Double occurrence words with the same alternance graph

Laurence Ghier

Département de Mathématiques et Informatique
Université du Maine
72017 Le Mans Cédex
France

Abstract. Let m be a double occurrence word (i.e. each letter occuring in m occurs precisely twice). An alternance of m is a nonordered pair vw of distinct letters such that we meet alternatively $\ldots v \ldots w \ldots v \ldots w \ldots$ when reading m. The alternance graph $A(m)$ is the simple graph whose vertices are the letters of m and whose edges are the alternances of m. We define a transformation of double occurrence words such that whenever $A(m) = A(n)$, m and n are related by a sequence of these transformations.

1. Introduction

A simple graph is called a *circle graph* if it is isomorphic to the intersection graph of a finite collection of chords of a circle. Without loss of generality, we may assume that no two chords share a common endpoint. Thus if we attach a letter to each chord, and if we write this letter near each end of the chord, we construct a word m by turning around the circle and recording the successive letters. Each letter occurs precisely twice in m, so we say that m is a *double occurrence word.* We do not distinguish any two double occurrence words which are *cyclically equivalent* (such that one of these words is obtained from the other one by a cyclic permutation of the sequence of the letters eventually followed by a reversion). The reverse construction which associates a chord configuration to a double occurrence word is easy to perform.

An *alternance* of m is a pair vw of distinct letters such that we meet alternatively $\ldots v \ldots w \ldots v \ldots w \ldots$ when reading m. We notice that two chords labelled by letters v and w intersect if and only if vw is an alternance of m. The alternance graph $A(m)$ is the simple graph whose vertices are the letters of m and whose edges are the alternances of m.

Clearly the class of circle graphs is equal to the class of alternance graphs, but from a combinatorial point of view it is easier to handle double occurrence words than chord configurations. A study of circle graphs can be found in Golumbic's book "Algorithmic theory and perfect graphs" [5].

If μ is a word on the set V, we denote by $\tilde{\mu}$ the word obtained by reversing the sequence of the letters of μ.

Let m be a double occurrence word on the set V. A *split of* m is a bipartition $\{V', V''\}$ of V such that $|V'|, |V''| \geq 2$ and $m = m'_1 m''_1 m'_2 m''_2$ where m'_1 and m'_2 (m''_1 and m''_2) are subwords of m which only use letters in $V'(V'')$. The replacement of m by one of the following words $m'_1 \tilde{m}''_1 m'_2 \tilde{m}''_2$, $\tilde{m}'_1 m''_1 \tilde{m}'_2 m''_2$, $\tilde{m}'_1 \tilde{m}''_1 \tilde{m}'_2 \tilde{m}''_2$ is called a *turnaround in* m w.r.t. $\{V', V''\}$.

(1.1) Remark: We may only consider the first of these turnarounds. The second one is similar to the first one after exchanging the roles of V' and V''. The third one is the composition of the two first ones.

In a chord configuration associated to m, let $C'(C'')$ be the block of the chords whose endpoints are labelled by $V'(V'')$. Then we can interprete a turnaround in m as the reversal of C', or C'', or both.

A *split of a simple graph* G is a bipartition $\{V', V''\}$ of the vertex-set $V(G)$ satisfying the two following conditions:

(i) $|V'|, |V''| \geq 2$

(ii) there exist subsets $W' \in P(V')$ and $W'' \in P(V'')$ such that:
$\{v'v'' \in E(G) : v' \in V', v'' \in V''\} = \{v'v'' : v' \in W', v'' \in W''\}$.

A graph is said to be *prime* if it has no split and at least three vertices. The following property is easy to verify.

(1.2) Property: If $\{V', V''\}$ is a split of a double occurrence word m, then $\{V', V''\}$ is a split of the alternance graph $A(m)$. Moreover, if n is another double occurrence word obtained through a turnaround in m w.r.t. $\{V', V''\}$ then $A(m) = A(n)$. ∎

The first part of the proposition has no converse. For example, with $m = acbdbcad$, $V' = \{a, b\}$, $V'' = \{c, d\}$, $\{V', V''\}$ is a split of $A(m)$ whereas it is not a split of m. On the other side the second part of the proposition implies that a sequence of turnarounds does not modify an alternance graph. The main result of this paper states that the converse actually holds.

(1.3) Theorem. *If n and m are two double occurrence words such that $A(n) = A(m)$, then, if $A(n)$ is connected there exists a sequence of turnarounds transforming n into m.*

The theorem will be proved in Section 6 but we notice that, for $A(n) = A(m)$ prime, A. Bouchet proved in [1] that there exists a single double occurrence word (up to cyclic equivalence) which realizes $A(n) = A(m)$. So $n = m$ (up to cyclic equivalence). Thus we shall suppose in the sequel that $A(n) = A(m)$ has some split $\{V', V''\}$.

2.Turnarounds and 4-cocycles

To prove the theorem, we associate to any double occurrence word m on a set V an ordered pair (G, T) where G is a 4-regular graph on the vertex-set V and T is an Euler tour of G. To construct (G, T) we consider a cycle T of length $2|V|$ whose vertices are labelled by the successive letters of m, and we identify each pair of vertices labelled by a same letter $v \in V$ into a single vertex which we naturally identify to v. Conversely if (G, T) is given m is equal to the sequence of the successive vertices of T. Thus the mapping $m \to (G, T)$ is bijective (up to an isomorphism). Now to interprete a turnaround in m in terms of (G,T) we recall some definitions.

If G is a (nondirected) graph with possible loops and/or multiple edges, then it will be convenient to decompose each edge e into two *half-edges* h' and h'', having one end each, the ends of e being the ends of h' and h''. A *transition* of G is a pair of distinct half-edges $\{h,k\}$ with a same end, and we denote this common end by $\sigma(h,k)$. A *closed trail* (also called briefly a *tour*) is a sequence of pairwise distinct half-edges $T = (h_0 h_1 h_2 \dots h_{2l-2} h_{2l-1})$, $l > 0$, such that $\{\{h_{i-1}, h_i\}, \{h_i, h_{i+1}\}\}$ is composed of one edge and of one transition for every $i = 0, 1, \dots, 2l-1$ (with the convention $i+1 = 0$ if $i = 2l-1$ and $i-1 = 2l-1$ if $i = 0$). T is an *Euler tour* if each half-edge appears in T. Supposing that $\{h_0, h_1\}$ is a transition, the *vertex-sequence* of T is $(\sigma(h_0, h_1)\sigma(h_2, h_3)\dots\sigma(h_{2l-2}, h_{2l-1}))$. A subsequence of T, say $P = h_i h_{i+1} \dots h_{i+2p-1}$ is called a *subpath* if $\{h_i, h_{i+1}\}, \{h_{i+2}, h_{i+3}\}, \dots \{h_{i+2p-2}, h_{i+2p-1}\}$ are transitions of G, and we call $(\sigma(h_i, h_{i+1})\sigma(h_{i+2}, h_{i+3})\dots\sigma(h_{i+2p-2}, h_{i+2p-1}))$ the *vertex-sequence* of P.

Consider the pair (G,T) associated to m, where T is equal to the sequence of the half-edges defined above. The vertex-sequence of T is equal to m. Consider a split $\{V', V''\}$ of m and the associated decomposition $m = m'_1 m''_1 m'_2 m''_2$ where m'_1 and m'_2 (m''_1 and m''_2) are subwords of m which only use letters in $V'(V'')$. This induces a decomposition $T = P'_1 P''_1 P'_2 P''_2$ where P'_1, P''_1, P'_2, P''_2 are subpaths whose vertex-sequences are respectively m'_1, m''_1, m'_2, m''_2.

We denote by P'_{1-} and P'_{1+} the first and the last half-edges of P'_1, and we define the similar notations for P''_1, P'_2, P''_2. Then $\{\{P'_{1+}, P''_{1-}\}, \{P''_{1+}, P'_{2-}\}, \{P'_{2+}, P''_{2-}\}, \{P''_{2+}, P'_{1-}\}\}$ is a subset of edges. Moreover it is equal to the cocycle $\delta V' = \delta V''$ because the paths P'_1 and P'_2 are incident to vertices in V' only when P''_1 and P''_2 are incident to vertices in V'' only. Now if we consider some turnaround $n = m'_1 \bar{m}''_1 m'_2 \bar{m}''_2$, the corresponding pair (H,U) can be constructed by replacing in G the four edges of $\delta V'$ by $\{\{P'_{1+}, P''_{1+}\}, \{P''_{1-}, P'_{2-}\}, \{P'_{2+}, P''_{2+}\}, \{P''_{2-}, P'_{1-}\}\}$, which yields the graph H and transforms the Euler tour T into $U = P'_1 \bar{P}''_1 P'_2 \bar{P}''_2$ with its corresponding path decomposition (see Figure 1).

3. Connectivity

Let $A = (V,E)$ be a simple graph and $V' \in P(V)$. The *cut-matrix* w.r.t. V' is the binary matrix $\Pi = (\Pi_{v'v''}: v' \in V', v'' \in V \backslash V')$ such that $\Pi_{v'v''} = 1$ if and only if $v'v'' \in E$. Let $C(V') = \text{rank}(\Pi)$. The mapping C is called the *connectivity function*.

Let us consider now a pair (G,T) with a 4-regular graph G and an Euler tour T, and suppose that A is the alternance graph of the vertex-sequence of T. Let us consider a bipartition $\{V', V''\}$ of V and let $\Gamma'(\Gamma'')$ be the set of the components of the induced subgraph $G[V'](G[V''])$. Finally let us define the binary matrix $B = (b_{c'c''}: c' \in \Gamma', c'' \in \Gamma'')$ such that $b_{c'c''}$ is the parity of the number of edges

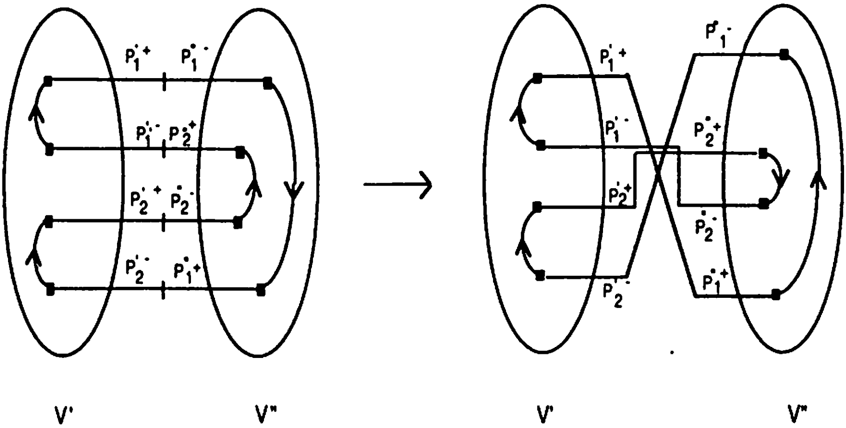


Figure 1

The preceding operation will be called a turnaround of (G, T) w.r.t. the cocycle $\delta V' = \delta V''$.

that join the component c' to the component c''. The following property is proved by A. Bouchet [3] by using the theory of isotropic systems.

(3.1) Lemma. *The connectivity function satisfies* $C(V') = |\delta V'|/2 - |\Gamma'| - |\Gamma''| + rank(B) + 1$. ∎

To have another expression of $C(V')$, let us define for each component c of $\Gamma' \cup \Gamma''$ the *excess* $e(c)$ as the number of edges of $\delta V'$ incident to c decreased by 4, and let $e = 1/4 \sum(e(c) : c \in \Gamma' \cup \Gamma'')$.

A simple computation shows that $e = 1/2|\delta V'| - |\Gamma'| - |\Gamma''|$. Thus Lemma (3.1) implies the following formula:

$$C(V') = e + \text{rank}(B) + 1. \tag{3.2}$$

∎

4. Ring configuration

Let G be a 4-regular graph and $\{V', V''\}$ a bipartition of $V = V(G)$. G has a *ring configuration* w.r.t. $\{V', V''\}$ if V' is partitionned into $V'_1, V'_2 \ldots V'_k$ and V'' is partitioned into $V''_1, V''_2 \ldots V''_k$ in such a way that the following properties hold:

(i) $G[V'_i]$ is a component of $G[V']$ and $G[V''_i]$ is a component of $G[V'']$ for $i = 1, 2 \ldots k$

(ii) for $i = 1, 2 \ldots k$, the edges which join $V'_i(V''_i)$ to $V/V'_i(V/V''_i)$ compose a 4-cocycle made of two edges between V'_i and $V''_i(V''_i$ and $V'_i)$ and two edges between V'_i and $V''_{i-1}(V''_i$ and $V'_{i+1})$ with the convention $V''_0 = V''_k(V'_{k+1} = V'_1)$.

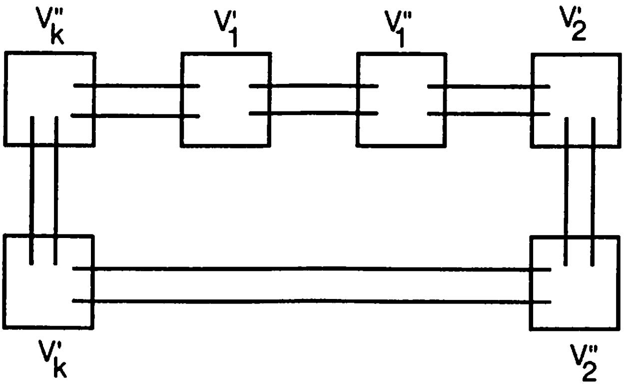
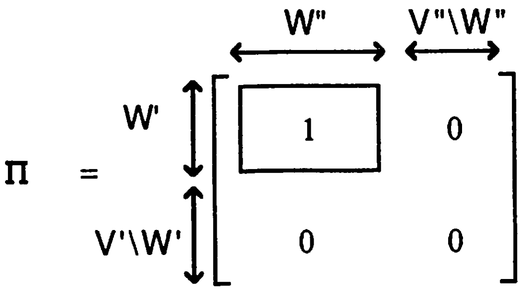


Figure 2

Example:

(4.1) Lemma. *Let (G,T) be the ordered pair associated to a double occurrence word m and let $A(m)$ be the alternance graph of m. If $\{V', V''\}$ is a split of $A(m)$, then either $G[V']$ and $G[V'']$ are connected or G has a ring configuration w.r.t. $\{V', V''\}$.*

Proof: Let TT be cut-matrix of $A(m)$ w.r.t. $\{V', V''\}$. Since $\{V', V''\}$ is a split of $A(m)$, TT has the following structure:



W' and W'' are not empty otherwise $A(m)$ will not be connected. So $C(V') =$ rank$(TT) = 1$. Formula (3.2) implies

$$C(V') = e + \text{rank}(B) + 1,$$

and so

$$e + \text{rank}(B) = 0.$$

We claim that $e(c) \geq 0$ for every component c of $\Gamma' \cup \Gamma''$. We consider the different cases:

if $e(c) = -4$, then the number of edges of $\delta V'$ incident to c is equal to zero, which is impossible since G is connected, if $e(c) = -3$ or -1, then there would be a vertex of odd degree in c, which is impossible because G is 4-regular,

if $e(c) = -2$, then the number of edges of $\delta V'$ incident to c is equal to 2; the double occurrence word m is the sequence of the successive vertices of T, we can decompose m into $m = m_1 m_2$ where m_1 only has vertices of c when m_2 only has vertices of $V \backslash c$, so there is no alternance in m composed of a vertex of c and a vertex of $V \backslash c$, and so $A(m)$ is not connected, that is impossible.

So the claim is proved, which implies $e \geq 0$. Since rank($B \geq 0$), we have $e = \text{rank}(B) = 0$. Since $\text{rank}(B) = 0$ the number of edges between any two components of $\Gamma' \cup \Gamma''$ is equal to 2 or 4. In the first case G has a ring configuration w.r.t. $\{V', V''\}$. In the second case $G[V']$ and $G[V'']$ are connected. ∎

5. Turnaround of (G,T)

(5.1) Lemma. *If G is a 4-regular graph with a ring configuration w.r.t. a bi-partition $\{V', V''\}$ of $V(G)$, then there exists a sequence of turnarounds making $G[V']$ and $G[V'']$ connected with $|\delta V'| = |\delta V''| = 4$.*

Proof: We use the same notation as in Section 4 for a ring configuration. We suppose that $k > 2$. We notice that $\delta(V_1'' \cup V_2')$ is a 4-cocycle. We partition this 4-cocycle into two pairs $\{\{h', h''\}, \{k', k''\}\}$ and $\{\{l', l''\}, \{m', m''\}\}$ in such a way that $\{\{h', h''\}, \{l', l''\}\}$ is the pair of edges joining V_1' and V_1'', and $\{\{k', k''\}, \{m', m''\}\}$ is the pair of edges joining V_2' and V_2'' (see figure 3). We make the turnaround as described at the end of Section 2. Then if we let $W_2' = V_1' \cup V_2'$ and $W_2'' = V_1'' \cup V_2''$ we have a new ring configuration with the following partitions $\{V' = W_2' \cup V_3' \cup \cdots \cup V_k', V'' = W_2'' \cup V_3'' \cup \cdots \cup V_k''\}$ where k, the 'half-length', is replaced by $k - 1$ (see Figure 3).

If we suppose $k = 2$ we do the same construction as above then $G[W_2']$ and $G[W_2'']$ are connected and $|\delta W_2'| = |\delta W_2''| = 4$, which proves the lemma. If $k > 2$ we use the construction to make an induction on k.

6. Proof of Theorem (1.2)

We prove the theorem by induction on $p = |V|$.

If $p = 2$, it is true.

We suppose that the theorem is true for $p > 2$. We know this result if $A(m) = A(n)$ is prime [1], so we suppose that $A(m) = A(n)$ has a split $\{V', V''\}$.

According to Section 2, we associate an ordered pair (G, T) with m and another ordered pair (H, U) with n.

By Lemma (4.1), we know that G and H have a ring configuration w.r.t. $\{V', V''\}$.

By Lemma (5.1), there exists a sequence of turnarounds transforming G into a graph G_1 such that $G_1[V']$ and $G_1[V'']$ are connected and verify $|\delta V'| = |\delta V''| = 4$. The Euler tour T is transformed into T_1 which yields a pair (G_1, T_1), and we

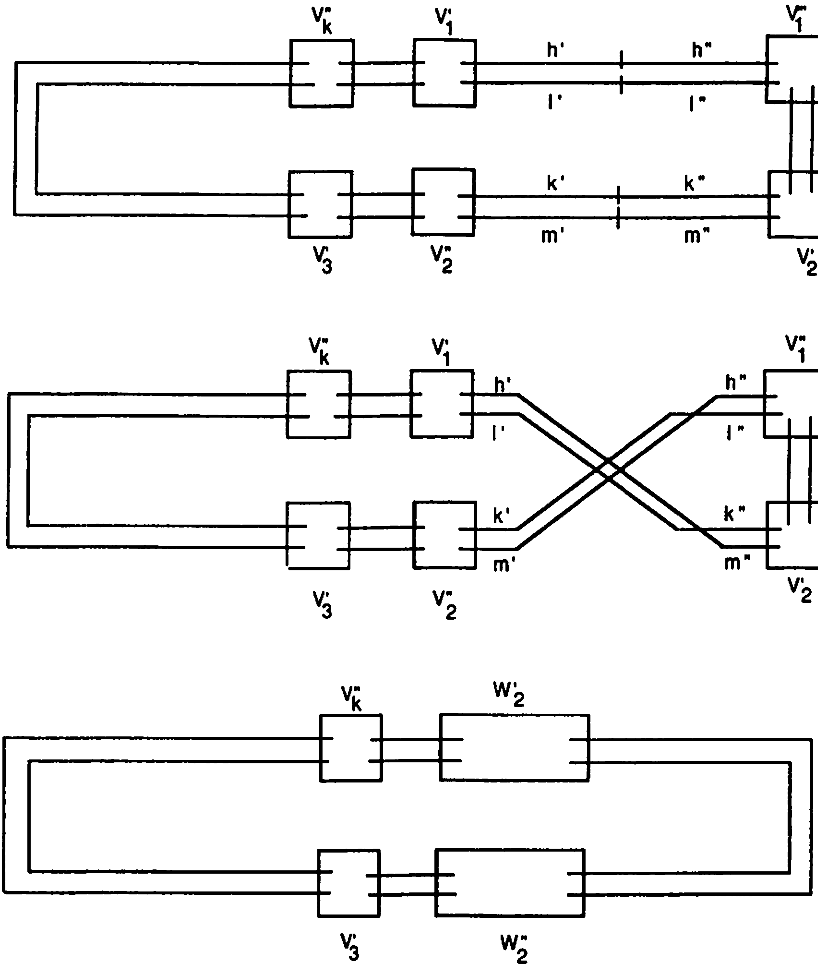


Figure 3

denote by μ the vertex-sequence of T_1. Similarly we derive a pair (H_1, U_1) from (H, U), such that $H_1[V']$ and $H_1[V'']$ are connected and verify $|\delta V'| = |\delta V''| = 4$, moreover U_1 is an Euler tour in H_1 and we denote by η the vertex-sequence of U_1. We have $A(\mu) = A(m) = A(n) = A(\eta)$. We consider the following decompositions of μ and η w.r.t. $\{V', V''\}$: $\mu = \mu'_1\mu''_1\mu'_2\mu''_2$ and $\eta = \eta'_1\eta''_1\eta'_2\eta''_2$ where μ'_1, μ'_2, η'_1 and η'_2 only have letters of V', whereas $\mu''_1, \mu''_2, \eta''_1$ and η''_2 only have letters of V''.

In order to use the induction on $p = |V|$, we consider a new element $v \notin V$

and we construct two words $\mu' = \mu'_1 v \mu'_2 v$ and $\mu'' = \mu''_1 v \mu''_2 v$ associated to the decomposition of μ. Similarly we construct two words $\eta' = \eta'_1 v \eta'_2 v$ and $\eta'' = \eta''_1 v \eta''_2 v$ associated to the decomposition of η. The words μ' and η' are defined on the same set of letters $V' + v$ whereas μ'' and η'' are defined on $V'' + v$. We have $A(\mu') = A(\eta')$ and $A(\mu'') = A(\eta'')$; because $\{V', V''\}$ is a split of the graph $F = A(\mu) = A(\eta)$, if we consider in F a vertex v'' of V'' such that $F[V' + v'']$ is connected then $A(\mu')$ and $A(\eta')$ are isomorphic images of $F[V' + v'']$ in the bijection i: $V' + v'' \to V' + v$ defined by $i(v') = v'$ for every v' of V' and $i(v'') = v$. Similarly for $A(\mu'')$ and $A(\eta'')$.

By induction there exists a sequence of turnarounds which transforms μ' into η' and another one which transforms μ'' into η''. We prove that each of these turnarounds, denoted by $\to$, can be associated to a turnaround on μ. And so the theorem will be proved.

We suppose for example that $\to$ is applied to μ'. It is associated with a split $\{V'_1, V'_2\}$ of μ'. We may suppose that $v \in V'_2$ without loss of generality, so that the decomposition of μ' w.r.t. $\{V'_1, V'_2\}$ can be written $\mu' = B'_1 A'_1 C'_1 v B'_2 A'_2 C'_2 v$ with $\mu'_1 = B'_1 A'_1 C'_1$ and $\mu'_2 = B'_2 A'_2 C'_2$ where A'_1 and A'_2 are words on V'_1 while $B'_1, C'_1 B'_2$ and C'_2 are words on $V'_2 - v$. By Remark (1.1) we may suppose that the result of $\to$ on μ' is the word $B'_1 \tilde{A}'_1 C'_1 v B'_2 \tilde{A}'_2 C'_2 v$. Then the transformation which changes the word $\mu = \mu'_1 \mu''_1 \mu'_2 \mu''_2$ into the word $B'_1 \tilde{A}'_1 C'_1 v B'_2 \tilde{A}'_2 C'_2 v$ is a turnaround w.r.t. the split $\{V'_1, V'' \cup V'_2 - v\}$, which is the expected turnaround of μ. ∎

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