The spectrum for 2-perfect 8-cycle systems

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ABSTRACT: A decomposition of K_v into 2-perfect 8-cycles is shown to exist if and only if $v \equiv 1$ (modulo 16).

1 Introduction

There has been a lot of work done recently on *cycle systems*; see [8] for an excellent survey. An *m-cycle system of order v* is an ordered pair (V, C) where V is the vertex set of K_v and C is a set of edge-disjoint m-cycles which partition the edge set of K_v.

Additional structure may be required of the decomposition of K_v into m-cycles. One such property is that of being *i-perfect*. Suppose we have an m-cycle system of K_v so that when, for each cycle, we take the graph formed by joining all vertices in the cycle at distance i, we again have a decomposition of K_v. Then the decomposition of K_v is called an *i-perfect m-cycle system*.

Previous work in this area has largely concentrated on 2-perfect m-cycle systems; see [9, 6, 10, 7, 5] for instance, although [1] deals with 3-perfect cycle systems.

Clearly for $m = 3$, the decomposition will be a Steiner triple system, and for $m = 4$, no 4-cycle system can be 2-perfect. Apart from $m = 6$ (see [6], and also [2] for the corresponding decomposition of λK_v), all work has been on 2-perfect m-cycle systems where m is odd.

In this paper we determine the spectrum of 2-perfect 8-cycle systems. In particular, we prove:

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MAIN THEOREM

The necessary and sufficient condition for the existence of a 2-perfect 8-cycle decomposition of K_v is $v \equiv 1$ (mod 16).

First we verify the necessary conditions for a 2-perfect 8-cycle system to exist. Certainly the number of edges of K_v, namely $v(v-1)/2$, must be divisible by 8. Moreover, the degree of each vertex, $v-1$, must be even, and so v must be odd. These requirements mean that v must be 1 modulo 16.

Subsequently we shall show that for all $v \equiv 1$ (modulo 16) there exists a 2-perfect 8-cycle decomposition of K_v.

We shall need several "starting" cases; we give some smaller ones here as examples, and others in the Appendix. Henceforth we shall denote the m-cycle consisting of the edges $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \ldots, \{x_{m-1}, x_m\}, \{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, \ldots, x_m)$ or $(x_1, x_m, x_{m-1}, \ldots, x_2)$.

EXAMPLE 1.1 *There is a 2-perfect 8-cycle decomposition of K_{17}.*

With element set Z_{17}, the starter 8-cycle $(0, 16, 1, 7, 3, 10, 13, 5)$ generates (mod 17) a suitable decomposition.

EXAMPLE 1.2 *There is a 2-perfect 8-cycle decomposition of $K_{4,4,4,4}$.*

We take as element set $\bigcup_{i=0}^{3}\{(0,i),(1,i),(2,i),(3,i)\}$. Then the following three 8-cycles are starters modulo 4, where the second component is fixed and the first component is cycled modulo 4.

$$((0,0),(2,2),(0,1),(2,3),(1,0),(3,1),(0,2),(3,3))$$
$$((0,0),(1,2),(2,1),(3,3),(1,0),(0,2),(0,3),(0,1))$$
$$((0,0),(1,1),(1,2),(3,3),(3,0),(2,1),(1,3),(0,2)).$$

EXAMPLE 1.3 *There is a 2-perfect 8-cycle decomposition of K_{33}.*

Let the vertices be Z_{33}. Then the following two 8-cycles are suitable starters modulo 33:

$$(0,7,1,31,3,17,6,29), \quad (0,13,4,21,9,8,16,31).$$

EXAMPLE 1.4 *There is a 2-perfect 8-cycle decomposition of K_{49}.*

Let the vertices be $\mathbf{Z}_{49}$. Then the following three 8-cycles are suitable starters modulo 49:

$$(0,17,1,25,3,34,6,44),\ (0,41,4,11,9,39,16,10),$$
$$(0,14,11,40,25,16,12,48).$$

EXAMPLE 1.5 *There is a* 2*-perfect* 8*-cycle decomposition of* K_{81}.

Let the vertices be $\mathbf{Z}_{81}$. Then the following five 8-cycles are suitable starters modulo 81:

$$(0,67,1,77,3,50,6,58),\quad (0,69,4,40,9,12,16,35),$$
$$(0,32,11,62,23,14,36,63),\quad (0,1,14,57,29,35,46,56),$$
$$(0,73,18,35,37,77,57,33).$$

EXAMPLE 1.6 *There is a* 2*-perfect* 8*-cycle decomposition of* K_{97}.

Let the vertices be $\mathbf{Z}_{97}$. Then the following six 8-cycles are suitable starters modulo 97:

$$(0,41,1,49,3,58,6,68),(0,93,4,27,9,48,16,70),$$
$$(0,16,11,48,23,89,36,50),(0,94,14,16,29,36,46,76),$$
$$(0,15,24,52,53,19,25,86),(0,24,43,76,96,21,47,59).$$

2 The case $n \equiv 1$ modulo 3

We consider K_v where $v = 16n+1$ and $n = 3m+1$. Thus $v = 48m+17 = 4(12m+4)+1$. So we take, as vertex set of K_v, the elements

$$\{\infty\} \cup \{(i,j) \mid 1 \le i \le 12m+4, \quad 1 \le j \le 4\}.$$

When $m = 0$, we have $v = 17$, and a decomposition of K_{17} is given in Example 1.1. So now we assume that $m \ge 1$.

There is a resolvable balanced incomplete block design (BIBD) on $12m+4$ elements with block size 4 and $\lambda = 1$. Take such a design on the set $\{(i,j) \mid 1 \le i \le 12m+4\}$, and let one parallel class be fixed. For each block $\{(i_1,j),(i_2,j),(i_3,j),(i_4,j)\}$ in that fixed parallel class, we place a copy of the decomposition of K_{17} (see Example 1.1) on the vertex set

$$\{\infty\} \cup \{(i_1,j),(i_2,j),(i_3,j),(i_4,j) \mid 1 \le j \le 4\}.$$

For all other blocks $\{(x,j),(y,j),(z,j),(w,j)\}$ of the resolvable BIBD, we place a copy of our decomposition of $K_{4,4,4,4}$ (see Example 1.2) on the vertex set

$$\begin{aligned}\{(x,j) \mid 1 \le j \le 4\} \ &\cup \ \{(y,j) \mid 1 \le j \le 4\} \\ &\cup \ \{(z,j) \mid 1 \le j \le 4\} \cup \{(w,j) \mid 1 \le j \le 4\}.\end{aligned}$$

The result is a 2-perfect 8-cycle decomposition of K_v where $v = 16n+1$ and n is 1 modulo 3; this construction covers all cases with $n \equiv 1$ modulo 3, without exception.

3 The case $n \equiv 0$ modulo 3

Now we consider K_v where $v = 16n+1$ and $n = 3m$, so $v = 48m+1$.

Let the vertices of K_v be

$$\{\infty\} \cup \{(i,j) \mid 1 \le i \le 12m, \quad 1 \le j \le 4\}.$$

Now there exists a group divisible design (GDD) on $12m$ elements with block size 4 and group size 12 (and $\lambda = 1$) for all $m \ge 4$ (or $m = 1$). (See [4, Theorem 6.3].)

For $m = 1$ and 2, decompositions for K_{49} and K_{97} are given in Examples 1.4 and 1.6.

For $m = 3$, a suitable decomposition of K_{145} is given in the Appendix.

For $m \ge 4$, a construction is as follows:

Take the GDD described above, on the set $\{(i,j) \mid 1 \le i \le 12m\}$. For each group $\{(i_1,j),(i_2,j),\ldots,(i_{12},j)\}$, place on the vertex set

$$\{\infty\} \cup \{(i_1,j),(i_2,j),\ldots,(i_{12},j) \mid 1 \le j \le 4\}$$

a copy of the decomposition of K_{49} given in Example 1.4. For each block of the GDD, $\{(x,j),(y,j),(z,j),(w,j)\}$, place a copy of our decomposition of $K_{4,4,4,4}$ on the vertex set

$$\begin{aligned}\{(x,j) \mid 1 \le j \le 4\} \ &\cup \ \{(y,j) \mid 1 \le j \le 4\} \\ &\cup \ \{(z,j) \mid 1 \le j \le 4\} \cup \{(w,j) \mid 1 \le j \le 4\}.\end{aligned}$$

The result is a 2-perfect 8-cycle decomposition of K_v where $v = 48m+1$ and $m \ge 4$.

4 The case $n \equiv 2$ modulo 3

4.1 $n \equiv 2$ modulo 6

Here $v = 16n + 1$ where $n = 6m + 2$. So $v = 4(24m + 8) + 1$.

Let the vertex set of K_v be

$$\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 24m + 8, \quad 1 \leq j \leq 4\}.$$

A decomposition of K_{33} is given in Example 1.3. So now let $m \geq 1$.

Now there exists a GDD on $24m + 8$ elements, with group size 8, block size 4 (and $\lambda = 1$) for all $24m + 8 \geq 32$ or $m = 0$, that is, for all $m \geq 0$. (See [4, Theorem 6.3].)

Take this GDD on the set

$$\{(i,j) \mid 1 \leq i \leq 24m + 8\}.$$

For each group $\{(i_s, j) \mid 1 \leq s \leq 8\}$, place on the vertex set $\{\infty\} \cup \{(i_s, j) \mid 1 \leq s \leq 8, \quad 1 \leq j \leq 4\}$ a copy of the decomposition of K_{33} given in Example 1.3.

Then for each block $\{(x,j),(y,j),(z,j),(w,j)\}$ of the GDD, take the vertex set

$$\begin{aligned}\{(x,j) \mid 1 \leq j \leq 4\} \quad &\cup \quad \{(y,j) \mid 1 \leq j \leq 4\} \\ &\cup \quad \{(z,j) \mid 1 \leq j \leq 4\} \cup \{(w,j) \mid 1 \leq j \leq 4\}\end{aligned}$$

and place on this a copy of the decomposition of $K_{4,4,4,4}$ given in Example 1.2.

This completes the case $n \equiv 2 \mod 6$, with no exceptions.

4.2 $n \equiv 5$ modulo 6

Here $v = 16n + 1$ where $n = 6m + 5$, and so $v = 16(6m + 5) + 1$.

Let the vertex set of K_v be

$$\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 6m + 5, \quad 1 \leq j \leq 16\}.$$

A decomposition of K_{81} (when $m = 0$ above) is given in Example 1.5. The case $m = 1$ (corresponding to K_{177}) is given in the Appendix. The general construction here requires $m \geq 3$, so we now deal with $m = 2$.

EXAMPLE 4.1 *There exists a* 2*-perfect* 8*-cycle decomposition of* K_{273}.

Note firstly that there exists a group divisible design (GDD) on 48 elements with group size 8 and block size 3 which is resolvable. This is given explicitly in the Appendix of [11]. The 16 starter blocks given there are developed modulo 20. Adjoin to each starter block the element $(0,3)$ and develop this also modulo 20 (on the first component). The result is a GDD on 68 elements with block size 4, one group of size 20 (namely the 20 new elements $\{(i,3) \mid 0 \leq i \leq 19\}$) and six groups of size 8.

Now we are ready to give our decomposition of K_{273}.

Let the elements be

$$\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 68,\ 1 \leq j \leq 4\}.$$

On the set $\{(i,j) \mid 1 \leq i \leq 68\}$ we place the above-described GDD. Without loss of generality let its group of size 20 be $\{(i,j) \mid 1 \leq i \leq 20\}$, with the six groups of size 8 on the remaining 48 elements. Then on $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 20,\ 1 \leq j \leq 4\}$ we place a copy of our decomposition of K_{81} (see Example 1.5). Also for each group $\{(i_x,j) \mid 1 \leq x \leq 8\}$ of size 8 of the GDD, place on $\{(i_x,j) \mid 1 \leq x \leq 8,\ 1 \leq j \leq 4\} \cup \{\infty\}$ a decomposition of K_{33} (see Example 1.3). Finally, for each block of the GDD of size 4, say $\{(i_x,j) \mid 1 \leq x \leq 4\}$, place on $\bigcup_{x=1}^{4}\{(i_x,j) \mid 1 \leq j \leq 4\}$ a decomposition of $K_{4,4,4,4}$. The result is a 2-perfect 8-cycle decomposition of K_{273}.

Now let $m \geq 3$. It is known ([3, Theorem 4]) that there exists a GDD on $6m+5$ elements with blocks of size 4 and groups of size 2, and exactly one group of size 5 (and $\lambda = 1$) provided $m \neq 1, 2$. So take such a GDD on the set $\{(i,j) \mid 1 \leq i \leq 6m+5\}$, and without loss of generality let $\{(i,j) \mid 1 \leq i \leq 5\}$ be the group of size 5, and $\{(2i,j),(2i+1,j)\}$ $(3 \leq i \leq 3m+2)$ the remaining groups of size 2. Then on $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 5,\ 1 \leq j \leq 16\}$ place a decomposition of K_{81} (see Example 1.5). And on $\{\infty\} \cup \{(2i,j),(2i+1,j) \mid 1 \leq j \leq 16\}$, for each i with $3 \leq i \leq 3m+2$, place a decomposition of K_{33} (see Example 1.3).

Finally, for each block $\{(x,j) \mid 1 \leq s \leq 4\}$ of the GDD, place on $\bigcup_{s=1}^{4}\{(x_s,j) \mid 1 \leq j \leq 16\}$ a copy of the decomposition of $K_{16,16,16,16}$ given in the Appendix.

5 Conclusion

Combining the results in the previous sections, we have proved

MAIN THEOREM

There exists a decomposition of K_v into 2*-perfect* 8*-cycles if and only if* $v \equiv 1$ *mod* 16.

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APPENDIX

A 2-perfect 8-cycle decomposition of $K_{16,16,16,16}$:
Elements are

$$\{(i,j) \mid 0 \leq i \leq 3,\ 1 \leq j \leq 4\} \cup \{(i,j) \mid 0 \leq i \leq 3,\ 5 \leq j \leq 8\}$$
$$\cup \{(i,j) \mid 0 \leq i \leq 3,\ 9 \leq j \leq 12\} \cup \{(i,j) \mid 0 \leq i \leq 3,\ 13 \leq j \leq 16\}.$$

The graph has 6×16^2 edges, and so we need 4×48 8-cycles. We fix the second component and cycle the first modulo 4. We then have 48 starter 8-cycles (modulo (4, –)):

((0,1),(2,9),(0,5),(2,13),(1,1),(3,5),(0,9),(3,13)),
((0,1),(2,10),(0,6),(2,13),(1,2),(3,7),(0,12),(3,14)),
((0,1),(2,11),(0,7),(2,13),(1,3),(3,8),(0,11),(3,15)),
((0,1),(2,12),(0,8),(2,13),(1,4),(3,6),(0,10),(3,16)),
((0,2),(2,10),(0,5),(2,14),(1,4),(3,8),(0,12),(3,16)),
((0,2),(2,11),(0,6),(2,14),(1,3),(3,6),(0,9),(3,15)),
((0,2),(2,12),(0,7),(2,14),(1,2),(3,5),(0,10),(3,14)),
((0,2),(2,9),(0,8),(2,14),(1,1),(3,7),(0,11),(3,13)),
((0,3),(2,11),(0,5),(2,15),(1,2),(3,6),(0,11),(3,14)),
((0,3),(2,12),(0,6),(2,15),(1,1),(3,8),(0,10),(3,13)),
((0,3),(2,9),(0,7),(2,15),(1,4),(3,7),(0,9),(3,16)),
((0,3),(2,10),(0,8),(2,15),(1,3),(3,5),(0,12),(3,15)),
((0,4),(2,12),(0,5),(2,16),(1,3),(3,7),(0,10),(3,15)),
((0,4),(2,9),(0,6),(2,16),(1,4),(3,5),(0,11),(3,16)),
((0,4),(2,10),(0,7),(2,16),(1,1),(3,6),(0,12),(3,13)),
((0,4),(2,11),(0,8),(2,16),(1,2),(3,8),(0,9),(3,14)),

((0,1),(1,9),(2,5),(3,13),(1,1),(0,9),(0,13),(0,5)),
((0,1),(1,10),(2,6),(3,13),(1,2),(0,11),(0,16),(0,6)),
((0,1),(1,11),(2,7),(3,13),(1,3),(0,12),(0,15),(0,7)),
((0,1),(1,12),(2,8),(3,13),(1,4),(0,10),(0,14),(0,8)),
((0,2),(1,10),(2,5),(3,14),(1,4),(0,12),(0,16),(0,8)),
((0,2),(1,11),(2,6),(3,14),(1,3),(0,10),(0,13),(0,7)),
((0,2),(1,12),(2,7),(3,14),(1,2),(0,9),(0,14),(0,6)),
((0,2),(1,9),(2,8),(3,14),(1,1),(0,11),(0,15),(0,5)),
((0,3),(1,11),(2,5),(3,15),(1,2),(0,10),(0,15),(0,6)),
((0,3),(1,12),(2,6),(3,15),(1,1),(0,12),(0,14),(0,5)),
((0,3),(1,9),(2,7),(3,15),(1,4),(0,11),(0,13),(0,8)),
((0,3),(1,10),(2,8),(3,15),(1,3),(0,9),(0,16),(0,7)),
((0,4),(1,12),(2,5),(3,16),(1,3),(0,11),(0,14),(0,7)),
((0,4),(1,9),(2,6),(3,16),(1,4),(0,9),(0,15),(0,8)),
((0,4),(1,10),(2,7),(3,16),(1,1),(0,10),(0,16),(0,5)),
((0,4),(1,11),(2,8),(3,16),(1,2),(0,12),(0,13),(0,6)),

((0,1),(1,5),(1,9),(3,13),(3,1),(2,5),(1,13),(0,9)),
((0,1),(1,6),(1,10),(3,13),(3,2),(2,7),(1,16),(0,10)),
((0,1),(1,7),(1,11),(3,13),(3,3),(2,8),(1,15),(0,11)),
((0,1),(1,8),(1,12),(3,13),(3,4),(2,6),(1,14),(0,12)),
((0,2),(1,6),(1,9),(3,14),(3,4),(2,8),(1,16),(0,12)),
((0,2),(1,7),(1,10),(3,14),(3,3),(2,6),(1,13),(0,11)),
((0,2),(1,8),(1,11),(3,14),(3,2),(2,5),(1,14),(0,10)),
((0,2),(1,5),(1,12),(3,14),(3,1),(2,7),(1,15),(0,9)),
((0,3),(1,7),(1,9),(3,15),(3,2),(2,6),(1,15),(0,10)),
((0,3),(1,8),(1,10),(3,15),(3,1),(2,8),(1,14),(0,9)),
((0,3),(1,5),(1,11),(3,15),(3,4),(2,7),(1,13),(0,12)),
((0,3),(1,6),(1,12),(3,15),(3,3),(2,5),(1,16),(0,11)),
((0,4),(1,8),(1,9),(3,16),(3,3),(2,7),(1,14),(0,11)),
((0,4),(1,5),(1,10),(3,16),(3,4),(2,5),(1,15),(0,12)),
((0,4),(1,6),(1,11),(3,16),(3,1),(2,6),(1,16),(0,9)),
((0,4),(1,7),(1,12),(3,16),(3,2),(2,8),(1,13),(0,10)).

A 2-perfect 8-cycle decomposition of K_{145}, based on Z_{145}:

(0,53,10,123,62,122,111,98), (0,38,89,120,48,30,133,68),
(0,62,81,127,22,144,29,140), (0,93,44,144,8,12,113,88),
(0,14,42,45,140,82,72,16), (0,1,3,9,17,24,39,121),
(0,12,53,124,34,144,16,70), (0,20,57,31,140,54,21,48),
(0,21,99,30,8,114,35,64).

A 2-perfect 8-cycle decomposition of K_{177}, based on $\mathbf{Z}_{177}$:

(0, 148, 93, 12, 121, 167, 32, 144), (0, 100, 114, 138, 154, 36, 94, 20),
(0, 101, 30, 143, 135, 123, 53, 132), (0, 166, 50, 67, 175, 171, 54, 85),
(0, 83, 49, 96, 24, 50, 174, 142), (0, 18, 156, 42, 92, 54, 106, 28),
(0, 5, 11, 4, 13, 23, 8, 56), (0, 13, 34, 7, 144, 51, 29, 120),
(0, 1, 90, 113, 156, 4, 55, 19), (0, 2, 138, 63, 19, 106, 160, 49),
(0, 3, 100, 38, 148, 75, 45, 82).