# The binary matroids having an element which is in every four-wheel minor

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**Abstract.** The binary matroids with no three- and four-wheel minors were characterized by Brylawski and Oxley, respectively. The importance of these results is that, in a version of Seymour's Splitter Theorem, Coullard showed that the three- and four-wheel matroids are the basic building blocks of the class of binary matroids. This paper determines the structure of a class of binary matroids which almost have no four-wheel minor. This class consists of matroids $M$ having a four-wheel minor and an element $e$ such that both the deletion and contraction of $e$ from $M$ have no four-wheel minor.

## 1. Introducton

The matroid terminology used mostly follows Oxley [13], Truemper [15], and Welsh [17]. Let $M$ be a matroid. The ground set of $M$ is denoted by $E(M)$. Let $X \subseteq E(M)$. The rank of $X$ in $M$ is denoted by either $rk\ X$ or $rk_M X$. The deletion and contraction of $X$ from $M$ are denoted by $M\backslash X$ and $M/X$, respectively. The restriction of $M$ to $X$ is denoted by $M \mid X$. Three-element circuits and cocircuits of $M$ are called *triangles* and *triads*, respectively.

Let $M_1$ and $M_2$ be matroids on $E(M) \cup e_1$ and $E(M) \cup e_2$, respectively, such that $M_1\backslash e_1 = M$ and $M_2/e_2 = M$. We say that $M_1$ is an *addition* to $M$ and $M_2$ is an *expansion* of $M$. We say that $M_1$ is a *non-trivial* addition to $M$ if $e$ is neither a loop nor a coloop of $M_1$ and $e$ is not in a 2-element circuit of $M_1$.

Let $N$ be a minor of $M$. A minor of $M$ which is isomorphic to $N$ is called an $N$*-minor*. Let $e \in E(M)$. We say that $e$ is *in every* $N$*-minor* of $M$ if neither $M\backslash e$ nor $M/e$ has an $N$-minor. Evidently $e$ is in every $N$-minor of $M$ if and only if it is in every $N^*$-minor of $M^*$. We say that $e$ *avoids* some $N$-minor of $M$ if there exists an $N$-minor of $M$ whose ground set does not contain $e$. Define $\tilde{M/e}$ to be a matroid obtained from $M/e$ by deletion of the elements of each parallel class except for one representative of each class. Similarly derive $\tilde{M\backslash e}$ from $M\backslash e$ by contracting the elements of each series class except for one representative of each class [15].

If $k$ is a positive integer, then a bipartition $(A, B)$ of $E(M)$ is a $k$*-separation* of $M$ if $|A| \geq k$, $|B| \geq k$, and $rk\ A + rk\ B - rk\ M \leq k - 1$ [16]. For an integer $n \geq 2$, $M$ is $n$*-connected* if it has no $k$-separations for any $k < n$. In a

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3-connected matroid with at least four elements there are no circuits or cocircuits with fewer than three elements.

Let $A$ be a matrix with entries in the field $GF(2)$. The dependence matroid on the columns of $A$ is denoted by $D(A)$. We say that $A$ and $D(A)$ are *binary*. If column $e$ is adjoined to $A$, then $A+e$ denotes the resulting matrix. If $e$ is a column of $A$, then $A\backslash e$ denotes the matrix obtained by deleting $e$ from $A$. Suppose that $f$ is a column of $A$ whose sole non-zero entry is a one in row $i$. Then $A/f$ denotes the matrix obtained by removing row $i$ and column $f$ from $A$.

An example of a matrix $A$ in standard form together with its associated dual matrix $A^*$ is given in Figure 1. The $r$ by $r$ identity matrix is denoted by $I_r$.

$$A = \begin{array}{c} e_1 \ldots e_r, e_{r+1} \ldots e_n \\ \hline I_r, B \end{array} \qquad A^* = \begin{array}{c} e_{r+1} \ldots e_n, e_1 \ldots e_r \\ \hline I_{n+r}, B^T \end{array}$$

Suppose that $E$ and $F$ are binary matrices such that one can be obtained from the other by interchanging columns and performing elementary row operations. Then we say that $E$ and $F$ are *equivalent*. The unique representability of binary matroids is used throughout the paper [5,(3.7)].

We next give the binary case of a result of Coullard [6,(8.10)]. This result is a version of the Splitter Theorem [14]. It indicates the central role played by the three- and four-wheel matroids in the class of binary matroids. The wheel-matroid of rank $r$ is denoted by $W_r$.

**1.1 Theorem.** *Let $N$ be a 3-connected proper minor of a 3-connected binary matroid $M$ such that $|E(N)| \geq 4$ and $M$ is not a wheel. Suppose that if $N \cong W_3$, then $M$ has no $W_4$-minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong N, M_n = M$ and, for each $i$ in $\{1,2,\ldots,n\}$, $M_i$ is an addition to or expansion of $M_{i-1}$.*

The sequence of matroids in Theorem 1.1 is said to be a *chain* of 3-connected matroids from $N$ to $M$.

For each $r \geq 4$ let $H_r$ be the binary matrix given in Figure 2.

$$\begin{array}{c|cccc|ccc} a_1 a_2 \ldots a_r & b_1 & b_2 & \ldots & b_r & c_r & d_r & e_r \\ & 0 & 1 & \ldots & 1 & 1 & 0 & 1 \\ & 1 & 0 & \ldots & 1 & 1 & 1 & 0 \\ I_r & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\ & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\ & 1 & 1 & \ldots & 0 & 1 & 0 & 1 \end{array}$$

**Figure 2:** $H_r$

Let $A_r$ and $B_r$ denote the matrices $H_r \backslash b_r$ and $H_r \backslash d_r, e_r$, respectively. Let $Y_r = D(A_r)$ and $Z_r = D(B_r)$. Evidently $Z_r^* \cong Z_{r+1} \backslash b_{r+1}, c_{r+1}$ as the matrices $B_r^*$ and $B_{r+1} \backslash b_{r+1}, c_{r+1}$ have the same columns. The matroids $Y_r \backslash b_{r-1}, d_r$ and $Y_r \backslash c_r, d_r$ are self-dual. This can be seen by replacing row $r$ in $A_r \backslash b_{r-1}, d_r$ and $A_r \backslash c_r, d_r$ by row 1+ row $r$ and then interchanging the first and last columns. Each of the two resulting matrices has the same columns as its dual matrix. For $r \geq 5$ the matroids $(Y_{r-1} \backslash d_{r-1})^*$ and $Y_r \backslash b_{r-1} c_r, d_r$ are isomorphic. This can be seen by replacing row $i$ of $(A_{r-1} \backslash d_{r-1})^*$ by row $i$+ row $r$ for each $i$ in $\{2, 3, \ldots, r-1\}$. Then replace row $r$ of the resulting matrix by row 1+ row $r$. A suitable reordering of the columns produces $A_r \backslash b_{r-1}, c_r, d_r$.

Brylawki [3] stated that a binary matroid has no $W_3$-minor if and only if it is a series-parallel network (for graphs see [1], [8], [9]). The next theorem forms the core of a complete decomposition of the binary matroids with no $W_4$-minor given by Oxley [11,(2.1)].

**1.2 Theorem.** *Let $M$ be a binary matroid with $rk\ M \geq 4$ and $rk\ M^* \geq 4$. Then $M$ is 3-connected and has no $W_4$-minor if and only if $M \cong Z_r, Z_r^*, Z_r \backslash b_r$, or $Z_r \backslash c_r$ for some $r \geq 4$.*

The following result characterizes binary matroids which almost have no four-wheel minor. This is the main result of the paper.

**1.3 Theorem.** *Let $M$ be a 3-connected binary matroid with $rk\ M \geq 7$ and $rk\ M^* \geq 7$. Then the number of elements of $M$ which are in every $W_4$-minor exceeds one if and only if for some $r \geq 7$ either $M$ or $M^*$ is isomorphic to $Y_r \backslash X$ where $X$ is a possibly empty subset of $\{b_{r-1}, c_r, d_r\}$. The only elements of $Y_r \backslash X$ which are in every $W_4$-minor are $a_1, a_r, b_1$, and $e_r$ when $d_r \in X$, and $a_1, a_r$, and $b_1$ when $d_r \notin X$.*

The previous theorem states that a 3-connected binary matroid $M$ with rank and corank at least seven has exactly $0, 1, 3$, or 4 elements which are in every four-wheel minor. Suppose that $M$ has exactly one such element $e$. Then, for some $m \geq 5$, there exists a chain of 3-connected matroids $M_0, M_1, \ldots, M_n$ from one of $Z_m, Z_m^*, Z_m \backslash b_m$, and $Z_m \backslash c_m$ to $M$. If $n > 1$, then $M_1$ is an expansion of $Z_m$ or $Z_m \backslash c_m$, or $M_1$ is an addition to $Z_m^*$ or $Z_m \backslash b_m$. Either $M/e$ or $M \backslash e$ is 3-connected and has no $W_4$-minor. The proof of these statements is similar to the proof of Theorem 1.3 and is omitted. The next result corresponds to Theorem 1.3 for regular matroids. The graph $K_5 - e$ is obtained by deleting an edge from the complete graph on five vertices.

**1.4 Theorem.** *Let $M$ be a 3-connected regular matroid. Then $M$ has an element which is in every $W_4$-minor if and only if $M \cong W_4, M(K_5 - e)$, or $M^*(K_5 - e)$.*

Let $F$ be a flat of a matroid $M$. we say that $F$ is a *modular flat* of $M$ if, whenever $E$ is a flat of $M$, $rk\ E + rk\ F = rk\ E \cup F + rk\ E \cap F$.

Let $M_1$ and $M_2$ be matroids whose ground sets meet in $T$ such that $M_1|T = M_2|T$. If $T$ is a modular flat of $M_1$, then the *generalized parallel connection* of $M_1$ and $M_2$ across $T$ is denoted by $P_T(M_1, M_2)$ [4,section 5]. This matroid has the property that for each of its flats $F$, $rkF = rk_{M_1}\ F \cap E(M_1) + rk_{M_2}\ F \cap E(M_2) - rk_{M_1}\ F \cap T$.

The next lemma gives the geometric connection between the binary matroids with no $W_4$-minor of Theorem 1.2, and those having elements in every $W_4$-minor of Theorem 1.3. It states that the matroid $Y_r$ is obtained by attaching a three-wheel matroid to $Z_{r-1}$ using a generalized parallel connection.

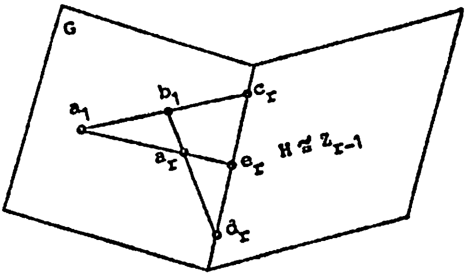


Figure 3

**1.5 Lemma.** *For $r \geq 5$ let $G$ and $H$ be the restrictions of $Y_r$ to $\{a_1, a_r, b_1, c_r, d_r, e_r\}$ and $E(Y_r)\backslash\{a_1, a_r, b_1\}$, respectively, and let $F = \{c_r, d_r, e_r\}$. Then $Y_r = P_F(G, H)$, $G \cong W_3$, and $H \cong Z_{r-1}$. Moreover, $H\backslash b_{r-1} \cong H\backslash d_r \cong Z_{r-1}\backslash b_{r-1}$, $H\backslash c_r \cong Z_{r-1}\backslash c_{r-1}$,* and *$H\backslash b_{r-1}, c_r \cong H\backslash c_r, d_r \cong Z^*_{r-2}$ if $r \geq 6$.*

Proof: The flat $F$ is modular in $G$ as it is a full line [4,(3.15)]. Thus $Y_r = P_F(G, H)$. It is easy to check that $G \cong W_3$. To show that $H \cong Z_{r-1}$ consider the matrix representation $A_r\backslash\{a_1, a_r, b_1\}$ for $H$. This matrix has identical first and last rows. Drop its last row and order its column by $[e_r\ a_2\ a_3 \ldots a_{r-1}\ d_r\ b_2\ b_3 \ldots b_{r-1}\ c_r]$. This is the matrix $B_{r-1}$ which represents $Z_{r-1}$. Thus $H \cong Z_{r-1}$ and by this ordering of columns, $H\backslash b_{r-1} \cong Z_{r-1}\backslash b_{r-1}$, $H\backslash c_r \cong Z_{r-1}\backslash c_{r-1}$, $H\backslash d_r \cong Z_{r-1}\backslash b_1$, $H\backslash b_{r-1}, c_r \cong Z_{r-1}\backslash b_{r-1}, c_{r-1}$, and $H\backslash c_r, d_r \cong Z_{r-1}\backslash b_1, c_{r-1}$. It is easy to check that $Z_{r-1}\backslash b_1 \cong Z_{r-1}\backslash b_{r-1}$ and $Z_{r-1}\backslash b_{r-1}, c_{r-1} \cong Z_{r-1}\backslash b_1, c_{r-1} \cong Z^*_{r-2}$ if $r \geq 6$. ∎

The following well-known lemma is frequently used in the proof of Theorem 1.3.

**1.6 Lemma.** [6] *Let $N$ be a simple minor of a matroid $M$. If $e$ is in a 2-element circuit (cocircuit) of $M$, then $M$ has an $N$-minor if and only if $M\backslash e$ ($M/e$) has an $N$-minor.*

For $r \geq 5$ and $g \in \{a_1, a_r, b_1\}$ suppose that $Y_r\backslash g$ or $Y_r/g$ has a $W_4$-minor. By using Lemmas 1.5 and 1.6 and observing Figure 3 we see that $Z_{r-1}$ has a $W_4$-minor. This contradicts Theorem 1.2. Thus

(1.7). $a_1, a_r$, *and* $b_1$ *are is every* $W_4$ *-minor of* $Y_r$.

## 2. The Proof

In this section the proofs of Theorems 1.3 and 1.4 are given. we begin with some preliminary lemmas.

**2.1 Lemma.** [2,(1)] *Let* $M$ *be a* 3 *-connected matroid and* $e \in E(M)$. *Then at least one of* $M/e$ *and* $M\backslash e$ *is 3-connected.*

The next lemma is well known (see, for example, [10,(2.1)]).

**2.2 Lemma.** *Let* $N$ *be a* 3 *-connected matroid with* $|E(N)| \geq 3$ *and* $M$ *be an addition to* $N$. *Then* $M$ *is* 3 *-connected if and only if* $M$ *is a non-trivial addition to* $N$.

**2.3 Lemma.** [11,(2.6,2.8)] *For* $r \geq 4$ *suppose that a non-empty set of columns* $X$ *is adjoined to either* $B_r\backslash b_r$ or $B_{r+1}\backslash b_{r+1}c_{r+1}$ *to give a representation of a simple binary matroid with no* $W_4$ *-minor. If the former holds, then* $X = \{(1,1,\ldots,1,0)^T\}$. *If the latter holds, then* $X \subseteq \{(1,1,\ldots,1,0)^T, (1,1,\ldots,1)^T\}$.

Several technical lemmas which are used in the proof of Theorem 1.3 are given next.

**2.4 Lemma.** *For some* $r \geq 4$ *suppose that column* $f = (f_1, f_2, \ldots, f_r)^T$ *is adjoined to* $B \in \{B_r, B_r\backslash b_r, B_r\backslash c_r, B_r\backslash b_r, c_r\}$ *to give a representaton of a simple binary matroid* $M$. *Suppose that* $f_i = f_j$ *or distinct* $i, j \in \{1, 2, \ldots, r\}$. *Further suppose that* $i \neq r$ *and* $j \neq r$ *if* $B$ *is* $B_r\backslash b_r$ *or* $B_r\backslash b_r, c_r$. *If* $x, y \in \{a_i, a_j, b_i, b_j\}$, *then there exists an automorphsm* $\eta$ *of* $M$ *such that* $\eta(x) = y$ *and* $\eta(f) = f$.

Proof: In $B+f$ replace row $k$ by row $k$+ row $i$+ row $j$ for each $k$ in $\{1, 2, \ldots, r\} = \{i, j\}$. A suitable reordering of the columns produces the matrix $B + f$. This induces an automorphism $\phi$ of $M$ such that $\phi(a_i) = b_j$, $\phi(a_j) = b_i$, and $\phi(f) = f$.
Interchanging rows $i$ and $j$ of $B + f$ induces and automorphism $\psi$ of $M$ such that $\psi(a_i) = a_j$, $\psi(b_i) = b_j$, and $\psi(f) = f$. The result follows from considering compositions of $\phi, \phi^{-1}, \psi$, and $\psi^{-1}$. ∎

**2.5 Lemma.** *For* $r \geq 4$ *let* $f$ *and* $g$ *be binary columns of length* $r$ *such that* $f$ *has exactly two zero-entries and* $g$ *has exactly two one-entries. Then there exists an isomorphism* $\lambda$: $D(B_r+g) \rightarrow D(B_r+f)$ *such that* $\lambda(g) = f$ *and* $\lambda(c_r) = c_r$.

Proof: By the symmetry of $B_r$ we may assume that $g = (1,1,0,\ldots,0)^T$. In $B_r + g$ replace row $k$ by row $k$+ row 1+ row $r$ for each $k$ in $\{2, 3, \ldots, r-1\}$. Column $g$ now has exactly two zero-entries. A suitable reordering of the rows and columns produces $B_r + f$. Column $g$ is transformed into $f$ by these operations, while column $c_r$ is unchanged. ∎

The next lemma will be used in the proof of Theorem 1.3 to show that $e_r$ is in every $W_4$-minor of $Y_r\backslash X$ when $d_r \in X$.

**2.6 Lemma.** *Let $M$ be a 3-connected binary matroid with $f \in E(M)$ such that $M\backslash f \cong Z_r$ for some $r \geq 4$. Then $M/f$ has no $W_4$-minor.*

Proof: By induction on $r$. If $r = 4$, then $rk\ M/f = 3$ and so $M/f$ has no $W_4$-minor. Suppose that $r > 4$ and the result holds for 3-connected binary additions to $Z_n$ when $n < r$. Assume that $M/f$ has a $W_4$-minor.

Adjoin a binary column $f = (f_1, f_2, \ldots, f_r)^T$ with at least two zero-entries and at least two one-entries to $B_r$ to obtain a representation for $M$. First suppose that $f$ has exactly two one-entries. By the symmetry of $B_r$ we may assume that $f = (1,1,0,\ldots,0)^T$. Then $\{a_1, a_2, f\}$ and $\{b_1, b_2, f\}$ are circuits of $M$ while $\{a_1, a_2, b_1, b_2\}$ is a cocircuit. It follows from Lemma 1. 6 that $M/f \cong M\backslash a_1, b_1/f$ has a $W_4$-minor. Since $\{a_2, b_2\}$ is a cocircuit of $M\backslash a_1, b_1/f$, the matroid $M\backslash a_1, b_1/a_2, f$ has a $W_4$-minor. The dependence of $\{a_1, f\}$ in $M/a_2$ implies that $M/a_2\backslash f$ has a $W_4$-minor. This contradicts Theorem 1.2 as $M\backslash f \cong Z_r$.

Next suppose that $f$ has exactly two zero-entries. Adjoin a binary column $g$ with exactly two one-entries to $B_r$. By Lemma 2.5 there exists an isomorphism $\lambda: D(B_r + g) \to M$ such that $\lambda(g) = f$. By the previous paragraph $D(B_r + g)/g$ has no $W_4$-minor. Thus $M/f$ has no $W_4$-minor; a contradiction. It follows that $f$ has at least three zero-entries and at least three one-entries. By the symmetry of $B_r$ we may assume that $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$.

Let $X, Y \subseteq E(M)$ be such that $M\backslash X/Y \cong W_4$ and $f \in Y$. Suppose that $M/c_r$ has a $W_4$-minor. The matroid $Z_r/c_r$ is graphic and forms a cycle on $r$ edges [11,(2.4)]. This matroid has corank one and $M\backslash f = Z_r$. Thus $M/c_r$, has both corank at most two and a $W_4$-minor; a contradiction. Hence $c_r \notin Y$.

Suppose that $a_i \in Y$ for some $i$ in $\{1, 2, \ldots, r\}$. The dependence of $\{b_i, c_r\}$ in $M/a_i$ implies that this set meets $X \cup Y$. Suppose that $g \in X \cap \{b_i, c_r\}$. Then $M/a_i\backslash f, g \cong Z_{r-1}$ and the column corresponding to $f$ in $(B_r + f)/a_i\backslash g$ has at least two zero-entries and at least two one-entries. Thus $M/a_i\backslash g$ is a non-trivial addition to $Z_{r-1}$. The induction hypothesis implies that $(M/a_i\backslash g)/f$ has no $W_4$-minor. However, $\{a_i, b_i, c_r\}$ is a circuit of $M$ and $M/a_i, f$ has a $W_4$-minor. Thus $M/a_i, f\backslash g$ has a $W_4$-minor; a contradiction. Hence $X \cap \{b_i, c_r\} = \phi$. It follows that $b_i$ is in $Y$ as $c_r$ is not. The element $c_r$ is a loop of $M/a_i, b_i$ and this matroid has a $W_4$-minor. Thus $M/c_r$ has a $W_4$-minor: a contradiction. It follows that $\{a_1, a_2, \ldots, a_r\} \cap Y = \phi$.

Now suppose that $b_j \in Y$ for some $j$ in $\{1, 2, \ldots, r\}$. Choose $i$ in $\{1, 2, \ldots, r\}$ so that the entries in rows $i$ and $j$ of $f$ agree. Then there exists an automorphism $\eta$ of $M$ such that $\eta(b_j) = a_i$ and $\eta(f) = f$ by Lemma 2.4. The $W_4$-minor in $M/b_j, f$ implies the existence of a $W_4$-minor in $M/a_i, f$. This contradicts the conclusion of the previous paragraph. Thus $\{b_1, b_2, \ldots, b_r\} \cap Y = \phi$ and $Y = \{f\}$.

Let $A = \{a_1, a_2, b_1, b_2\}$, $B = \{a_1, a_3, b_1, b_3\}$, $C = \{a_2, a_3, b_2, b_3\}$, $D = \{a_4, a_5, b_4, b_5\}$, $E = \{a_4, a_6, b_4, b_6\}$, and $F = \{a_5, a_6, b_5, b_6\}$. Then $A$, $B$, $C$, $D$, $E$, and $F$ are cocircuits of $M$. The dual of $M\backslash X/f \cong W_4$ must be simple.

Thus $X$ meets each of these cocircuits in at most one element. Each element of $A \cup B \cup C$ is in two of the sets $A, B$, and $C$. Thus $|(A \cup B \cup C) \cap X| \leq 1$. It follows that $|(A \cup B \cup C)\backslash X| \geq 5$, and likewise $|(D \cup E \cup F)\backslash X| \geq 5$. Thus $M\backslash X/f$ has at least ten elements: a contradiction. ∎

**2.7 Corollary.** *Let $M$ be a 3-connected binary matroid with $f \in E(M)$ such that $M\backslash f \cong Z_r, Z_r^*, Z_r\backslash b_r$, or $Z_r\backslash c_r$ for some $r \geq 5$. Then $M/f$ has no $W_4$-minor.*

Proof: If $M\backslash f \cong Z_r$, then the result follows from Lemma 2.6. Suppose that $M\backslash f \cong Z_r^*, Z_r\backslash b_r$, or $Z_r\backslash c_r$. Consider a representation for $M$ obtained by adjoining column $f$ to $B_{r+1}\backslash b_{r+1}, c_{r+1}, B_r\backslash b_r$ or $B_r\backslash c_r$. Then $M$ is a restriction of $D(B_{r+1}+f)$ or $D(B_r+f)$. By Lemma 2.6, $D(B_j+f)/f$ has no $W_4$-minor for $j \in \{r, r+1\}$. Thus $M/f$ has no $W_4$-minor. ∎

**2.8 Lemma.** *For some $r \geq 5$ let $M$ be a binary matroid reprsented by adjoining a binary column $f$ with at least two zero-entries and at least two one-entries to $B \in \{B_r, B_r\backslash c_r\}$. Then each element of $E(M)\backslash\{f\}$ avoids some $W_4$-minor of $M$.*

Proof: Choose $i$ in $\{1,2,\ldots,r\}$ so that the column corresponding to $f$ in $(B+f)/a_i$ has at least two zero-entries and at leat two one-entries. By Lemma 2.3, $M/a_i\backslash b_j$ has a $W_4$-minor for all $j$ in $\{1,2,\ldots,r\}$.

Let $k \in \{1,2,\ldots,r\}$. By Lemma 2.4 there exists an automorphism of $M$ mapping $b_k$ to $a_k$. By the previous paragraph $M\backslash b_k$ has a $W_4$-minor. Thus $M\backslash a_k$ also has a $W_4$-minor. If $B = B_r$, then $M\backslash c_r$ has a $W_4$-minor by Lemma 2.3. ∎

**2.9 Lemma.** *For some $r \geq 5$ let $M$ be a binary matroid represented by adjoining a binary column $f = (f_1, f_2, \ldots, f_r)^T$ with at least two zero-entries and at least two one-entries to $B \in \{B_r\backslash b_r, B_r\backslash b_r, c_r\}$. If there is an element of $E(M)\backslash\{f\}$ which is in every $W_4$-minor of $M$, then there exists $i \in \{1,2,\ldots,r-1\}$ such that either $f_i$ and $f_r$ are the only zero-entries of $f$, or $f_i$ and $f_r$ are the only one-entries of $f$.*

Proof: Suppose that if $f$ has exactly two zero-entries or exactly two one-entries, then neither of these two entries is $f_r$. Then the column corresponding to $f$ in the matrix $(B+f)/a_r$ has at least two zero-entries and at least two one-entries. For all $k$ in $\{1,2,\ldots,r-1\}$, $M/a_r\backslash b_k$ has a $W_4$-minor by Lemma 2.3. We may apply Lemma 2.4 as in the proof of Lemma 2.8 to obtain that $M\backslash x$ has a $W_4$-minor for all $x$ in $\{a_1, a_2, \ldots, a_{r-1}\}$. If $B = B_r\backslash b_r$, then $M\backslash c_r$, has a $W_4$-minor by Lemma 2.3. Thus each element of $E(M)\backslash\{f\}$ avoids some $W_4$-minor of $M$. ∎

**2.10 Lemma.** *For some $r \geq 5$ and $i \in \{1,2,\ldots,r-1\}$ suppose that the binary column $f = (f_1, f_2, \ldots, f_r)^T$ has either its only zero-entries being $f_i$ and $f_r$ or its only one-entries being $f_i$ and $f_r$. If $M$ is the dependence matroid of $(B_r\backslash b_r, c_r) + f$, then the only elements of $M$ which are in every $W_4$-minor* *are* $a_i, b_i, a_r$, *and* $f$. *Moreover,* $(B_r\backslash b_r, c_r) + f$ *is equivalent to* $A_r\backslash c_r, d_r$, *and* $(B_r\backslash b_r) + f$ *is equivalent to* $A_r\backslash d_r$. *Thus* $A_r\backslash d_r$ *and* $A_r\backslash e_r$ *are equivalent.*

Proof: Assume that $f_i$ and $f_r$ are the only one-entries of $f$. Let $k \in \{1, 2, \ldots, r\} \backslash \{i, r\}$. It follows from Lemma 2.3 that $M/a_k$ has a $W_4$-minor. By Lemma 2.4 there is an automorphism of $M$ mapping $a_k$ to $b_k$. Thus $M/b_k$ also has a $W_4$-minor. Hence each element of $E(M)\backslash\{a_i, b_i, a_r, f\}$ avoids some $W_4$-minor. By interchanging rows 1 and $i$, columns $a_1$ and $a_i$, and columns $b_1$ and $b_i$ of $(B_r\backslash b_r, c_r) + f$ we obtain $A_r\backslash c_r, d_r$. It follow from (1.7) and these operations that $a_i, b_i$, and $a_r$ are in every $W_4$-minor of $M$. The element $f$ is in every $W_4$-minor of $M$ by Corollary 2.7. From applying the same operations to $(B_r\backslash b_r) + f$ we obtain that this matrix is equivalent to $A_r\backslash d_r$.

Assume that $f_i$ and $f_r$ are the only zero-entries of $f$. Choose $k \in \{1, 2, \ldots, r\} \backslash \{i, r\}$ and replace row $\ell$ of $(B_r\backslash b_r, c_r) + f$ by row $i$+ row $k$+ row $\ell$ for each $\ell$ in $\{1, 2, \ldots, r\}\backslash\{i, k\}$. Next interchange rows $i$ and $k$ of the resulting matrix. Column $f$ now has a one in rows $i$ and $r$, and a zero in all other rows. The columns of $B_r\backslash b_r, c_r$ have been permuted with columns $a_i$ and $b_i$ interchanged and column $a_r$ unchanged. It now follows from the previous paragraph that $(B_r\backslash b_r, c_r) + f$ is equivalent to $A_r\backslash c_r, d_r$ and that only $a_i, b_i, a_r$, and $f$ are in every $W_4$-minor of $M$. The above operations can also be used to show that $(B_r\backslash b_r) + f$ is equivalent to $A_r\backslash d_r$. Moreover, as $A_r\backslash e_r = (B_r\backslash b_r) + d_r$, the matrix $A_r\backslash e_r$ is equivalent to $A_r\backslash d_r$. ∎

**2.11 Lemma.** *For some* $r \geq 5$ *let* $B$ *be a binary matrix obtained by adoining a non-empty set of binary columns* $X$ *to* $B_r\backslash b_r, c_r$ *so that* $M = D(B)$ *is simple and has more than one element in every* $W_4$*-minor. Then, for some* $i$ *in* $\{1, 2, \ldots, r-1\}$, $X$ *is obtained by taking a non-empty subset of* $\{d_r, e_r\}$ *and interchanging the first-entry and the* $i$*th-entry of every column in the subset, and then possibly adding* $c_r$ *to this subset. Thus* $B$ *is equivalent to* $A_r, A_r\backslash c_r, A_r\backslash d_r$ *or* $A_r\backslash c_r, d_r$.

Proof: Assuming $|X| = 1$. Let $f \in X$. Then column $f$ has at least two zero-entries and at least two one-entries since $M$ has a $W_4$-minor. It follows from Lemma 2.9 that $f$ is obtained from either column $d_r$ or $e_r$ by permuting entries 1 and $i$ for some $i < r$. Moreover, by Lemma 2.10, $B$ is equivalent to $A_r\backslash c_r, d_r$.

Assume $|X| \geq 2$. Suppose $b_r \in X$. Then, by Lemma 2.8, $M$ has at most one element which is in every $W_4$-minor; a contradiction. Thus each column of $X\backslash\{c_r\}$ has at least two zero-entries and at least two one-entries.

Assume that $X\backslash\{c_r\}$ contains two distinct columns $f = (f_1, f_2, \ldots, f_r)^T$ and $g = \{g_1, g_2, \ldots, g_r\}^T$. The dependence matroids of $(B_r\backslash b_r, c_r)+f$ and $(B_r\backslash b_r, c_r)+g$ have more than one element in every $W_4$-minor. By Lemma 2.9 there exist $i, j \in \{1, 2, \ldots, r-1\}$ such that $f_i = f_r$ and all other entries of $f$ differ from $f_i$, and $g_j = g_r$ and all other entries of $g$ differ from $g_j$. Suppose $i \neq j$. By Lemma 2.10 only $a_i, b_i, a_r$, and $f$ are in every $W_4$-minor of $D(B_r\backslash b_r, c_r + f)$, and only $a_j, b_j, a_r$, and $g$ are in every $W_4$-minor of $D(B_r\backslash b_r, c_r + g)$. Thus only

$a_r$ could be in every $W_4$-minor of $M$; a contradiction. It follow that $i = j$. Since $M$ is simple $X\backslash\{c_r\} = \{f, g\}$. The set $\{f, g\}$ is obtained from $\{d_r, e_r\}$ by interchanging the first entry and the $i$th entry of each of $d_r$ and $e_r$. Thus if $c_r \notin X$, then $|X| = 2$ and $B$ is equivalent to $A_r\backslash c_r$.

Assume $c_r \in X$. Suppose $|X\backslash\{c_r\}| = 1$. Let $f \in X\backslash\{c_r\}$. Then $B = B_r\backslash b_r, c_r + X = B_r\backslash b_r + f$. Lemma 2.9 implies that $X\backslash\{c_r\}$ is obtained from $d_r$ or $e_r$ by permuting the first and $i$th entry for some $i < r$. Lemma 2.10 implies that $B$ is equivalent to $A_r\backslash d_r$. Suppose $|X\backslash\{c_r\}| \geq 2$. The previous paragraph implies that $X\backslash\{c_r\}$ is obtained from $\{d_r, e_r\}$ by permuting the first and $i$th entry. Hence $B$ is equivalent to $A_r$.

**The proof of Theorem 1.3**

Suppose that $r \geq 7$. Let $X \subseteq \{b_{r-1}, c_r, d_r\}$. We first show that $Y_r\backslash X$ is 3-connected. We have previously noted that $Y_r\backslash b_{r-1}, c_r, d_r \cong (Y_{r-1}\backslash d_{r-1})$ and thus it suffices to show that $Y_{r-1}\backslash d_{r-1}$ is 3-connected. This follows as $Z^*_{r-2}$ is 3-connected [11] and $Y_{r-1}\backslash c_{r-1}, d_{r-1}, e_{r-1} \cong Z^*_{r-2}$.

The elements $a_1, a_r$, and $b_1$ are in every $W_4$-minor of $Y_r\backslash X$ by (1.7). Note that $A_r\backslash b_{r-1}, c_r, d_r/a_{r-1} = B_{r-1}\backslash b_{r-1}, c_{r-1} + e_{r-1}$. The elements $a_1, a_r, b_1$, and $e_r$ of $Y_r\backslash b_{r-1}, c_r, d_r$ respectively, correspond to the columns $a_1, a_{r-1}, b_1$, and $e_{r-1}$ in $B_{r-1}\backslash b_{r-1}, c_{r-1} + e_{r-1}$. Lemma 2.10 implies that only $a_1, a_{r-1}, b_1$, and $e_{r-1}$ can be in every $W_4$-minor of $D(B_{r-1}\backslash b_{r-1}, c_{r-1} + e_{r-1})$. Thus only $a_1, a_r, b_1$, and $e_r$ can be in every $W_4$-minor of $Y_r\backslash X$. Suppose $d_r \in X$. Then $e_r$ is in every $W_4$-minor of $Y_r\backslash X$ by Corollary 2.7. Suppose $d_r \notin X$. Then $Y_r\backslash e_r$ has a $W_4$-minor by Lemma 2.3. Thus the only elements of $Y_r\backslash X$ which are in every $W_4$-minor are $a_1, a_r, b_1$, and $e_r$ when $d_r \in X$, and $a_1, a_r$, and $b_1$ when $d_r \notin X$.

Assume that $M$ has at least two elements which are in every $W_4$-minor and neither $M$ nor $M^*$ is isomorphic to $Y_r\backslash X$ for $r \geq 7$. Let $e$ be in every $W_4$-minor of $M$. By Lemma 2.1 either $M/e$ or $M\backslash e$ is 3-connected. These two matroids have no $W_4$-minor and either rank or corank at least six.

Let $N$ be a largest 3-connected minor of $M$ which has no $W_4$-minor. Then, by Theorem 1.2, $N$ is isomorphic to $Z_{s-1}, Z^*_{s-1}, Z_s\backslash b_s$, or $Z_s\backslash c_s$ for some $s \geq 6$. By Theorem 1.1 there exists a chain of 3-connected matroids $M_0, M_1, M_2, \ldots, M_n$ from $N$ to $M$ with $N \cong M_0$ and $M_n = M$. The matroid $M_1$ must have a $W_4$-minor by the choice of $N$. By duality we may assume that $M_1$ is an addition to $M_0$. If $M_0 \cong Z_{s-1}$ or $Z_s\backslash c_s$, then, by Lemma 2.8, $M_1$ has at most one element in every $W_4$-minor; a contradiction. Moreover, as $Z_s\backslash c_s$, is self-dual, a largest 3-connected minor of $M$ which has no $W_4$-minor is not isomorphic to $Z_s\backslash c_s$. Thus $M_0 \cong Z^*_{s-1}$ or $Z_s\backslash b_s$. Consider a representation for $M$ obtained by adjoining a column to $B_s\backslash b_s, c_s$ or $B_s\backslash b_s$. Lemmas 2.9 and 2.10 imply that if the former holds, then $M_1 \cong Y_s\backslash c_s, d_s$, while if the latter holds, then $M_1 \cong Y_s\backslash d_s$. In either case $M_1 \neq M$ as either $s = 6$ and $rk\ M_1 < rk\ M$, or $s \geq 7$ and $M \ncong Y_s\backslash X$.

Assume that $M_0 \cong Z^*_{s-1}$ and $M_1 \cong Y_s\backslash c_s, d_s$. Then $M_1$ is self-dual and we may assume that $M_2$ is an addition to $M_1$. Both $Y_s$ and $Y_s\backslash d_s$ have a $Z_s\backslash b_s$ minor. The matroid $Z_s\backslash b_s$ is larger than $N$. Thus $M_i$ is isomorphic to neither $Y_s$ nor $Y_s\backslash d_s$. for $i \geq 2$. It follows from Lemma 2.11 that $M_2 \cong Y_s\backslash c_s$ and $M_3$ is an expansion of $M_2$.

Suppose $M_3/g = Y_s\backslash c_s$. The only elements of $M$ which may be in every $W_4$-minor are $a_1$, $a_s$ and $b_1$. When determining possible triangles and triads containing these elements in this proof it is often convenient to consult Figure 3. The only possible triangles of $M_3$ containing $a_1$ or $b_1$ are $\{a_1, a_s, e_s\}$ and $\{a_s, b_1, d_s\}$. The set $\{a_1, a_s, b_1\}$ is a triad of $M_3$ and any other triad of $M_3$ containing $a_1$ or $b_1$ would also contain $g$.

Suppose that $a_1$ is in every $W_4$-minor of $M_3$. The matroid $M_3/a_1$ has at least $|E(M_1)|$ elements. It cannot be 3-connected as otherwise the choice of $N$ would force it to have a $W_4$-minor. It follows from Lemma 2.1 that $M_3\backslash a_1$ is 3-connected. The choice of $N$ forces $M_3\backslash a_1$ to have at most $|E(M_0)|$ element. Thus $M_3\backslash a_1 \cong M_3\backslash a_1/b_1, g$. However $\{a_s, d_s\}$ is dependent in the latter matroid; a contradiction. Thus $a_1$ avoids some $W_4$-minor of $M_3$.

Suppose that $b_1$ is in every $W_4$-minor of $M_3$. The matroid $M_3/b_1$ has at least $|E(M_1)|$ elements and hence is not 3-connected. Thus $M_3\backslash b_1$ is 3-connected and has at most $|E(M_0)|$ elements. Hence $M_3\backslash b_1 \cong M_3\backslash b_1/a_1, g$. The set $\{a_s, e_s\}$ is dependent in the latter matroid; a contradiction. Thus $b_1$ avoids some $W_4$-minor of $M_3$. Hence only $a_s$ could be in every $W_4$-minor of $M_3$; a contradiction. Thus $M_0 \cong Z_s\backslash b_s$ and $M_1 \cong Y_s\backslash d_s$.

Assume that $M_2$ is an addition to $M_1$. Then, by Lemma 2.11, $M_2 \cong Y_s$ and $M_3$ is an expanion of $M_2$. Suppose $M_3/g = Y_s$. Consider a representation for $M_3^*$ obtained by adjoining a column $g = (g_1, g_2, \ldots, g_{s+2})^T$ to the matrix $A_s^*$. The only elements of $M_3$ which may be in every $W_4$-minor are $a_1$, $a_s$, and $b_1$. The set $\{a_1, a_s, b_1\}$ is a triad of $M_3$ and any other triad of $M_3$ containing $a_1$ or $a_s$ also contains $g$. The only possible triangles of $M_3$ containing $a_1$ or $a_s$ are $\{a_1, b_1, c_s\}$, $\{a_1, a_s, e_s\}$ and $\{a_s, b_1, d_s\}$. We next show that if $a_1$ or $a_s$ is in every $W_4$-minor of $M_3$, then $M_3 \cong Y_{s+1}\backslash b_s$. It will then follow that $M_3 \cong Y_{s+1}\backslash b_s$ as at least one of these elements is in every $W_4$-minor of $M_3$.

Assume that $a_1$ is in every $W_4$-minor of $M_3$. If $M_3\backslash a_1$ is 3-connected, then $M_3\backslash a_1 \cong M_3\backslash a_1/b_1, g$. However $\{a_s, d_s\}$ is dependent in the latter matroid; a contradiction. Thus $M_3/a_1$ is 3-connected with at most $|E(M_0)|$ elements. Hence $M_3/a_1 \cong M_3/a_1\backslash b_1, e_s$ and $\{a_1, b_1, c_s\}$ and $\{a_1, a_s, e_s\}$ are triangles of $M_3$.

Consider the matrix $(A_s^* + g)\backslash a_1/b_1, e_s$. Its dependence matroid has no $W_4$-minor. Lemma 2.3 implies that $(g_2, g_3, \ldots, g_{s+1})^T$ is $(1, 1, \ldots, 1, 0, 1)^T$ or $(1, 1, \ldots, 1)^T$. Suppose the former holds. Then $M_3^*\backslash a_1/b_1, e_s \cong Z_s\backslash c_s$. However, $M$ has no such minor. Thus the latter holds. Note that $g_1 = g_s = 1$ and

$g_{s+2} = 0$ as $\{a_1, b_1, c_s\}$ and $\{a_1, a_s, e_s\}$ are tringles of $M_3$. After interchanging rows $s$ and $s+1$ in $(A_s^* + g)^*$ we obtain the matrix $A_{s+1}\backslash b_s$. Thus $M_3 \cong Y_{s+1}\backslash b_s$.

Assume that $a_s$ is in every $W_4$-minor of $M_3$. If $M_3\tilde{\backslash} a_s$, is 3-connected, then $M_3\tilde{\backslash} a_s \cong M_3\backslash a_s/b_1, g$. The set $\{a_1, c_s\}$ is dependent in the latter matroid; a contradiction. Thus $M_3\tilde{/}a_s$ is 3-connected. It follows that $M_3\tilde{/}a_s \cong M_3/a_s\backslash d_s, e_s$ and $\{a_1, a_s, e_s\}$ and $\{a_s, b_1, d_s\}$ are triangles of $M_3$.

Consider the matrix $(A_s^* + g)\backslash a_s/d_s, e_s$. Its dependence matroid has no $W_4$-minor. Lemma 2.3 implies that $(g_1, g_2, \ldots, g_s)^T$ is $(1, 1, \ldots, 1, 0)^T$ or $(1, 1, \ldots, 1)^T$. If the former case occurs, then $M_3^*\backslash a_s/d_s, e_s \cong Z_s\backslash c_s$; a contradiction. Thus the latter holds. Note that $g_{s+2} = 0$ and $g_{s+1} = g_1 = 1$ as $\{a_1, a_s, e_s\}$ and $\{a_s, b_1, d_s\}$ are triangles of $M_3$. After interchanging rows $s$ and $s+1$ of $(A_s^* + g)^*$ we obtain the matrix $A_{s+1}\backslash b_s$. Thus $M_3 \cong Y_{s+1}\backslash b_s$.

We next show that $M_4$ can be neither an addition to nor an expansion of $Y_{s+1}\backslash b_s$. It will then follow from this and duality that $M_3 \not\cong Y_{s+1}\backslash b_s$ and $M_3^* \not\cong Y_{s+1}\backslash b_s$.

Assume that $M_4$ is an expansion of $M_3$. Suppose $M_4/h = Y_{s+1}\backslash b_s$. Then the only elements of $M_4$ which may be in every $W_4$-minor are $a_1, a_{s+1}$, and $b_1$. The only possible triangles of $M_4$ containing $a_1$ or $a_s$ are $\{a_1, b_1, c_{s+1}\}$, $\{a_{s+1}, b_1, d_{s+1}\}$, and $\{a_1, a_{s+1}, e_{s+1}\}$. The set $\{a_1, a_{s+1}, b_1\}$ is a triad of $M_4$ and any other triad of $M_4$ containing $a_1$ or $a_{s+1}$ also contains $h$. Both $a_1$ and $a_{s+1}$ are in at most two triangles and at most two triads of $M_4$. Thus each of $M_4\tilde{/}a_1$, $M_4\tilde{\backslash} a_1$, $M_4\tilde{/}a_s$, and $M_4\tilde{\backslash} a_s$, have at least $|E(M_1)|$ elements. It follows from Lemma 2.1 and the choice of $N$ that both $a_1$ and $a_{s+1}$ avoid some $W_4$-minor of $M_4$; a contradiction. Thus $M_4$ is isomorphic to an addition to $Y_{s+1}\backslash b_s$.

Consider a representation for $M_4$ obtained by adoining a column $h = (h_1, h_2, \ldots, h_{s+1})^T$ to $A_{s+1}\backslash b_s$. The only elements of $M_4$ which may be in every $W_4$-minor are $a_1, a_{s+1}$, and $b_1$. The sets $\{a_1, b_1, c_{s+1}\}$, $\{a_{s+1}, b_1, d_{s+1}\}$, and $\{a_1, a_{s+1}, e_{s+1}\}$ are triangles of $M_4$ and any other triangle of $M_4$ containing $a_1$ or $a_{s+1}$ also contains $h$. The only possible triad of $M_4$ containing $a_1$ or $a_{s+1}$ is $\{a_1, a_{s+1}, b_1\}$.

The matrix $[(A_{s+1}\backslash b_s) + h]/a_s\backslash h$, which represents $M_4/a_s\backslash h$, is equal to $A_s$. Thus $M_4/a_s$ is an addition to $Y_s$. It cannot be 3-connected by Lemma 2.11. Thus, by Lemma 2.2, there is an element $x$ for which $\{h, x\}$ is dependent in $M_4/a_s$. Thus $\{a_s, h, x\}$ is a triangle of $M_4$.

Assume that $a_1$ is in every $W_4$-minor of $M_4$. Then $M_4\tilde{\backslash} a_1$ has at least $|E(M_2)|$ elements and is not 3-connected. Thus $M_4\tilde{/}a_1$ is 3-connected with at most $|E(M_0)|$ elements. It follows that $M_4\tilde{/}a_1 \cong M_4/a_1\backslash c_{s+1}, e_{s+1}, h$ and for some element $y$ the set $\{a_1, h, y\}$ is a triangle of $M_4$. From considering the element of $M_4$ as columns in the matrix $A_{s+1}\backslash b_s + h$ we obtain the equation $x + y = a_1 + a_s = (1, 0, \ldots, 0, 1, 0)^T$. It is easy to check that the only simple solution for $x$ and $y$ is $x = a_1$ and $y = a_s$. Thus $h = (1, 0, \ldots, 0, 1, 0)^T$. From considering the matrix $[(A_{s+1}\backslash b_s) + h]/a_{s+1}\backslash b_1$ we obtain that $M_4/a_{s+1}\backslash b_1$ has a $W_4$-minor by Lemma 2.3. Hence only $a_1$ can be in every $W_4$-minor of $M_4$; a contradiction.

Thus $a_1$ avoids some $W_4$-minor of $M_4$.

Assume that $a_{s+1}$ is in every $W_4$-minor of $M_4$. Then $M_4\tilde{\backslash}a_{s+1}$ has at least $|E(M_2)|$ elements and is not 3-connected. Thus $M_4\tilde{/}a_{s+1}$ is 3-connected with at most $|E(M_0)|$ elements. It follows that $M_4\tilde{/}a_{s+1} \cong M_4/a_{s+1}\backslash d_{s+1}, e_{s+1}, h$ and for some element $y$ the set $\{a_{s+1}, h, y\}$ is a triangle of $M_4$. From considering the elements of $M_4$ as columns in the matrix $A_{s+1}\backslash b_s + h$ we obtain the equation $x + y = a_s + a_{s+1} = (0,\ldots,0,1,1)^T$. It is easy to check that the only simple solution in $x$ and $y$ is $x = a_{s+1}$ and $y = a_s$. Thus $h = (0,\ldots,0,1,1)$. From considering the matrix $(A_{s+1}\backslash b_s + h)/a_1\backslash b_1$ we obtain that $M_4/a_1\backslash b_1$ has a $W_4$-minor by Lemma 2.3. Hence only $a_{s+1}$ is in every $W_4$-minor of $M_4$; a contradiction. Thus $a_{s+1}$ avoids some $W_4$-minor of $M_4$. It follows that only $b_1$ may be in every $W_4$-minor of $M_4$; a contradiction. Thus

(2.12). *$M_3 \ncong Y_{s+1}\backslash b_s$, and by duality $M_3^* \ncong Y_{s+1}\backslash b_s$.*

It follows that $M_2$ is an expansion of $M_1$ where $M_1 \cong Y_s\backslash d_s$.

Recall that $(Y_s\backslash d_s)^* \cong Y_{s+1}\backslash b_s, c_{s+1}, d_{s+1}$. Consider a representation for $M_2^*$ obtained by adjoining a column $g = (g_1, g_2, \ldots, g_{s+1})^T$ to the matrix $A_{s+1}\backslash b_s, c_s, d_{s+1}$. We shall first show that $g \in \{c_{s+1}, d_{s+1}\}$. It will then follow that $M_2^* \cong Y_{s+1}\backslash b_s, c_{s+1}$ or $Y_{s+1}\backslash b_s, d_{s+1}$. Moreover, if $M_3^*$ is an addition to $M_2^*$, then $M_3^* \cong Y_{s+1}\backslash b_s$.

The only elements of $M_2^*$ which may be in every $W_4$-minor are $a_1, a_{s+1}, b_1$, and $e_{s+1}$. By considering the matrix $(A_{s+1}\backslash b_s, c_{s+1}, d_{s+1} + g)/a_2\backslash g$ we see that $M_2^*/a_s\backslash g \cong Y_s\backslash c_s, d_s$. Thus $M_2^*/a_s$ is an additon to $Y_s\backslash c_s, d_s$.

Suppose that $M_2^*/a_s$ is not 3-connected. From applying Lemma 2.2 as before we obtain that $\{a_s, g, x\}$ is a triangle of $M_2^*$ for some element $x$. By the symmetry of $A_{s+1}\backslash b_s, c_{s+1}, d_{s+1}$ induced by interchanging any two of rows 2 through $s-1$ we may assume that $x \in \{a_1, a_2, a_{s+1}, b_1, b_2, e_{s+1}\}$. Thus $g \in \{(1,0,\ldots,0,1,0)^T, (0,1,0,\ldots,0,1,0)^T, (0,\ldots,0,1,1)^T, (0,1,\ldots,1,0,1)^T, (1,0,1,\ldots,1,0,1)^T, (1,0,\ldots,0,1,1)^T\}$. If $g \neq (0,\ldots,0,1,1)^T$, then $M_2^*/a_{s+1}\backslash e_{s+1} \cong M_2^*/a_{s+1}\backslash a_1$ has a $W_4$-minor by Lemma 2.3. Thus only $b_1$ can be in every $W_4$-minor of $M_2^*$; a contradiction. Hence $g = (0,\ldots,0,1,1)^T$. It follows that $\{a_1, a_{s+1}, e_{s+1}\}$ and $\{a_s, a_{s+1}, g\}$ are triangles of $M_2^*$ and $\{a_s, a_{s+1}, e_{s+1}\}$ is a triad. Thus $M_2^*\tilde{/}a_1$ is not 3-connected. Hence $M_2^*\tilde{\backslash}a_1 = M_2^*\backslash a_1$ is 3-connected with $|E(M_1)|$ elements. It follows that $M_2^*\backslash a_1$ has a $W_4$-minor. The element $b_1$ is in no triangle or triad of $M_2^*$. Thus, by Lemma 2.1, $b_1$ avoids some $W_4$-minor of $M_2^*$. By Lemma 2.3, $M_2^*/a_s\backslash a_{s+1}$ has a $W_4$-minor. Hence only $e_{s+1}$ could be in every $W_4$-minor of $M_2^*$;a contradiction. Thus $M_2^*/a_s$ is 3-connected.

The matrix $(A_{s+1}\backslash b_s, c_{s+1}, d_{s+1} + g)/a_s$ can be obtained by adding the columns $(1,0,\ldots,0,1)^T$ and $(g_1,\ldots,g_{s-1},g_{s+1})^T$ to $B_s\backslash b_s, c_s$. It follows from Lemma 2.11 that $(g_1,\ldots,g_{s-1},g_{s+1})^T$ is $(1,1,\ldots,1)^T$ or $(0,1,\ldots,1,0)^T$. Suppose the former holds. If $g_s = 0$, then $M_2^*/a_{s+1}\backslash e_{s+1} \cong Z_s\backslash c_s$; a contradiction. Thus $g_s = 1$ and $g = c_{s+1}$. Suppose the latter holds. If $g_s = 0$, then

$M_2^*/a_{s+1}\backslash a_1 \cong M_2^*/a_{s+1}\backslash e_{s+1}$ has a $W_4$-minor by Lemma 2.3. Hence only $b_1$ could be in every $W_4$-minor of $M_2^*$ a contradiction. Thus $g_s = 1$ and $g = d_{s+1}$. If follows that $M_2^* \cong Y_{s+1}\backslash b_s, c_{s+1}$ or $Y_{s+1}\backslash b_s, d_{s+1}$.

Suppose $M_3^*$ is an addition to $M_2^*$. Then $M_3^* \cong Y_{s+1}\backslash b_s$. This contradicts (2.12). Thus is an expansion of $M_2^*$. Suppose $M_2^* \cong Y_{s+1}\backslash b_s, d_{s+1}$. This matroid is self-dual. Hence $M_3$ is isomorphic to an addition to $Y_{s+1}\backslash b_s, d_{s+1}$. By the previous arguments $M_3 \cong Y_{s+1}\backslash b_s$. This contradicts (2.12). Thus $M_3^*$ is an expansion of $Y_{s+1}\backslash b_s, c_{s+1}$.

Assume that $M_3^*/h = Y_{s+1}\backslash b_s, c_{s+1}$. The only elements of $M_3^*$ which can be in every $W_4$-minor are $a_1, a_{s+1}$ and $b_1$. The only possible triangles containing $a_1$ or $b_1$ are $\{a_1, a_{s+1}, e_{s+1}\}$ and $\{a_{s+1}, b_1, d_{s+1}\}$. The only possible triads containing $a_1$ or $b_1$ are $\{a_1, a_{s+1}, b_1\}$ and possibly some containing $h$. Suppose $a_1$ is in every $W_4$-minor of $M_3^*$. Then $\tilde{M_3^*}/a_1$ has at least $|E(M_1)|$ elements and is not 3-connected. Thus $\tilde{M_3^*}\backslash a_1$ is 3-connected with $|E(M_0)|$ elements. Hence $M_3^*\backslash a_1 \cong M_3^*\backslash a_1/b_1, h$. However $\{a_{s+1}, d_{s+1}\}$ is dependent in $M_3^*\backslash a_1/b_1, h$. Thus $a_1$, and by a similar argument $b_1$, avoids some $W_4$-minor of $M_3^*$; a contradiction. ∎

**The proof of Theorem 1.4.**

It is easy to check that each of $W_4$, $M(K_5 - e)$, and $M^*(K_5 - e)$ have an element which is in every $W_4$-minor. The last two matroids have three such elements. Suppose $M$ has an element which is in every $W_4$-minor and $M \not\cong W_4$, $M(K_5 - e)$, or $M^*(K_5 - e)$. Each element of $W_5$, $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, and $M^*(K_5)$ avoids some $W_4$-minor. Thus $M$ has no minor isomorphic to one of these matroids. It follows from Tutte's excluded minor characterizations of the regular and graphic matroids [17, sections 10.4 and 10.5] that $M$ is graphic. By Oxley's characterization of the regular matroids with no $W_5$-minor [12,Table 1] and Theorem 1.1, $M$ has a minor isomorphic to the cycle matroid of the graph $J$ given in Figure 4.

J

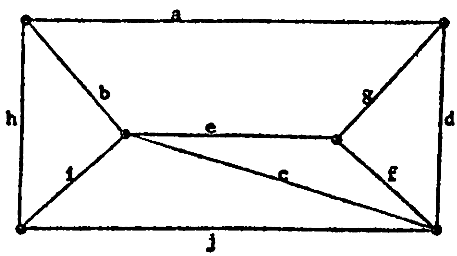


**Figure 4**

The following isomorphisms show that each element of $M(J)$ avoids some $W_4$-minor. $J/a\backslash b \cong J/a\backslash c \cong J/h\backslash b \cong J/g\backslash d \cong J/e\backslash f \cong J/j\backslash i \cong W_4$. Thus $M$ has no element which is in every $W_4$-minor; a contradiction. ∎

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