On the Toughness of Some Generalized Petersen Graphs

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Abstract

Upper and lower bounds are established for the toughness of the generalized Petersen graphs $G(n,2)$ for $n \geq 5$, and all non-isomorphic disconnecting sets that achieve the toughness are presented for $5 \leq n \leq 15$. These results also provide an infinite class of $G(n,2)$ for which the toughness equals $\frac{5}{4}$, namely when $n \equiv 0 \pmod 7$.

1 Introduction

Generalized Petersen graphs $G(n,k)$ are defined as follows for $n \geq 3$ with $1 \leq k \leq n-1$ and $2k \neq n$: $G(n,k)$ has vertex set $V = \{u_1, ..., u_n, v_1, ..., v_n\}$, and edge set $E = \{(u_i, u_{i+1}) | 1 \leq i \leq n\} \cup \{(u_i, v_i) | 1 \leq i \leq n\} \cup \{(v_i, v_{i+k}) | 1 \leq i \leq n\}$, where all subscripts are taken modulo n. We restrict our attention to the cases where $n \geq 5$ and $k = 2$. It will be convenient to refer to a subgraph of $G(n,k)$ induced by $V = \{u_i, ..., u_{i+m-1}, v_i, ..., v_{i+m-1}\}$ for some i and some $m \leq n$, as an m-section. The toughness $t(G)$ of a graph is defined [1] as ∞ if $G = K_n$, else $t(G) = \min\left\{\frac{|S|}{\omega(G-S)}\right\}$ taken over all disconnecting subsets S of $V(G)$ such that $\kappa(G) \leq |S| < |V(G)|$, where $\omega(G-S)$ denotes the number of components in the subgraph induced by $G-S$. We will sometimes find it convenient to refer to a disconnecting set S and $|S|$ both by S and refer to $\omega(G-S)$ by just ω, but their meanings should be clear in context. We will also use $c(G-S)$ to denote the set of components induced by $G-S$. All other terms can be found in [2].

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2 Toughness

The following results were inspired by the following conjecture made in [3].

Conj α: If $n \geq 5$ and $n \equiv 1 \pmod 4$ then $t(G(n,2)) = \frac{4}{3}$.

The authors had noted that $t(G(5,2)) = t(G(9,2)) = \frac{4}{3}$ and wondered if perhaps this was the sign of an infinite class of graphs for which toughness equals $\frac{4}{3}$. However, for $n \geq 11$ we have $t(G(n,2)) < \frac{4}{3}$ as shown by the following theorem.

2.1 Upper Bounds

Theorem 1 *If* $n \equiv 0 \pmod 7$ *and* $n \geq 7$ *then* $t(G(n,2)) \leq \frac{5}{4}$.

If $n \equiv 1 \pmod 7$ *and* $n \geq 15$ *then* $t(G(n,2)) \leq \frac{5n+16}{4n+10}$.

If $n \equiv 2 \pmod 7$ *and* $n \geq 9$ *then* $t(G(n,2)) \leq \frac{5n+11}{4n+6}$.

If $n \equiv 3 \pmod 7$ *and* $n \geq 10$ *then* $t(G(n,2)) \leq \frac{5n+6}{4n+2}$.

If $n \equiv 4 \pmod 7$ *and* $n \geq 11$ *then* $t(G(n,2)) \leq \frac{5n+8}{4n+5}$.

If $n \equiv 5 \pmod 7$ *and* $n \geq 12$ *then* $t(G(n,2)) \leq \frac{5n+3}{4n+1}$.

If $n \equiv 6 \pmod 7$ *and* $n \geq 13$ *then* $t(G(n,2)) \leq \frac{5n-2}{4n-3}$.

Proof: (by construction)
We will specify the disconnecting set S by partitioning the vertices of $G(n,2)$ into sections and describing which vertices of each section belong to S. We start with an m-section of length $m \in \{7,8,9,10,4,12,6\}$ according to $n \equiv m \pmod 7$. Then we can complete the partition with $\frac{n-m}{7}$ 7-sections. The sections we will be using are shown in Figures 1-7, where the circled points specify the vertices in S and the boxed/shaded regions specify the components left behind. For each m-section we will denote the number of vertices in S and the number of components left behind by S_m and ω_m, respectively.

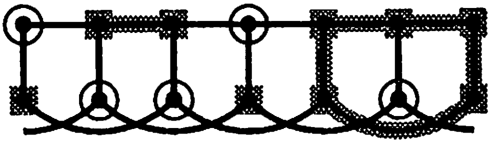


Figure 1 $m = 7$ with $S_7 = 5$ and $\omega_7 = 4$

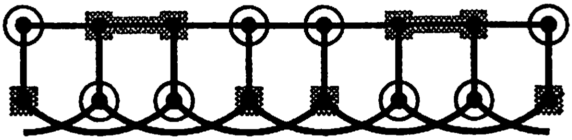


Figure 2 $m = 8$ with $S_8 = 8$ and $\omega_8 = 6$

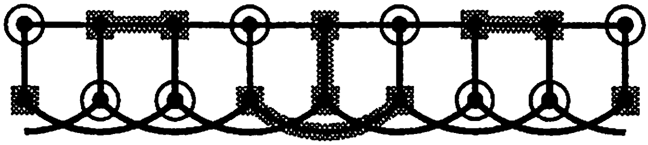


Figure 3 $m = 9$ with $S_9 = 8$ and $\omega_9 = 6$

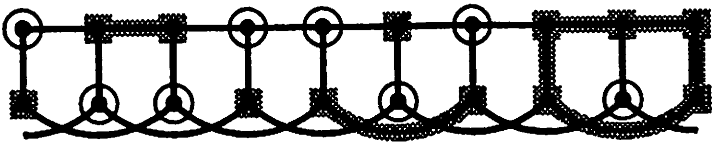


Figure 4 $m = 10$ with $S_{10} = 8$ and $\omega_{10} = 6$

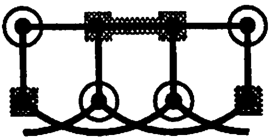


Figure 5 $m = 4$ with $S_4 = 4$ and $\omega_4 = 3$

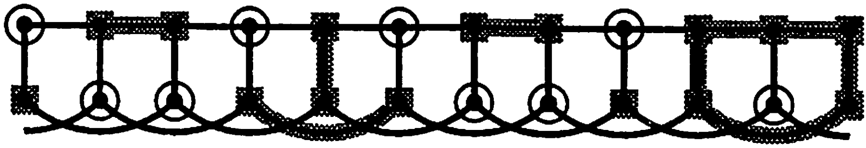


Figure 6 $m = 12$ with $S_{12} = 9$ and $\omega_{12} = 7$

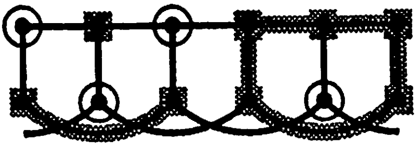


Figure 7 $m = 6$ with $S_6 = 4$ and $\omega_6 = 3$

Note that components from different sections will not combine in $G(n,2)-S$ since the first vertex of the top row, the second vertex of the second row, and the second to last vertex of the second row of the sections are always in S. Therefore since $S_7 = 5$ and $\omega_7 = 4$ we have

$$|S| = S_m + 5\left(\frac{n-m}{7}\right) = \frac{5n + 7S_m - 5m}{7}$$

and

$$\omega(G(n,2) - S) = \omega_m + 4\left(\frac{n-m}{7}\right) = \frac{4n + 7\omega_m - 4m}{7}.$$

Thus we have a disconnecting set S such that

$$t(G(n,2)) \le \frac{|S|}{\omega(G(n,2) - S)} = \frac{5n + 7S_m - 5m}{4n + 7\omega_m - 4m},$$

the stated upper bound when the appropriate values of m, S_m, and ω_m are substituted. □

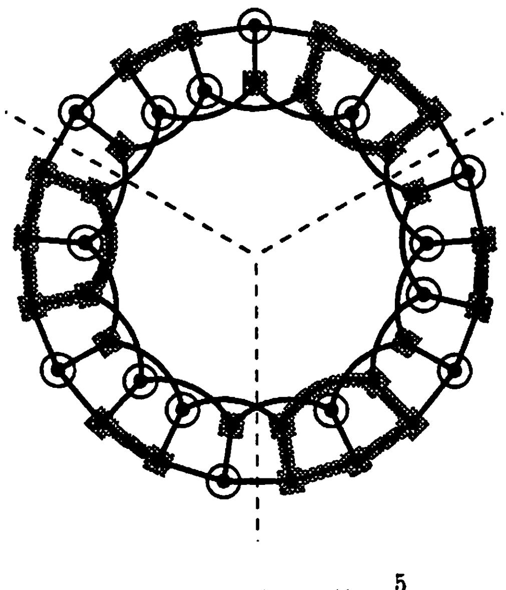


Figure 8 $t(G(21,2)) \le \frac{5}{4}$

We have followed the construction for $G(21,2)$ in Figure 8. To complement the results of Theorem 1. we have exhaustively calculated the toughness of $G(n,2)$ for all $5 \leq n \leq 15$ (via computer) and present in Appendix A all the non-isomorphic ways (with respect to dihedral symmetries) that the toughness can be achieved. Also note in appendix A that Figures 10b-10e are all isomorphic. The results in 6, 8, 10b, and 12a suggest a different construction that may work for n even. However, this construction in which $S = \{u_i \mid 1 \leq i \leq n-1 \text{ and } i \text{ odd}\} \cup \{v_i \mid 1 \leq j \leq \left\lceil \frac{n}{4} \right\rceil \text{ and } i = 4j-2\}$ yields $|S| = \frac{n}{2} + \left\lceil \frac{n}{4} \right\rceil$ and $\omega(G(n,2) - S) = \frac{n}{2} + 1$. So though it is better than our construction would be for $G(6,2)$ and $G(8,2)$, it does not beat our construction for $G(10,2)$ and $G(12,2)$, and it is worse than our construction for $n \geq 14$. Thus for $n \geq 5$ we have the following upper bounds for $t(G(n,2))$ as shown in Table 1, where equality is known for the numbers in bold print. (Equality beyond $n = 15$ will be explained in the next section.)

n	0	1	2	3	4	5	6
+0						$\mathbf{\frac{4}{3}}$	$\mathbf{\frac{5}{4}}$
+7	$\mathbf{\frac{5}{4}}$	$\mathbf{\frac{6}{5}}$	$\mathbf{\frac{4}{3}}$	$\mathbf{\frac{4}{3}}$	$\mathbf{\frac{9}{7}}$	$\mathbf{\frac{9}{7}}$	$\mathbf{\frac{9}{7}}$
+14	$\mathbf{\frac{5}{4}}$	$\mathbf{\frac{13}{10}}$	$\frac{13}{10}$	$\frac{13}{10}$	$\frac{14}{11}$	$\frac{14}{11}$	$\frac{14}{11}$
+21	$\mathbf{\frac{5}{4}}$	$\frac{9}{7}$	$\frac{9}{7}$	$\frac{9}{7}$	$\frac{19}{15}$	$\frac{19}{15}$	$\frac{19}{15}$

Table 1 Upper bounds for $t(G(n,2))$

2.2 Lower Bounds

The reason $\frac{5}{4}$ was of such interest in Theorem 1 is that we shall show it is a sharp lower bound in the sense that there is an infinite subclass of $G(n,2)$, namely when $n \equiv 0 \pmod 7$, for which $t(G(n,2)) = \frac{5}{4}$. So in light of the case where $n = 8$ in Table 1, we have the following theorem that we wish to prove.

Theorem 2 *If* $n \geq 5$ *and* $n \neq 8$ *then* $t(G(n,2)) \geq \frac{5}{4}$.

The proof of Theorem 2 will procede by contradiction by supposing a minimal counterexample exists. but first we need to present some preliminaries which will be used in its proof.

Proposition 3 *If* $\frac{S}{\omega} \leq 1$ *and* $a \geq b$, *where* $\omega > b \geq 0$, *then* $\frac{S-a}{\omega - b} \leq \frac{S}{\omega}$.

Lemma 4 *Let* $S \subseteq V(G)$ *with* $t(G) = \frac{|S|}{\omega(G-S)}$.
If $v \in S$ *then* v *is adjacent to at least 2 components of* $G - S$.

Proof: (by contradiction)
Suppose $v \in S$ and v is adjacent to at most 1 component of $G - S$.
Let $T = S - \{v\}$. Then

$$\frac{|T|}{\omega(G-T)} \leq \frac{|S|-1}{\omega(G-S)} < \frac{|S|}{\omega(G-S)} = t(G),$$

which contradicts S being a set that gives $t(G)$. □

Lemma 5 *Let* S *be a subset of* $V(G(n,2))$ *of minimum cardinality such that* $t(G(n,2)) = \frac{|S|}{\omega(G(n,2)-S)} \leq 1$. *Then no vertices of* S *are adjacent in* $G(n,2)$.

Proof: (by contradiction)
Suppose $u, v \in S$ are adjacent in $G(n,2)$. Let $T = S - \{v\}$. By Lemma 4, v must be adjacent to exactly 2 components of $G(n,2) - S$. So $|T| = |S| - 1$ and $\omega(G(n,2) - T) = \omega(G(n,2) - S) - 1$. By Proposition 3,

$$\frac{|T|}{\omega(G(n,2)-T)} \leq \frac{|S|}{\omega(G(n,2)-S)},$$

which contradicts S being a minimum cardinality set that gives $t(G(n,2))$.
□

In the spirit of bootstrapping, we will first prove a weaker version of Theorem 2.

Theorem 6 *If* $n \geq 5$ *then* $t(G(n,2)) > 1$.

Proof:

As summarized in Table 1, we have shown via exhaustive methods that Theorem 6 holds for $5 \leq n \leq 15$. From here we will procede by contradiction. Suppose we have a smallest $m \geq 16$ such that $t(G(m,2)) \leq 1$, and let S be a set of minimum cardinality for which $t(G(m,2)) = \dfrac{|S|}{\omega(G(m,2)-S)}$. We will now look at what this disconnecting set S may look like locally within any given 6-section of $G(m,2)$. We are initially confronted with 2^{12} possible local configurations of the S set. However, with the help of our earlier lemmas we can cut down the cases considerably. By Lemma 5, we can eliminate all cases where the 6-section contains adjacent vertices in S. Note that this eliminates more than half the cases since a 6-section can have at most 5 vertices in S. By Lemma 4, we can eliminate all cases containing configurations appearing in Figure 9, as they contain a vertex of S adjacent to only 1 component of $G-S$.

Figure 9 forbidden by Lemma 4

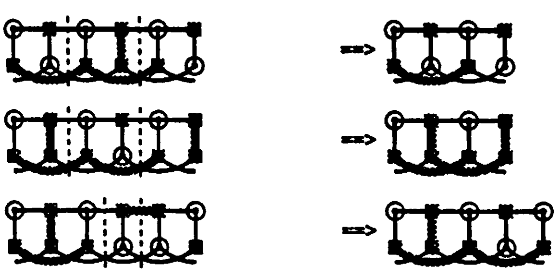
For the remaining cases we will obtain the desired contradiction by finding an $h < m$ such that $t(G(h,2)) \leq 1$. We will accomplish this by removing some r-section from what we know of this $G(m,2)$ and reconnecting our graph to form $G(h,2)$, where $h = m - r$. Our contradiction will then be obtained in the following way. The removal of the r-section and the reconnecting of the graph to form $G(h,2)$ will cause some number $a \geq 0$ of elements of S and some number $b \geq 0$ of components of $G(m,2) - S$ to be lost. If these numbers a and b satisfy the hypothesis of Proposition 3, then we have found a new disconnecting set S' such that

$$t(G(h,2)) \leq \frac{|S'|}{\omega(G(h,2)-S')} = \frac{|S|-a}{\omega(G(m,2)-S)-b} \leq 1,$$

a contradiction. So next we will eliminate all cases containing configurations appearing in Figure 10 as all of these cases will have the same r-section removed with the same results. The r-section to be removed will be the 2-section in the center of each configuration. It is easily verifyable that $\omega(G-S)$ will not be lowered via this extraction, thus making $t(G(h,2)) \leq t(G(m,2))$ for our desired contradiction in these cases.

Figure 10 simplest extractions

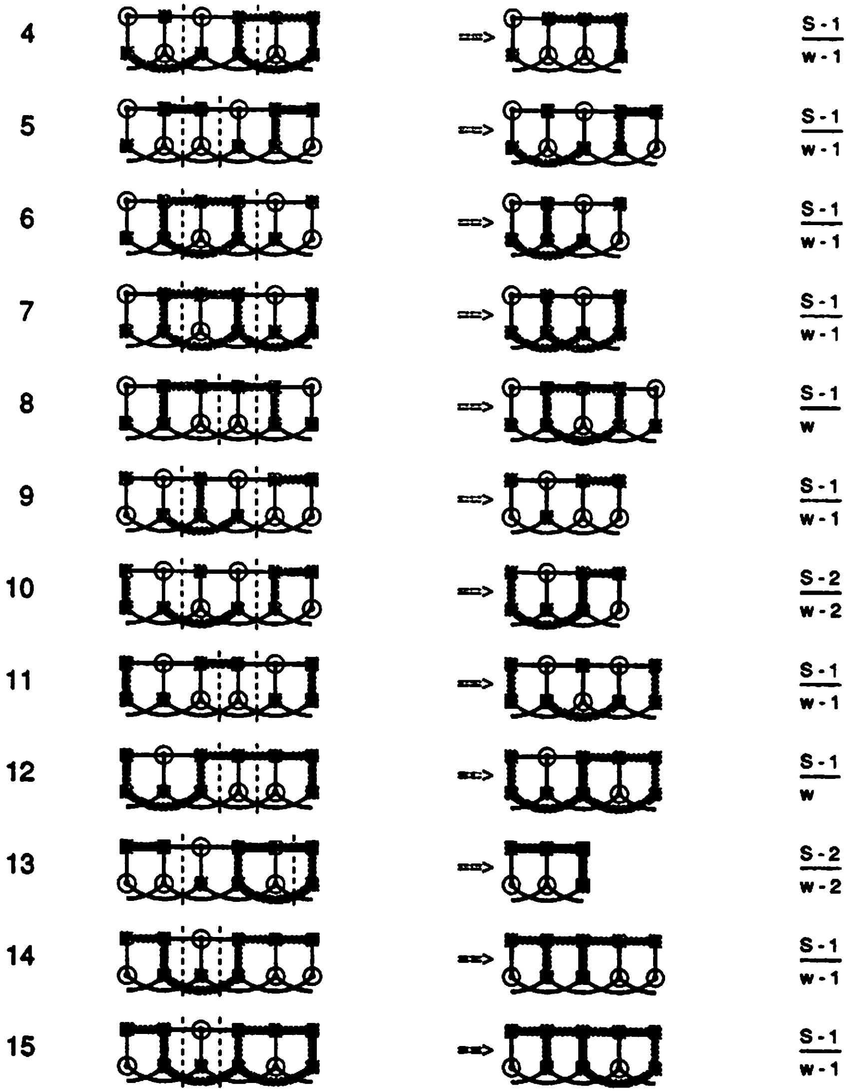
We accomplish the elimination of cases by Lemma 5, Lemma 4, and simple extractions via computer and are left with 15 remaining cases (excluding mirror images) which we will attack by hand. For these cases, as mentioned before, we will have to find an appropriate r-section to remove from $G(m,2)$. Then counting the number a of elements of S and the number b of components of $G(m,2) - S$ lost in this process, we will obtain our desired contradiction using Proposition 3. Counting a is very simple, and counting b is for the most part straightforward. Contributions to b may be made either by the loss of components that are completely isolated within the r-section removed or by the gluing together of two previously separate components via the process of reconnection to form $G(h,2)$. However, in counting b we must always take into account all possible cases of how $G(m,2)$ might be affected by the removal of the r-section. Since we are looking only at a subsection of $G(m,2)$, we can not be completely sure of what components look like if they run off the ends. The possibility exists that a component running off one end may be the same as a component running off the other end, though it is not attached within our field of view. Also a single component running off both ends may again meet itself outside of our field of view. Whenever the first of these two possibilities exists, we must assume that it does not take place, and whenever the second of these two possibilities exists we must assume that it does take place. By making these assumptions we will be considering the situtations that are worst for us in trying to find an r-section to remove for our contradiction. With this in mind we now present the 15 remaining cases pictorially. All cases will have pictures of the 6-section both before and after the removal of the r-section which will be denoted between the dotted lines. The fraction at the end of each case will represent how the counts a and b are to be used in Proposition 3 to obtain a contradiction.



1 $\frac{S-1}{W-1}$

2 $\frac{S-2}{W-2}$

3 $\frac{S-1}{W-1}$



Thus establishing Theorem 6. □

Proposition 7 *If* $\frac{S}{w} < \frac{5}{4}$ *and* $4a \geq 5b$*, where* $\omega > b \geq 0$*, then* $\frac{S-a}{w-b} < \frac{5}{4}$.

Lemma 8 *Let* $S \subseteq V(G)$ *with* $t(G) = \frac{|S|}{\omega(G-S)}$, $H \in c(G-S)$, *and* $T \subseteq V(H)$. *If* $\omega(H-T) > 1$ *then* $\frac{|T|}{\omega(H-T)-1} \geq t(G)$.

Proof: (by contradiction)
Suppose $\omega(H-T) > 1$ and $\frac{|T|}{\omega(H-T)-1} < t(G) = \frac{|S|}{\omega(G-S)}$. Then we can construct a new disconnecting set $S \cup T$ such that

$$\frac{|S \cup T|}{\omega(G-(S \cup T))} = \frac{|S|+|T|}{\omega(G-S)+\omega(H-T)-1} < \frac{|S|}{\omega(G-S)} = t(G),$$

which contradicts S being a set that gives $t(G)$. □

Corollary 9 *Let* $S \subseteq V(G)$ *with* $t(G) = \frac{|S|}{\omega(G-S)} > 1$.
If $H \in c(G-S)$ *then* H *is a block.*

Proof: (by contradiction)
Suppose $H \in c(G-S)$ and H contains a cutpoint v. Let $T = \{v\}$. Then

$$\frac{|T|}{\omega(H-T)-1} \leq \frac{1}{2-1} = 1 < t(G),$$

which contradicts Lemma 8. □

We now return to the theorem of main interest which we will prove in a manner similar to Theorem 6.

Theorem 2 If $n \geq 5$ and $n \neq 8$ then $t(G(n,2)) \geq \frac{5}{4}$.

Proof:
We have already shown that Theorem 2 holds for $5 \leq n \leq 15$. Now suppose we have a smallest $m \geq 16$ such that $t(G(m,2)) < \frac{5}{4}$, and let S be a set such that $t(G(m,2)) = \frac{|S|}{\omega(G-S)}$. Again we will look at all possible 6-sections of $G(m,2)$ and are initially confronted with 2^{12} possible local configurations of the S set. We can now use all of our preliminary work to eliminate most of the cases. By Lemma 4, we can eliminate all cases containing configurations appearing in Figure 11 (and their mirror images), as

they contain a vertex of S adjacent to at most 1 component of $G-S$.

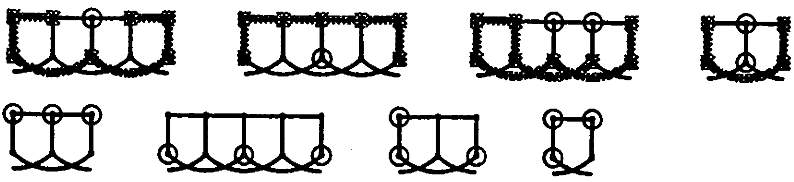


Figure 11   forbidden by Lemma 4

Now that we have established Theorem 6, by Corollary 9 we can eliminate all cases containing configurations appearing in Figure 12 (and their mirror images), as they contain a cut-vertex.

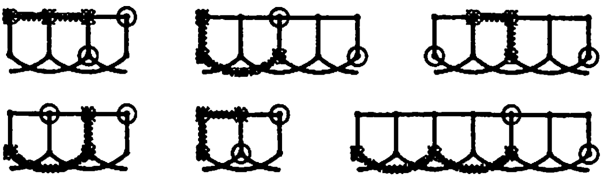
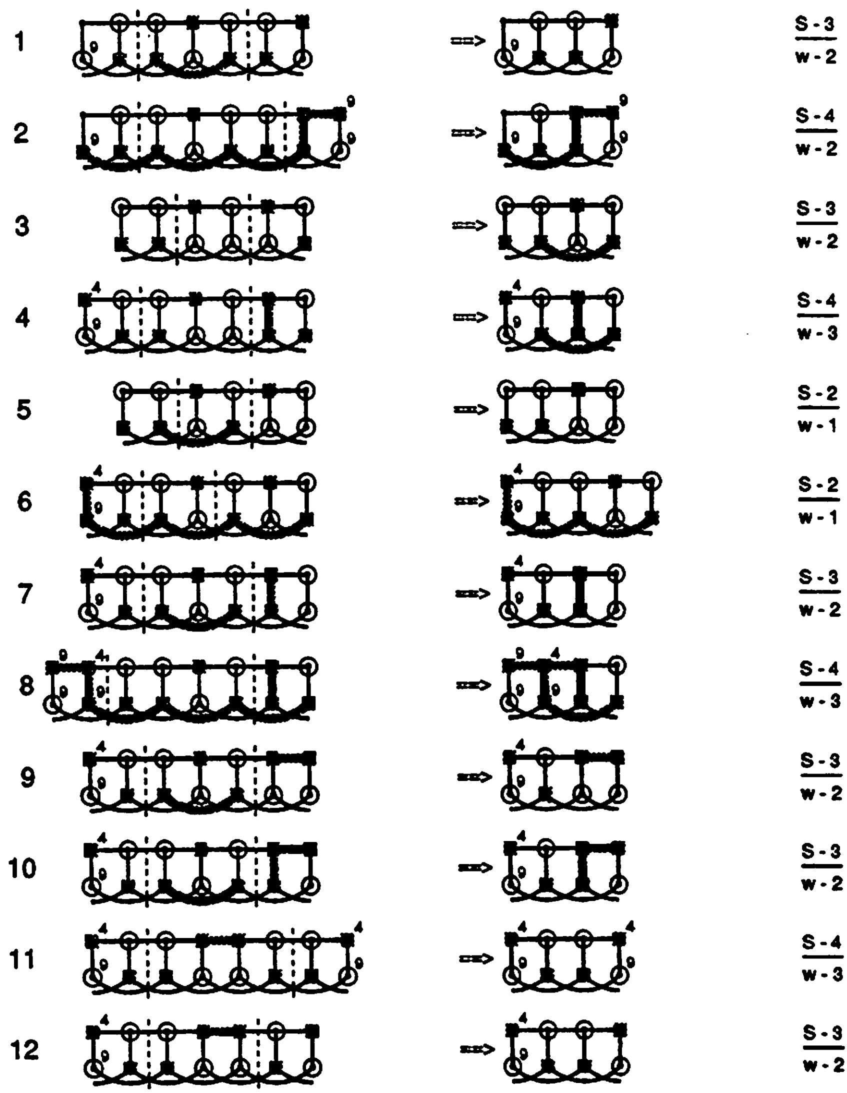
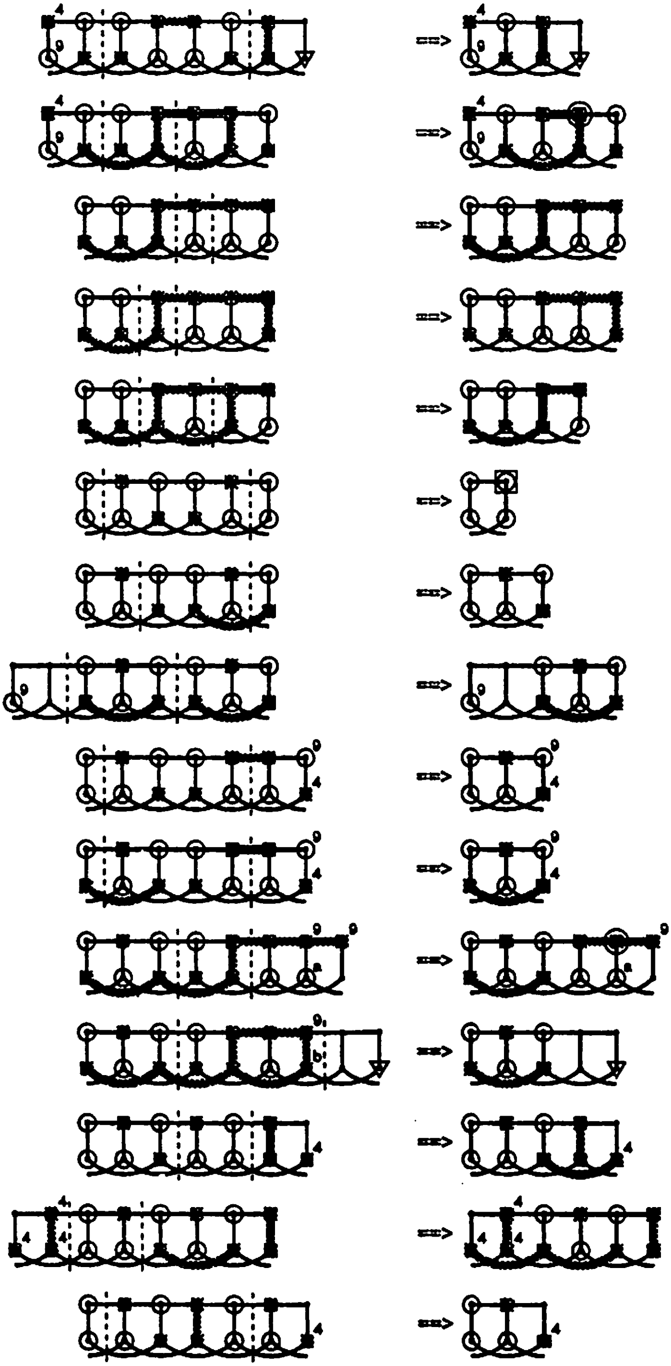


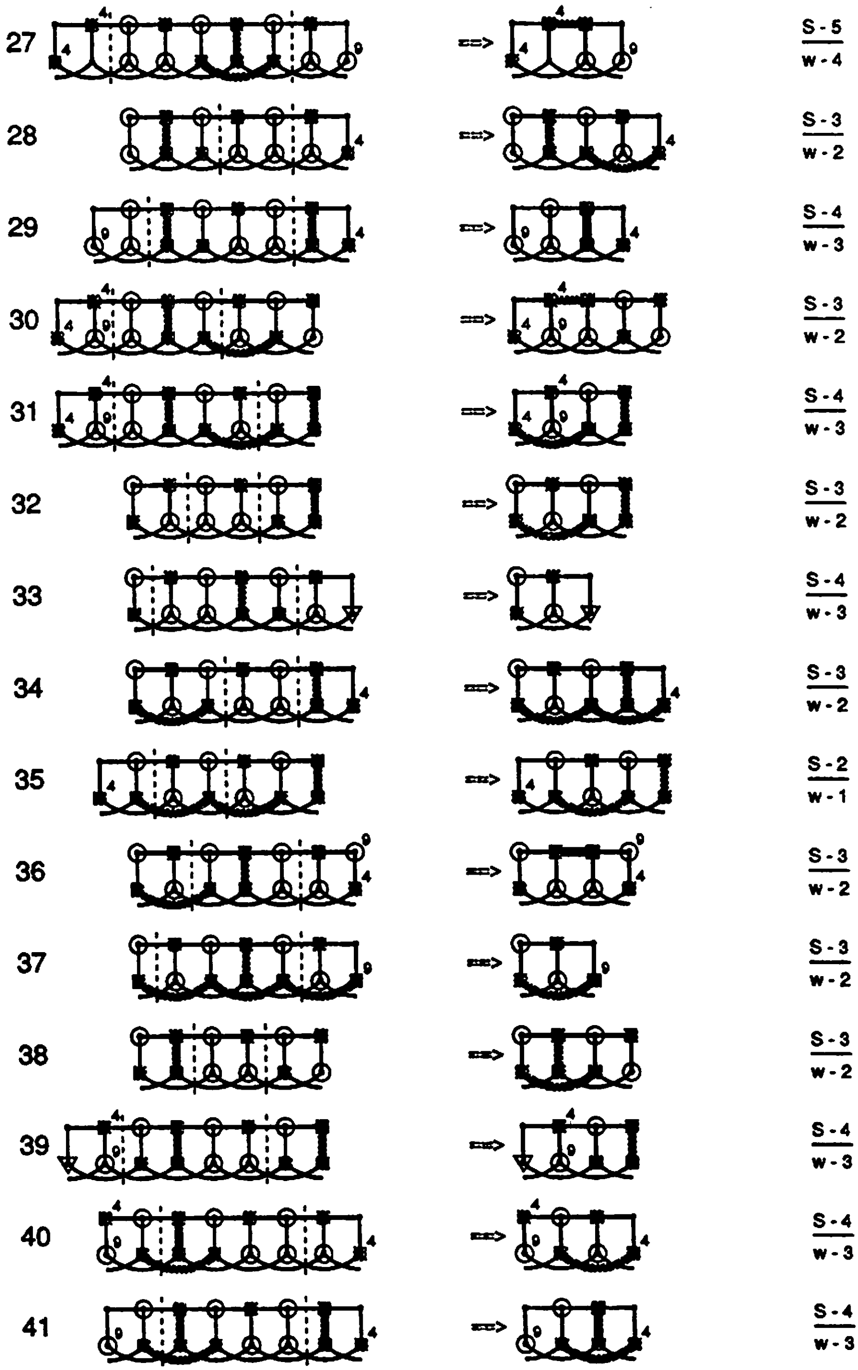
Figure 12   forbidden by Theorem 6 and Corollary 9

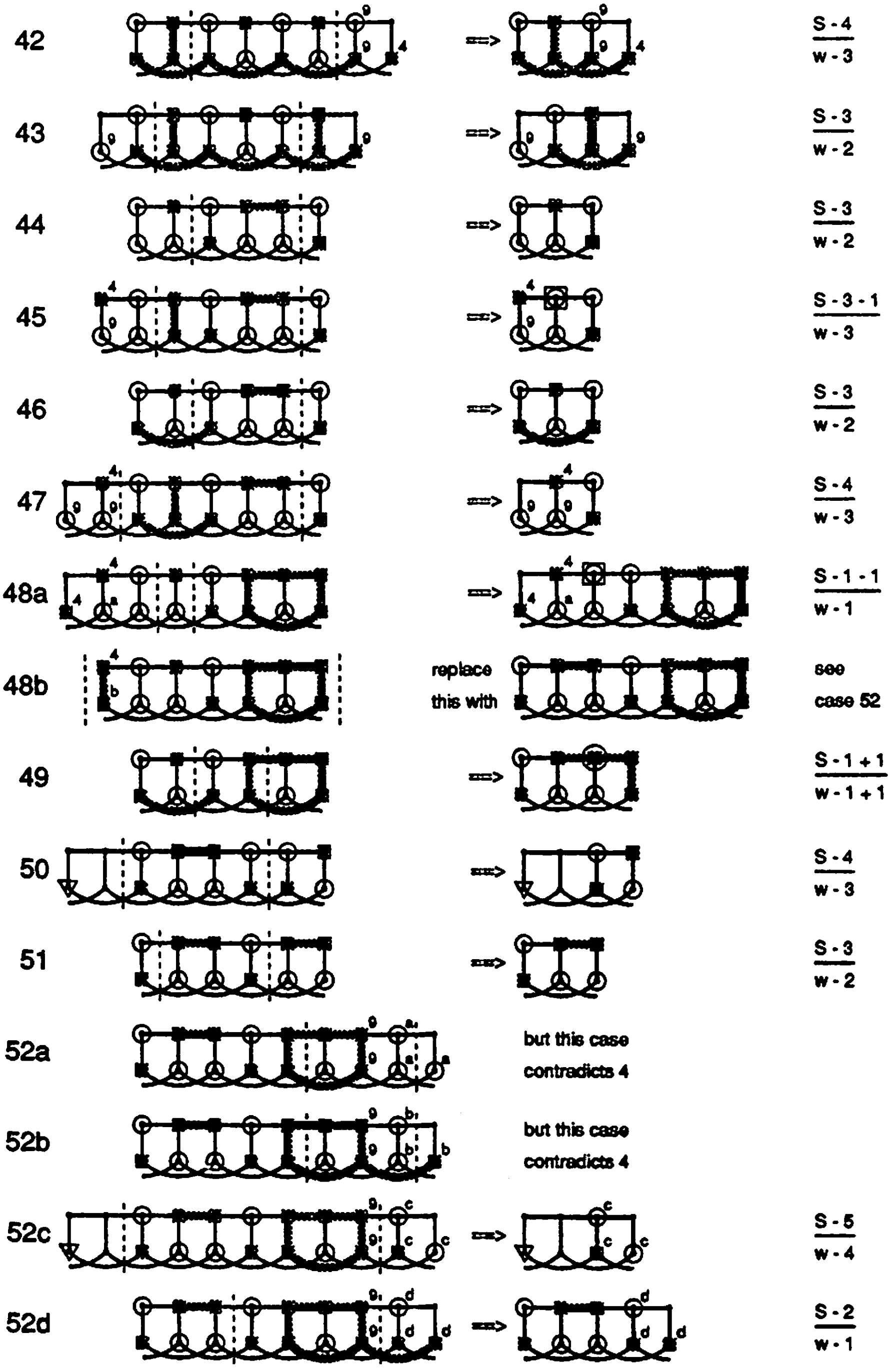
As in the proof of Theorem 6, we are looking for an r-section to remove from the remaining cases. However, we will now be using Proposition 7 instead of Propostion 3 to obtain our contradiction. But we can again eliminate all cases containing configurations appearing in Figure 10, the simplest extractions. We accomplish the elimination of cases by Lemma 4, Corollary 9, and simple extractions via computer and are left with 86 remaining cases (excluding mirror images) which we will attack by hand. These cases will be presented pictorially much the same as in the proof of Theorem 6. In addition, Lemma 4 and Corollary 9 will often determine the state of certain points beyond the 6-section. When this is the case, the points in question will have superscripts containing the number of the corollary that tells us their state. Note that in cases with more than one of these additional points, there is sometimes a specific order that the states of these points become known to us. Also some use of subcases is required and will be handled in two different forms. First, any lettered subcases will have the points relative to the subcase appropriately marked. Second, any unlettered cases containing a triangle around one of the points will denote that we must consider both the case when that point is in S and the case when that point is not in S, but the fraction at the end will be the same for both cases (though for different reasons). Also in some cases we will have to make some additional adjustments to $G(h,2)$ after the r-section is removed and the graph is reconnected. Points to be changed in the after picture of the corresponding cases will be additionally marked with either a bigger circle around any point to be added to S or a bigger box around any point to be removed from S. Thus in these cases, the fraction at the end will also be changed to account for these adjustments. Any other variations in the representation of a case will be dealt with in an accompanying description as they are encountered. So we present the final 86 cases.

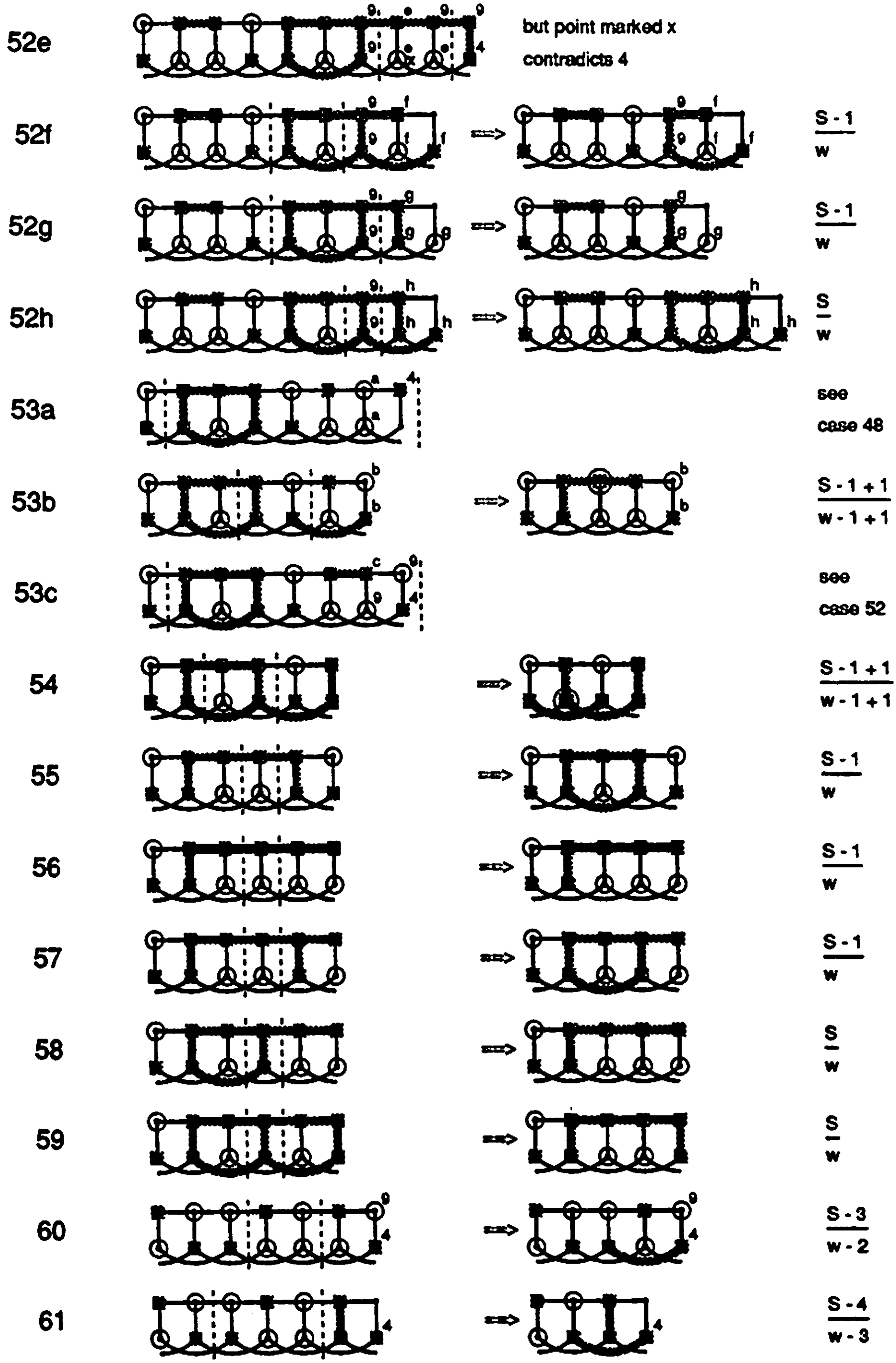


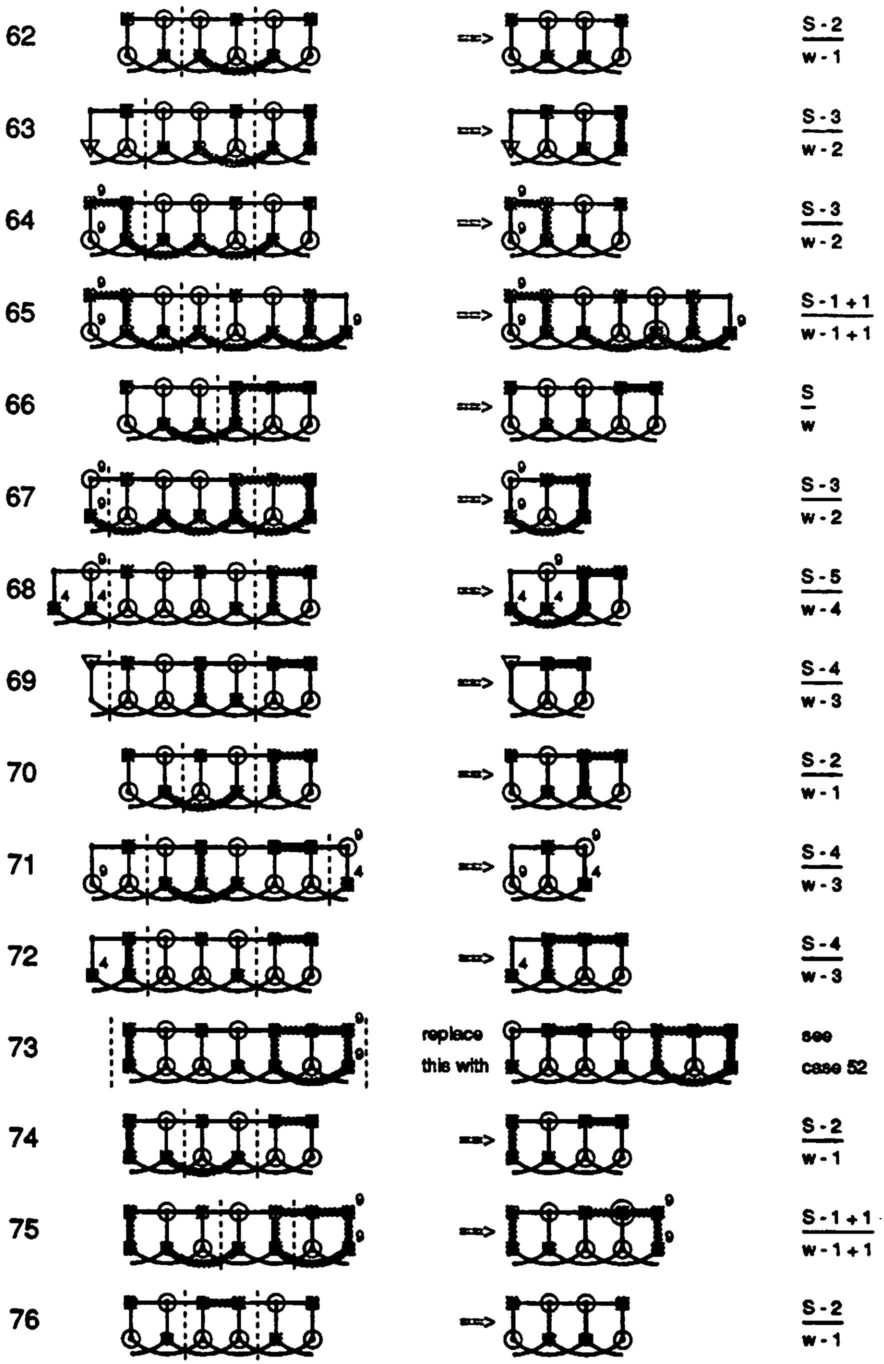


13	$\frac{S-4}{w-3}$
14	$\frac{S-1+1}{w-1+1}$
15	$\frac{S-1}{w}$
16	$\frac{S}{w}$
17	$\frac{S-1}{w}$
18	$\frac{S-4-1}{w-4}$
19	$\frac{S-3}{w-2}$
20	$\frac{S-3}{w-2}$
21	$\frac{S-4}{w-3}$
22	$\frac{S-4}{w-3}$
23a	$\frac{S-1+1}{w-1+1}$
23b	$\frac{S-2}{w-1}$
24	$\frac{S-3}{w-2}$
25	$\frac{S-3}{w-2}$
26	$\frac{S-4}{w-3}$

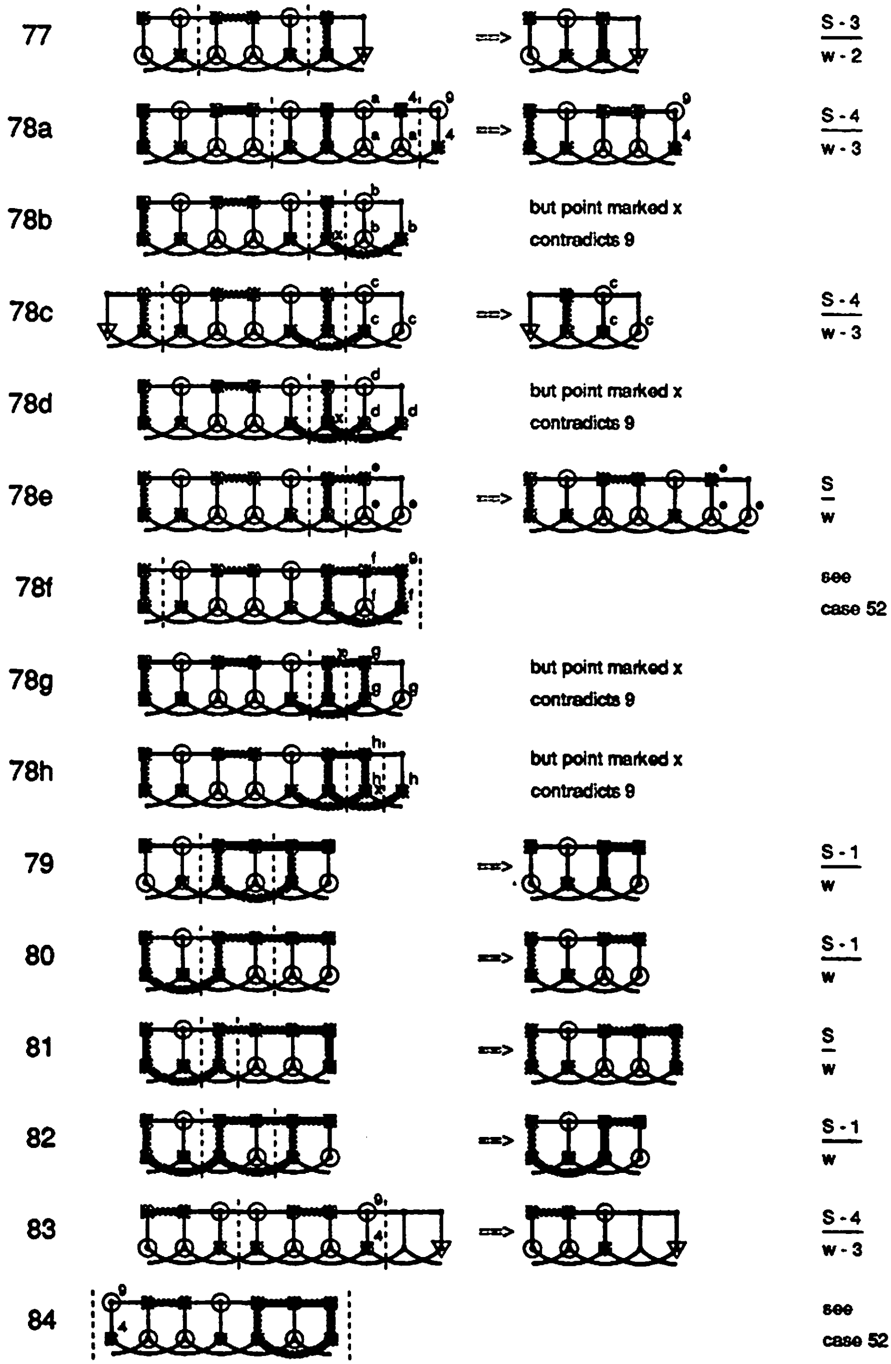

27 ==> S-5/w-4
28 ==> S-3/w-2
29 ==> S-4/w-3
30 ==> S-3/w-2
31 ==> S-4/w-3
32 ==> S-3/w-2
33 ==> S-4/w-3
34 ==> S-3/w-2
35 ==> S-2/w-1
36 ==> S-3/w-2
37 ==> S-3/w-2
38 ==> S-3/w-2
39 ==> S-4/w-3
40 ==> S-4/w-3
41 ==> S-4/w-3


42
S - 4 / w - 3
43
S - 3 / w - 2
44
S - 3 / w - 2
45
S - 3 - 1 / w - 3
46
S - 3 / w - 2
47
S - 4 / w - 3
48a
S - 1 - 1 / w - 1
48b
replace this with
see case 52
49
S - 1 + 1 / w - 1 + 1
50
S - 4 / w - 3
51
S - 3 / w - 2
52a
but this case contradicts 4
52b
but this case contradicts 4
52c
S - 5 / w - 4
52d
S - 2 / w - 1


52e
but point marked x
contradicts 4
52f
S - 1 / w
52g
S - 1 / w
52h
S / w
53a
see
case 48
53b
S - 1 + 1 / w - 1 + 1
53c
see
case 52
54
S - 1 + 1 / w - 1 + 1
55
S - 1 / w
56
S - 1 / w
57
S - 1 / w
58
S / w
59
S / w
60
S - 3 / w - 2
61
S - 4 / w - 3



62 ==> S - 2 / w - 1
63 ==> S - 3 / w - 2
64 ==> S - 3 / w - 2
65 ==> S - 1 + 1 / w - 1 + 1
66 ==> S / w
67 ==> S - 3 / w - 2
68 ==> S - 5 / w - 4
69 ==> S - 4 / w - 3
70 ==> S - 2 / w - 1
71 ==> S - 4 / w - 3
72 ==> S - 4 / w - 3
73 replace this with see case 52
74 ==> S - 2 / w - 1
75 ==> S - 1 + 1 / w - 1 + 1
76 ==> S - 2 / w - 1


77 ==> S-3 / w-2
78a ==> S-4 / w-3
78b but point marked x contradicts 9
78c ==> S-4 / w-3
78d but point marked x contradicts 9
78e ==> S / w
78f see case 52
78g but point marked x contradicts 9
78h but point marked x contradicts 9
79 ==> S-1 / w
80 ==> S-1 / w
81 ==> S / w
82 ==> S-1 / w
83 ==> S-4 / w-3
84 see case 52

85  $\frac{S}{W}$

86  $\frac{S-2}{W-1}$

Thus establishing Theorem 2. □

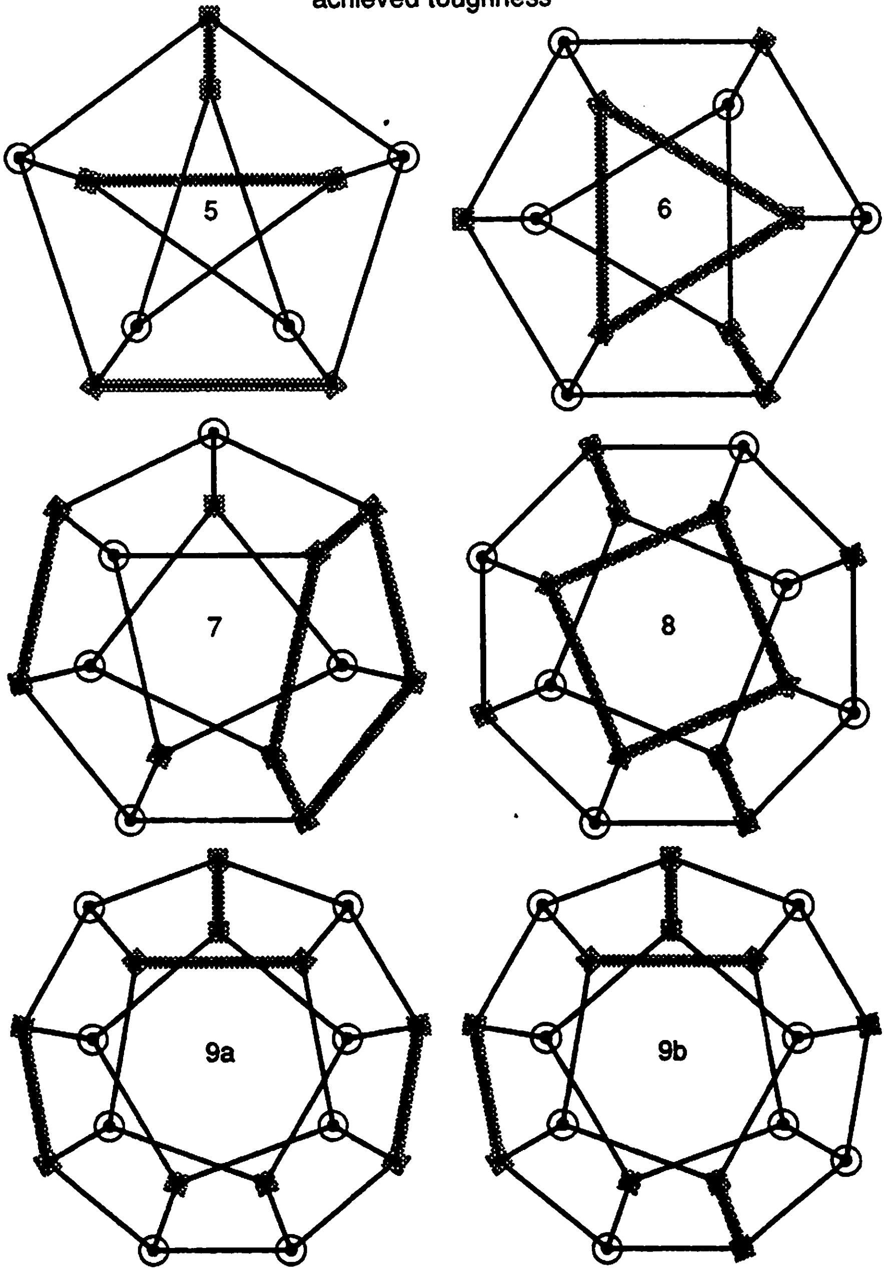
3 Conclusions

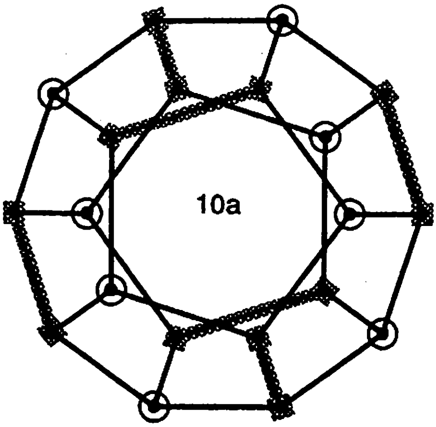
From Theorem 1 we note that conjecture α fails for $n \geq 13$ and that $t(G(n,2)) < \frac{4}{3}$ for all $n \geq 11$. However, combining the results of Theorem 1 and Theorem 2, we do have an infinite class of graphs for which toughness equals $\frac{5}{4}$.

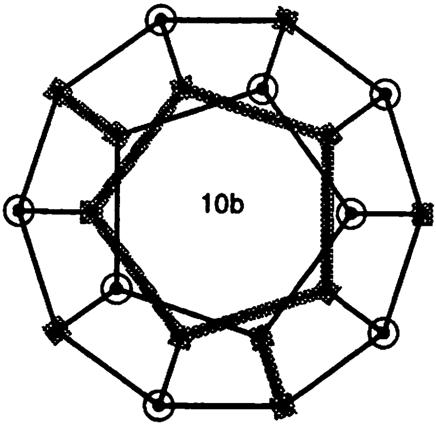
Corollary 10 *If $n \equiv 0 \ (mod\ 7)$ and $n \geq 7$ then $t(G(n,2)) = \frac{5}{4}$.*

Thus we have established equality for the first instance of n modulo 7 in Theorem 1. Also notice that all of the upper bounds in Theorem 1 converge to our lower bound of $\frac{5}{4}$ and suggest that our construction may be optimal. We in fact conjecture equality for all of the inequalities in Theorem 1. We have already established the base cases of this conjecture by the results in Appendix A. We also believe that knowing all of the different ways that the toughness can be achieved in these base cases may prove helpful in establishing our conjecture, and so they are presented for your inspiration.

APPENDIX A
achieved toughness




10a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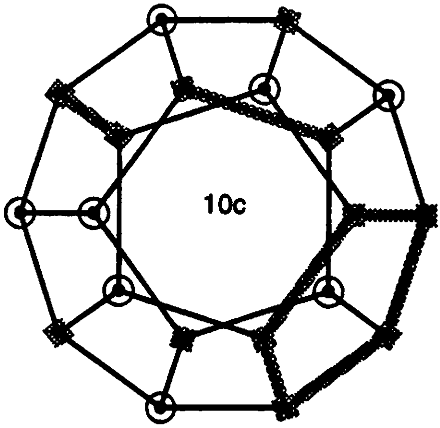
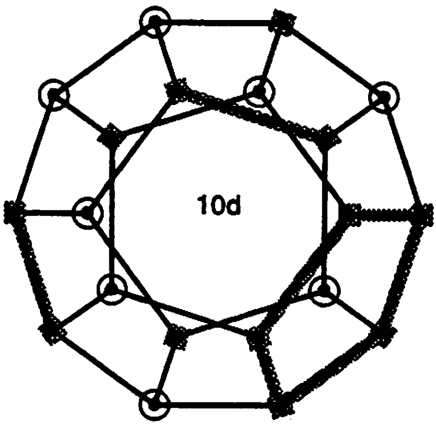
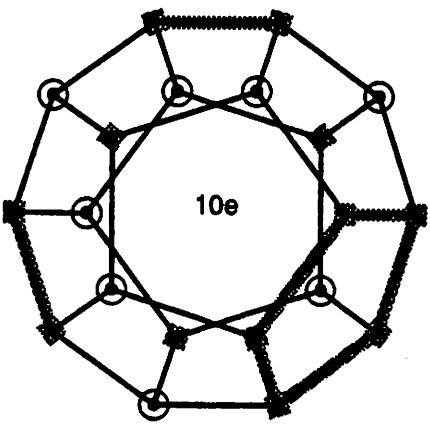

10c


10d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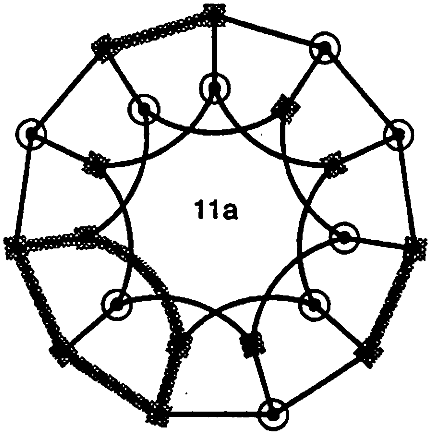
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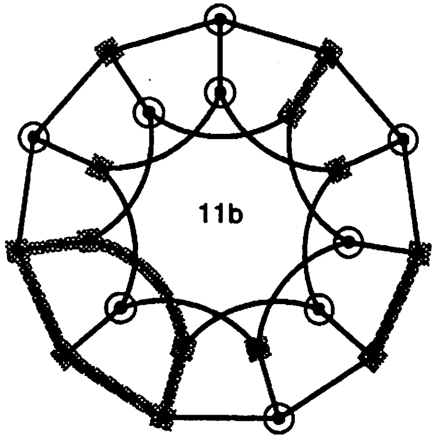
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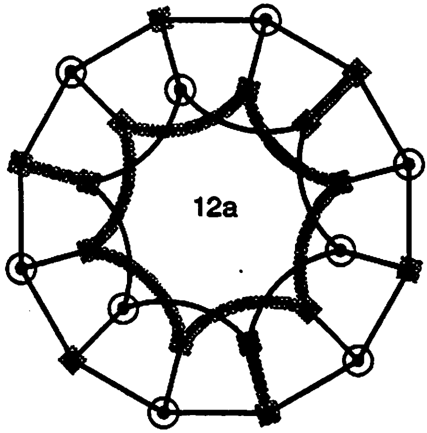
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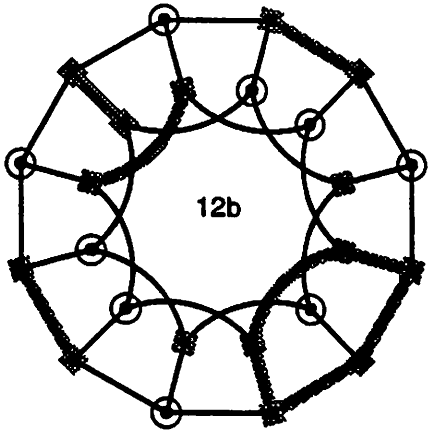
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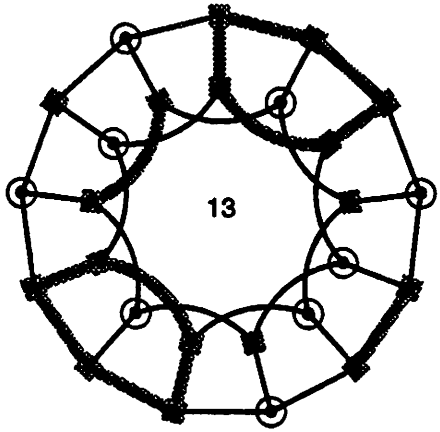
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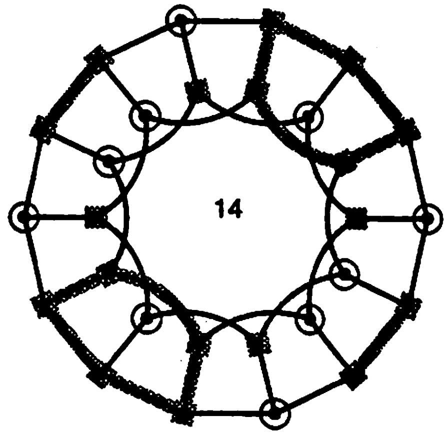
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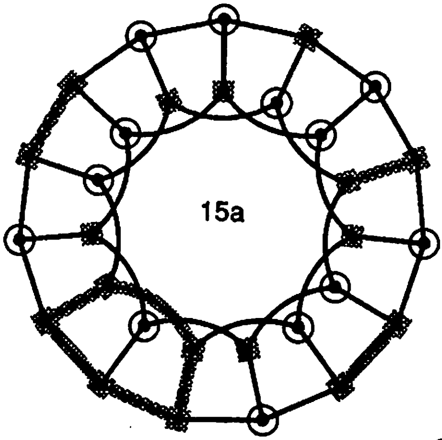
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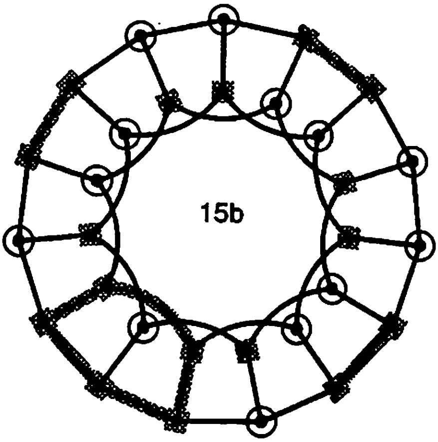
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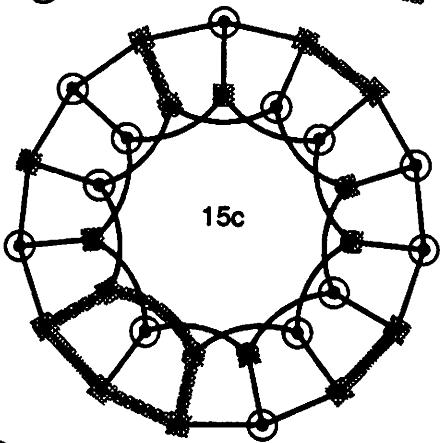
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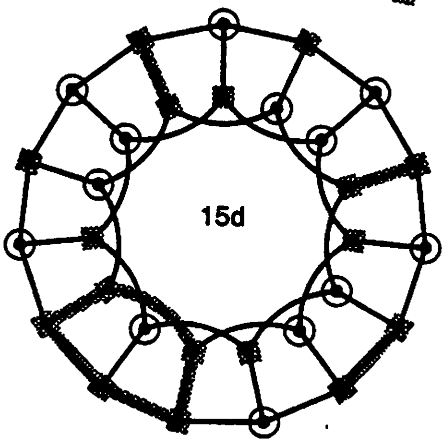
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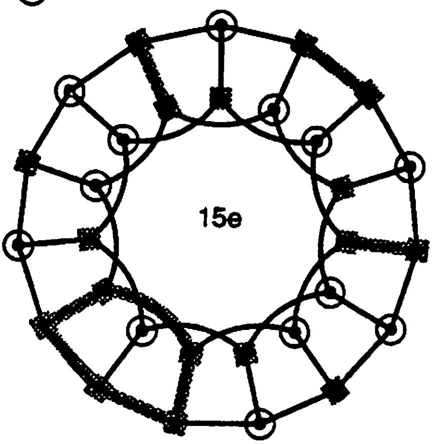
15e

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