1}$, we obtain a path Q'.

If $q < \beta - 1$, repeat this procedure for Q' and $\{v_{j(i)+q+1}, \ldots, v_{j(i)+\beta-1}\}$ in place of Q and $\{v_{j(i)+1}, \ldots, v_{j(i)+\beta-1}\}$ until we have inserted $\{v_{j(i)+1}, \ldots, v_{j(i)+\beta-1}\}$ into Q.

Obviously $V(f^{\beta}_{j(i)}(Q)) = V(Q) \cup \{v_{j(i)+1}, \ldots, v_{j(i)+\beta-1}\}$. So $0 < \mu(i) < j(i+1) - j(i)$ by the maximality of P.

We set $I_{h-1} = \{v_{j(1)+\mu(1)}, \ldots, v_{j(h-1)+\mu(h-1)}\}$.

Lemma 2. *For any* $i, r(0 < i < r < h), \beta(0 < \beta < \mu(i))$ *and* $\gamma(0 < \gamma < \mu(r))$*, we have*

(1) $v_{j(i)+\beta}v_{j(r)+\gamma} \notin E(G)$ *and*
(2) $g(v_{j(i)+\beta}) \cap g(v_{j(r)+\gamma}) = \emptyset$.

Proof: If it is not true, without loss of generality we suppose γ and β are the smallest integers satisfying either $v_{j(i)+\beta}v_{(r)+\gamma} \in E(G)$ or $g(v_{j(i)+\beta}) \cap g(v_{j(r)+\gamma}) \neq \emptyset$.

If $v_{j(i)+\beta}v_{j(r)+\gamma} \in E(G)$, then the v_1-path $f^{\gamma}_{j(r)}(f^{\beta}_{j(i)}(Q))$ is longer than P, where

$$Q = v_1 v_2 \ldots v_{j(i)} x_i H x_r v_{j(r)} v_{j(r)-1} \ldots v_{j(i)+\beta} v_{j(r)+\gamma} v_{j(r)+\gamma+1} \ldots v_t.$$

It contradicts the maximality of P.

If $\{v_p, v_{p+1}\} \in g(v_{j(i)+\beta}) \cap g(v_{j(r)+\gamma})$ and $p < j(i)$, then the v_1-path $f^{\gamma}_{j(r)}(f^{\beta}_{j(i)}(Q))$ is longer than P, where

$$Q = v_1 v_2 \ldots v_p v_{j(i)+\beta} v_{j(i)+\beta+1} \ldots v_{j(r)} x_r H x_i v_{j(i)} v_{j(i)-1} \ldots$$
$$v_{p+1} v_{j(r)+\gamma} v_{j(r)+\gamma+1} \ldots v_t.$$

It contradicts the maximality of P. Similarly, we reach a contradiction for $p > j(r)$ or $j(i) \leq p \leq j(r)$.

By Lemma 2, we know $f^{\beta}_{j(i)}(f^{\gamma}_{j(r)}(Q))$ is well-defined for all β $(0 < \beta \leq \mu(i))$ and γ $(0 < \gamma \leq \mu(r))$.

Lemma 3. *For all* i *and* r $(0 < i < r < h)$*, we have*

(1) $v_{j(i)+\mu(i)}v_{j(r)+\mu(r)} \notin E(G)$ *and*
(2) $N(v_{j(i)+\mu(i)}) \cap N(v_{j(r)+\mu(r)}) = \emptyset$.

Proof: The assertion $v_{j(i)+\mu(i)}v_{j(r)+\mu(r)} \notin E(G)$ results directly from Lemma 2.

If there exists $y \in N(v_{j(i)+\mu(i)}) \cap N(v_{j(r)+\mu(r)})$,then there exists an integer p such that $y = v_p \in V(P)$, otherwise the v_1-path $f^{\mu(i)}_{j(i)}(f^{\mu(r)}_{j(r)}(Q))$ is longer than P, where

$$Q = v_1 v_2 \ldots v_{j(i)} x_i H x_r v_{j(r)} v_{j(r)-1} \ldots v_{j(i)+\mu(i)} y v_{j(r)+\mu(r)} v_{j(r)+\mu(r)+1} \ldots v_t,$$

which contradicts the maximality of P.

If $v_p \in N(v_{j(i)+\mu(i)}) \cap N(v_{j(r)+\mu(r)}) \cap V(P)$, then $p \neq j(r)$ by Lemma 3. Without loss of generality, we suppose $p \leq j(i)$. By the choice of $\mu(i)$ and $\mu(r)$, we have $v_{p-1} \notin N(v_{j(i)+\mu(i)}) \cup N(v_{j(r)+\mu(r)})$. Thus the induced subgraph $G[v_p, v_{p-1}, v_{j(i)+\mu(i)}, v_{j(r)+\mu(r)}]$ is a claw, which contradicts the maximality of P.

If $j(i) + \mu(i) < p < j(r)$, we have $v_{p+1}v_{j(i)+\mu(i)} \notin E(G)$, otherwise the v_1-path $f_{j(i)}^{\mu(i)}(f_{j(r)}^{\mu(r)}(Q))$ is longer than P, where

$$Q = v_1 v_2 \ldots v_{j(i)} x_i H x_r v_{j(r)} v_{j(r)-1} \ldots v_{p+1} v_{j(i)+\mu(i)} v_{j(i)+\mu(i)+1} \ldots$$
$$v_p v_{j(r)+\mu(r)} v_{j(r)+\mu(r)+1} \ldots v_t,$$

which contradicts the maximality of P. And by the choice of $\mu(r)$, the induced subgraph $G[v_p, v_{p+1}, v_{j(i)+\mu(i)}, v_{j(r)+\mu(r)}]$ is a claw, a contradiction. Similarly, we obtain a contradiction for $p > j(r) + \mu(r)$.

Let $I_h = I_{h-1} \cup \{x_0\}$, then

Lemma 4. *The set I_h is an independent set of vertices in $V(G)$.*

Lemma 5. *For all $i, r(1 < i, r < h)$, $v_{j(i)}v_{j(r)+\mu(r)} \notin E(G)$.*

Proof: Obviously, $v_{j(i)}v_{j(i)+\mu(i)} \notin E(G)$ for all i $(2 \le i < h)$. Otherwise by the choice of $\mu(i)$, we have $v_{j(i)-1}v_{j(i)+\mu(i)} \notin E(G)$, then the subgraph $G[v_{j(i)}, x_i, v_{j(i)-1}, v_{j(i)+\mu(i)}]$ is a claw, a contradiction.

If we assume that $v_{j(i)}v_{j(r)+\mu(r)} \in E(G)$ and $r \neq i$, then because neither $G[v_{j(i)}, x_i, v_{j(r)+\mu(r)}, v_{j(i)+1}]$ nor $G[v_{j(i)}, x_i, v_{j(r)+\mu(r)}, v_{j(i)-1}]$ is a claw, we have that $v_{j(i)+1}, v_{j(i)-1} \in N(v_{j(r)+\mu(r)})$, and $\{v_{j(i)+1}, v_{j(i)}\}$ or $\{v_{j(i)-1}, v_{j(i)}\}$ is an insertion-pair of $v_{j(r)+\mu(r)}$, which contradicts the choice of $\mu(r)$.

By the maximality of P and Lemma 5, we have

Lemma 6. *For all $i(1 < i < h)$, $N(x_0) \cap N(v_{j(1)+\mu(1)}) \subseteq \{v_1\}$ and $N(x_0) \cap N(v_{j(i)+\mu(i)}) = \emptyset$.*

Combining Lemmas 1-6, we obtain

Proposition 1. *The set I_h is an independent set of vertices in $V(G)$.*

Proposition 2 *The sets $N(x_0)\backslash\{v_1\}$, $N(v_{j(1)+\mu(1)})$, $N(v_{j(2)+\mu(2)})$, $\ldots$, $N(v_{j(h-1)+\mu(h-1)})$ and I_h are pairwise disjoint.*

Now we examine two cases according to whether $m < j(h)$ or $m \ge j(h)$.

Case 1. $m < j(h)$

In this case, we may choose $\mu(h)$ such that $v_{j(h)+\mu(h)}$ is the first non-insertible vertex of $P[v_{j(h)+1}, v_t]$ on $Q = v_1 v_2 \ldots v_{j(h)} x_h$. We also obtain

Claim 1.0.

(1) $0 < \mu(h) < t - j(h)$.
(2) $N(v_{j(h)+\mu(h)}) \cap I_h = \emptyset$.
(3) $N(v_{j(h)+\mu(h)}) \cap N(I_h) = \emptyset$.

Because $m < j(h)$, there exists an integer a $(0 \leq a < h)$ such that $j(a) < m < s < j(a+1)$ (let $j(0) = \mu(0) = 0$). Obviously $j(a) + \mu(a) < m - 1$. Choose s' such that $v_{s+s'}$ is the first non-insertible vertex of $v_s v_{s+1} \dots v_{j(a+1)-1}$ on $Q = v_1 v_2 \dots v_{s-1} v_t v_{t-1} \dots v_{j(a+1)} x_{a+1}$. By the maximality of P, we have $0 \leq s' < j(a+1) - s$ and the following claims:

Claim 1.1. *$N(v_{s+a}) \cap P[v_1, v_{m-1}] = \emptyset$ for all a $(m - s < a < s' + 1)$.*

Proof: It is true that $N(v_{s+a}) \cap P[v_1, v_{m-1}] = \emptyset$ for all a $(m - s < a \leq 0)$, as $N(z) \cap P[v_1, v_{m-1}] = \emptyset$ for all $z \in \{v_i : v_{i-1} v_t \in E(G)\}$. For $0 < a < s' + 1$, let a be the smallest integer such that there exists an integer p satisfying $v_p \in N(v_{s+a}) \cap P[v_1, v_{m-1}]$. Set $Q = v_1 v_2 \dots v_{s-1}\ v_t v_{t-1} \dots v_{s+a}$. Then the v_1-path $f_{s-1}^{a}(Q)$ is longer than P, a contradiction.

Claim 1.2. *$x_0 v_{s+s'} \notin E(G)$ and $N(x_0) \cap N(v_{s+s'}) = \emptyset$.*

Proof: Obviously, $x_0 v_{s+s'} \notin E(G)$ is true.

By the choice of P and H, $(N(x_0) \cap N(v_{s+s'})) \backslash V(P) = \emptyset$. If $v_{j(r)} \in N(v_{s+s'})$, then $r > a$. By the choice of s', we have $v_{j(r)+1} \notin N(v_{s+s'})$, and the induced subgraph $G[v_{j(r)}, x_r, v_{j(r)+1}, v_{s+s'}]$ is a claw, a contradiction.

Similarly to the proofs of Lemma 3 and Lemma 4, we get

Claim 1.3. *For all i $(0 < i \leq h)$, γ $(0 \leq \gamma < s')$ and β $(0 < \beta < \mu(i))$,*

(1) $v_{s+\gamma} v_{j(i)+\beta} \notin E(G)$ *and*
(2) $g(v_{j(i)+\beta}) \cap g(v_{s+\gamma}) = \emptyset$.

Claim 1.4. *For all i $(1 \leq i \leq h)$,*

(1) $v_{s+s'} v_{j(i)+\mu(i)} \notin E(G)$ *and*
(2) $N(v_{s+s'}) \cap N(v_{j(i)+\mu(i)}) = \emptyset$.

Let $I_{h+2} = I_h \cup \{v_{s+s'}, v_{j(h)+\mu(h)}\}$. By Claims 1.0-1.4, we obtain

Proposition 3. *The set I_{h+2} is an independent set of vertices in G.*

Proposition 4 *The sets $N(x_0)\backslash\{v_1\}$, $N(v_{j(1)+\mu(1)})$, $N(v_{j(2)+\mu(2)})$, $\dots$, $N(v_{j(h)+\mu(h)})$, $N(v_{s+s'})$ and I_{h+2} are pairwise disjoint.*

So, we obtain that

$$\begin{aligned}
\sigma_{k+2} \leq \sigma_{h+2} &\leq \sum_{x \in I_{h+2}} |N(x)| \\
&\leq |\cup_{x \in I_{h+2}} N(x)| + 1 \leq |V(P) \backslash I_{h+2}| + 1 \\
&\leq n - (h+2) + 1 = n - h - 1 \\
&\leq n - k - 1.
\end{aligned}$$

It contradicts $\sigma_{k+2} \geq n - k$.

Case 2. $m \geq j(h)$

First we have $m \neq j(h)$, for otherwise the subgraph $G[v_{j(h)}, x_h, v_{j(h)-1}, v_t]$ is a claw. Moreover, by the choice of P and H, $N(v)\backslash P[v_m, v_t] = \emptyset$ for all $v \in P[v_{m+1}, v_t]$. By the k-connectedness ($k \geq 2$) of G, there exists an edge joining $P[v_1, v_{m-1}]$ and $P[v_{m+1}, v_t]$, let $v_{j^*}v_l$ be such an edge and

(E) Under (D), the subscript l is as large as possible,

where $0 < j^* < m$ and $s < l < t$. So $N(v) \cap P[v_1, v_{m-1}] = \emptyset$ for any $v \in P[v_{l+1}, v_t]$.

We choose s' (resp. l') such that $v_{s+s'}$ (resp. $v_{l+l'}$) is the first non-insertible vertex of $P[v_s, v_l]$ (resp. $P[v_l, v_t]$) on $Q = v_1 v_2 \ldots v_{s-1} v_t v_{t-1} \ldots v_l$ (resp. $Q = v_1 \ldots v_l$). Then we reach the following claims.

Claim 2.1.

(1) $(N(v_{s+\beta}) \cup N(v_{l+\gamma}))\backslash P[v_m, v_t] = \emptyset$, *for all* β $(0 \leq \beta \leq s')$ *and* γ $(0 \leq \gamma \leq l')$.

(2) $N(v_{s+s'}) \cap P[v_{l+1}, v_{l+l'}] = \emptyset$ *and* $N(v_{l+l'}) \cap P[v_{m+1}, v_{s+s'}] = \emptyset$.

(3) $v_m \notin N(v_{s+s'}) \cup N(v_{l+l'})$.

Proof: The assertion (1) follows directly from the minimality of m and the choice of P, H and l.

(2) If $v_{s+s'}v_{l+\beta} \in E(G)$, $(1 \leq \beta \leq l')$, then the v_1-path $f_l^{\beta}(f_{s-1}^{s'}(Q))$ is longer than P, where $Q = v_1 v_2 \ldots v_{s-1} v_t v_{t-1} \ldots v_{l+\beta} v_{s+s'} v_{s+s'+1} \ldots v_l$, a contradiction. So we have $N(v_{s+s'}) \cap P[v_{l+1}, v_{l+l'}] = \emptyset$. Similarly, we reach $N(v_{l+l'}) \cap P[v_{s+1}, v_{s+s'}] = \emptyset$.

(3) If $v_m \in N(v_{s+s'})$, by the choice of s', $v_{m+1} \notin N(v_{s+s'})$ and $v_{m-1} \notin N(v_{s+s'})$, then the induced subgraph $G[v_m, v_{m+1}, v_{m-1}, v_{s+s'}]$ is a claw, a contradiction. Similarly, $v_m \notin N(v_{l+l'})$.

With the same argument used in the proof of Lemma 2, we obtain

Claim 2.2. *For all* β $(0 \leq \beta \leq s')$ *and* γ $(0 \leq \gamma \leq l')$

(1) $v_{s+\beta}v_{l+\gamma} \notin E(G)$ *and*

(2) $g(v_{s+\beta}) \cap g(v_{l+\gamma}) = \emptyset$.

Claim 2.3. *For all* i $(0 < i < h)$, *we have*

(1) $(N(v_{s+s'}) \cup N(v_{l+l'})) \cap (I_h \cup B) = \emptyset$,

(2) $N(v_{s+s'}) \cap N(v_{j(i)+\mu(i)}) = \emptyset$ *and*

(3) $N(v_{l+l'}) \cap N(v_{j(i)+\mu(i)}) = \emptyset$.

Proof: By Claim 2.1(2), (1) is true, $N(v_{l+l'})\backslash P[v_{s+s'+1}, v_t] = \emptyset$, $N(v_{s+s'})\backslash (P[v_{m+1}, v_{l-1}] \cup P[v_{l+l'+1}, v_t]) = \emptyset$. Moreover, $N(v_{j(i)+\mu(i)}) \cap (P[v_{m+1}, v_{s+s'}] \cup P[v_{l+1}, v_t]) = \emptyset$. So if $v \in N(v_{j(i)+\mu(i)}) \cap (N(v_{s+s'}) \cup N(v_{l+l'}))$, then $v \in P[v_{s+s'+1}, v_t]$.

If $v \in N(v_{s+s'}) \cap N(v_{j(i)+\mu(i)})$, then $v^+, v^- \notin N(v_{j(i)+\mu(i)})$ by the choice of $\mu(i)$. Because the induced subgraph $G[v, v^+, v^-, v_{j(i)+\mu(i)}]$ is not a claw, we have $v^+v^- \in E(G)$, and the v_1-path $f^{s'}_{s-1}(Q)$ is as long as P, where

$$Q = v_1 v_2 \ldots v_{s-1} v_t v_{t-1} \ldots v^+ v^- \ldots v_{s+s'} v,$$

which contradicts the minimality of m.

If $v \in N(v_{l+l'}) \cap N(v_{j(i)+\mu(i)})$, then $v^- \notin N(v_{l+l'}) \cup N(v_{j(i)+\mu(i)})$ by the choice of $\mu(i)$ and l'. So the induced subgraph $G[v, v^-, v_{j(i)+\mu(i)}, v_{l+l'}]$ is a claw, a contradiction.

Claim 2.4. *$v_{s+s'}v_{l+l'} \notin E(G)$ and $N(v_{s+s'}) \cap N(v_{l+l'}) = \emptyset$.*

Proof: The assertion $v_{s+s'}v_{l+l'} \notin E(G)$ results directly from Claim 2.3.

If $v \in N(v_{s+s'}) \cap N(v_{l+l'})$, then $v \in P[v_{s+s'+1}, v_{l-1}] \cup P[v_{l+l'+1}, v_t]$ by Claim 2.1. Let $v = v_p$.

If $s + s' < p < l$, then $v_{p+1}, v_{p-1} \notin N(v_{l+l'})$ by the choice of l'. As the induced subgraph $G[v_p, v_{p-1}, v_{p+1}, v_{l+l'}]$ is not a claw, we have $v_{p-1}v_{p+1} \in E(G)$, thus the v_1-path $f^{l'}_l(f^{s'}_{s-1}(Q))$ is as long as P, where

$$Q = v_1 v_2 \ldots v_{s-1} v_t v_{t-1} \ldots v_{l+l'} v_p v_{s+s'} v_{s+s'+1} \ldots v_{p-1} v_{p+1} v_{p+2} \ldots v_l,$$

which contradicts the minimality of m. Similarly, we also get a contradiction when $l + l' < p < t$.

If $v_t \in N(v_{s+s'}) \cap N(v_{l+l'})$, then $v_{s-1}v_{s+s'} \in E(G)$ since the induced subgraph $G[v_t, v_{s-1}, v_{s+s'}, v_{l+l'}]$ is not a claw. Hence $v_{s-2}v_{s+s'} \notin E(G)$ by the choice of s'. But $v_{s-2}v_{l+l'} \notin E(G)$ (by Claim 2.1) and $v_{s+s'}v_{l+l'} \notin E(G)$, then the subgraph $G[v_t, v_{s-2}, v_{s+s'}, v_{l+l'}]$ is a claw, a contradiction.

Let $I'_{h+2} = I_h \cup \{v_{s+s'}, v_{l+l'}\}$. By Claims 2.1-2.4 and Proposition 1 and 2 we obtain:

Proposition 5. *The set I'_{h+2} is an independent set of vertices in G.*

Proposition 6. *The sets $N(x_0)\backslash\{v_1\}$, $N(v_{j(1)+\mu(1)}), \ldots, N(v_{j(h-1)+\mu(h-1)})$, $N(v_{s+s'})$, $N(v_{l+l'})$ and I'_{h+2} are pairwise disjoint.*

So we also have

$$\begin{aligned}\sigma_{k+2} \le \sigma_{h+2} &\le \sum_{x \in I'_{h+2}} |N(x)| \\ &\le |\cup_{x \in I'_{h+2}} N(x)| + 1 \le |V(P)\backslash I'_{h+2}| + 1 \\ &\le n - (h+2) + 1 = n - h - 1 \\ &\le n - k - 1\end{aligned}$$

It contradicts $\sigma_{k+2} \geq n-k$. The proof of Theorem 7 is complete.

In the case of $k = 2$, let $\{x_1, x_2, x_3, x_4\}$ be an independent set of vertices in $V(G)$ with $\sigma_4 = \sum_{i=1}^{4} |N(x_i)|$, then we have

$$\begin{aligned}\sigma_4 &= \sum_{i=1}^{4} |N(x_i)| = \left|\bigcup_{i=1}^{4} N(x_i)\right| + \sum_{1\leq i<j\leq 4} |N(x_i) \cap N(x_j)| \\ &= |\cup_{i=1}^{4} N(x_i)| + \sum_{1\leq i<j\leq 4} (|N(x_i)| + |N(x_j)| - |N(x_i) \cup N(x_j)|) \\ &= |\cup_{i=1}^{4} N(x_i)| + 3\sum_{i=1}^{4} |N(x_i)| - \sum_{1\leq i<j\leq 4} |N(x_i) \cup N(x_j)| \\ &\leq n - 4 + 3\sigma_4 - 6NC(G),\end{aligned}$$

since G is claw-free. So $\sigma_4(G) \geq (6NC(G) + 4 - n)/2$. And if $NC(G) \geq (n-3)/2$ then $\sigma_4 \geq (2n-5)/2$, equivalently $\sigma_4 \geq n-2$. By Theorem 7 G is homogeneously traceable and Corollary 8 is true.

Remark

We note that Theorem 7 for $k = 2$ and Corollary 8 are best possible. The graph G illustrated in Fig. 1 is 2-connected, claw-free and has $n = 4p + 8$ vertices. Note that $\sigma_4 = 4p + 5 = n - 3$, and $NC(G) = 2p + 2 = (n-4)/2$. But G is not homogeneously traceable, because there exists no hamiltonian v-path.

G:

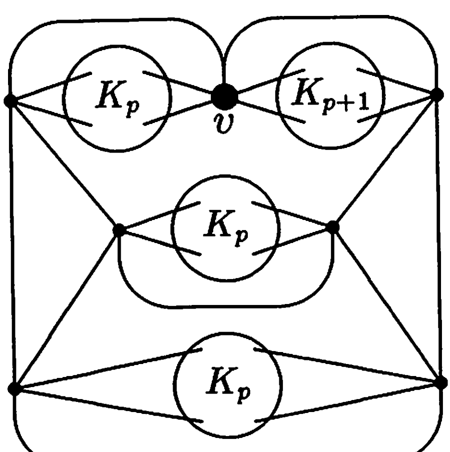

Figure 1

References

1. D. Bauer, G. Fan and H.J. Veldman, *Hamiltonian Properties of Graphs with Large Neighborhood Unions.* preprint.
2. J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", Macmillan London and Elsevier, New York, 1976.
3. G. Chartrand, R.J. Gould and S.F. Kapoor, *On Homogeneously Traceable non-Hamiltonian Graphs*, Annals of the New York Academy of Sciences **319** (1979), 130–135.
4. R.J. Faudree, R.J. Gould and T.E. Lindquester, *Hamiltonian Properties and Adjacency Conditions in Claw-Free Graphs*, Proc. 6th Int. Conf. on Theory and Applications of Graphs, Kalamazoo, Michigan (1988).
5. R.J. Gould, *Degree Sets for Homogeneously Traceable non-Hamiltonian Graphs*, Colloquium Mathematicum VOL. XLV (1981). FASC.1.
6. C.Q. Zhang, *Hamilton Cycles in Claw-Free Graphs*, J. Graph Theory **12** (1988), 209–216.