On Tournaments with a Prescribed Property

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Abstract. A round robin tournament on q players in which draws are not permitted is said to have property $P(n,k)$ if each player in any subset of n players is defeated by at least k other players. We consider the problem of determining the minimum value $f(n,k)$ such that every tournament of order $q \geq f(n,k)$ has property $P(n,k)$. The case $k = 1$ has been studied by Erdös, G. and E. Szekeres, Graham and Spencer, and Bollobás. In this paper we present a lower bound on $f(n,k)$ for the case of Paley tournaments.

1. Introduction

For our purposes graphs are finite and directed. Consider a round robin tournament T_q on q players $1, 2, \ldots, q$ in which there are no draws. It is very well known that such a tournament can be represented by a directed graph in which the vertices represent the players. If Player i defeats Player j then the graph contains the arc (i,j), and we say that vertex i *dominates* vertex j. Further, we say a set of vertices A dominates a set of vertices B if every vertex of A dominates every vertex of B. For convenience we refer to the graph of the tournament as T_q.

A tournament T_q is said to have property $P(n,k)$ if every subset of n vertices of T_q is dominated by at least k other vertices. An interesting problem is that of determining the smallest integer $f(n,k)$ such that T_q has property $P(n,k)$ whenever $q \geq f(n,k)$. This problem was posed to Erdös in 1962 by Schütte [3] for the particular case $k = 1$.

Using the probabilistic method, Erdös [3] proved that for sufficiently large n

$$2^{n+1} - 1 \leq f(n,1) \leq n^2 2^n(\log 2 + \epsilon)$$

for any $\epsilon > 0$. Szekeres and Szekeres [6] improved the lower bound to

$$f(n,1) \geq (n+2)2^{n-1} - 1$$

Graham and Spencer [4] defined the following class of tournaments. Let $p \equiv 3($ mod $4)$ be a prime. The directed graph D_p is defined as follows. The vertices of D_p are $\{0, 1, \ldots, p-1\}$ and D_p contains the arc (i,j) if and only if $i - j$ is a quadratic residue modulo p. The graph D_p is sometimes referred to as the *Paley tournament*. Graham and Spencer [4] proved, using results from number theory, that D_p has property $P(n, 1)$ whenever $p > n^2 2^{2n-2}$. Further, they observed that D_7 and D_{19} are the smallest Paley tournaments having property $P(2, 1)$ and $P(3, 1)$ respectively. They noted that D_{67} may be the smallest Paley tournament having property $P(4, 1)$. This is indeed the case and is a consequence of our work.

Bollobás [2] extended the results of Graham and Spencer to prime powers. More specifically, if $q \equiv 3(\mod 4)$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbf{F}_q$. Vertex a is joined to vertex b by an arc if and only if $a - b$ is a quadratic residue in $\mathbf{F}_q$. Bollobás noted that D_q has property $P(n, 1)$ whenever

$$q > \{(n-2)2^{n-1} + 1\}\sqrt{q} + n2^{n-1}.$$

In Section 3, we improve this bound to

$$q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + 2^n - 1.$$

In addition, we establish a lower bound on q so that D_q has property $P(n, k)$.

In the next section we present some preliminary results on finite fields which we make use of in the proofs of our main theorems.

2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbf{F}_q$ be a finite field of order q, where q is a prime power.

A *character* χ on $\mathbf{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbf{F}_q$, is a map from $\mathbf{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all x and with

$$\chi(xy) = \chi(x)\chi(y)$$

for any $x, y \in \mathbf{F}_q^*$. Since $\chi(1) = \chi(1)\chi(1)$ we have $\chi(1) = 1$.

Among the characters of $\mathbf{F}_q^*$ we have the *principal character* χ_0 defined by $\chi_0(x) = 1$ for all $x \in \mathbf{F}_q^*$; all other characters of $\mathbf{F}_q^*$ are called non-principal. A character χ is of *order* d if $\chi^d = \chi_0$ and d is the smallest positive integer with this property.

It will be convenient to extend the definition of non-principal character χ to the whole $\mathbf{F}_q$ by putting $\chi(0) = 0$.

The following lemma, due to Schmidt [5], is very useful in our work.

Lemma 2.1. *Let χ be a non-principal character on $\mathbf{F}_q$ of order $d > 1$. If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbf{F}_q$, then*

$$\left|\sum_{x \in \mathbf{F}_q} \chi\{(x - a_1)(x - a_2)\ldots(x - a_s)\}\right| \leq (s-1)\sqrt{q}.$$

Let q be a power of an odd prime. We define a *quadratic (residue) character* η on $\mathbf{F}_q$ by

$$\eta(a) = a^{\frac{q-1}{2}}, \quad \text{for all } a \in \mathbf{F}_q$$

Equivalently, η is 1 on squares, 0 at 0, and -1 otherwise. Therefore η is a non-principal character of order 2.

The following two lemmas are proved in [1].

Lemma 2.2. *Let η be a quadratic character on $\mathbf{F}_q$. If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbf{F}_q$ and s is even, then*

$$\sum_{x \in \mathbf{F}_q} \eta\{(x - a_1)(x - a_2) \ldots (x - a_s)\}$$
$$= -1 \pm \sum_{x \in \mathbf{F}_q} \eta\{(x + b_1)(x + b_2) \ldots (x + b_{s-1})\}$$

for some distinct elements $b_1, b_2, \ldots, b_{s-1}$ of $\mathbf{F}_q$. ∎

Lemma 2.3. *Let η be a quadratic character on $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$. Put*

$$g = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \eta(x - a)\} \prod_{b \in B} \{1 - \eta(x - b)\}.$$

As usual, an empty product is defined to be 1. Then

(a) $g \geq q - \{(t-3)2^{t-1} + 2\}\sqrt{q} - \{2^{t-1} - 1\}$ *where* $t = |A \cup B|$,

(b) $g \geq q - \{(2n-3)2^{2n-1} + 2\}\sqrt{q} - \{2^{2n-1} - 2n^2 - 1\}$

where $n = |A| = |B|$. ∎

We conclude this section by noting that if a and b are vertices of D_q, $q \equiv 3($ mod 4) a prime power, then

$$\eta(a - b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ -1, & \text{otherwise.} \end{cases}$$

Further, $\eta(-a) = -\eta(a)$ for any $a \in \mathbf{F}_q$.

3. Results

Our first result concerns Paley tournaments having property $P(n, k)$.

Theorem 3.1. *Let $q \equiv 3(\mod 4)$ be a prime power and k a positive integer. If*

$$q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + k2^n - 1, \tag{3.1}$$

then D_q has property $P(n,k)$.

Proof: Let A be any subset of n vertices of D_q. Then there are at least k other vertices each of which dominates A if and only if

$$h = \sum_{\substack{x \in F_q \\ x \notin A}} \prod_{a \in A} \{1 + \eta(x-a)\} > (k-1)2^n.$$

Let

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \eta(x-a)\}.$$

By Lemma 2.3(a) with B empty, we have

$$g \geq q - \{(n-3)2^{n-1} + 2\}\sqrt{q} - \{2^{n-1} - 1\}$$

Now

$$g - h = \sum_{x \in A} \prod_{i=1}^{n} \{1 + \eta(x - a_i)\}$$

where $A = \{a_1, a_2, \ldots, a_n\}$. If $g - h \neq 0$, then for some a_k the product

$$\prod_{i=1}^{n} \{1 + \eta(a_k - a_i)\} \neq 0. \tag{3.2}$$

For (3.2) to hold we must have $\eta(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k$, $\eta(a_k - a_i) = 1$. Hence a_k dominates all other vertices in A. Therefore a_k is unique and $g - h = 2^{n-1}$. Then, since $g - h$ could be 0 we conclude that

$$g - h \leq 2^{n-1}.$$

So

$$\begin{aligned} h &\geq g - 2^{n-1} \\ &\geq q - \{(n-3)2^{n-1} + 2\}\sqrt{q} - \{2^n - 1\}. \end{aligned}$$

Now if inequality (3.1) holds, then $h > (k-1)2^n$ as required. Since A is arbitrary this completes the proof. ∎

Some immediate corollaries of Theorem 3.1 are the following.

Corollary 1. *If $q = 4t + 3$ is a prime power, then D_q has property $P(2,k)$ for every $t \geq k$.* ∎

Corollary 2. *If $q \equiv 3(\mod 4)$ is a prime power and $q > (1 + 2\sqrt{2k})^2$, then D_q has property $P(3,k)$.* ∎

Corollary 3. *If $q \equiv 3(\mod 4)$ is a prime power and $q > (5 + 2\sqrt{4k+6})^2$, then D_q has property $P(4,k)$.* ∎

Corollary 4. *If $q \equiv 3(\mod 4)$ is a prime power, $n \geq 5$ and $q > ((n-3)2^{n-1} + 3)^2$, then D_q has property $P(n, 1)$.* ∎

<u>Remark 1</u> We have verified, using a computer, that D_7, D_{19}, and D_{67} are the smallest Paley tournaments having property $P(2,1)$, $P(3,1)$, and $P(4,1)$ respectively. Thus the bounds in Corollaries 1 and 2 are the best possible. Further, our computer analysis revealed that D_{103} does not have property $P(4,1)$ whilst D_{107} and D_{127} do and thus the bound of 131 given in Corollary 3 is fairly close to best possible.

<u>Remark 2</u> For $n = 3$ and any q there is always a set A for which $g - h = 4$. Expanding the g in the proof of Theorem 3.1 we get

$$\begin{aligned}
g &= \sum_{x \in \mathbf{F}_q} \prod_{i=1}^{3} \{1 + \eta(x - a_i)\} \\
&= \sum_{x \in \mathbf{F}_q} 1 + \sum_{x \in \mathbf{F}_q} \{\eta(x - a_1) + \eta(x - a_2) + \eta(x - a_3)\} \\
&\quad + \sum_{x \in \mathbf{F}_q} \{\eta((x - a_1)(x - a_2)) + \eta((x - a_1)(x - a_3)) \\
&\quad + \eta((x - a_2)(x - a_3))\} + \sum_{x \in \mathbf{F}_q} \eta((x - a_1)(x - a_2)(x - a_3)) \\
&= q - 3 + \sum_{x \in \mathbf{F}_q} \eta((x - a_1)(x - a_2)(x - a_3)).
\end{aligned}$$

(by Lemma 2.1 and 2.2)

Thus

$$\begin{aligned}
|g - q + 3| &= \left| \sum_{x \in \mathbf{F}_q} \eta((x - a_1)(x - a_2)(x - a_3)) \right| \\
&\leq 2\sqrt{q} \qquad \text{(by Lemma 2.1)}
\end{aligned}$$

Hence $g \leq q + 2\sqrt{q} - 3$. Consequently $h < 8k$ for $q < (-1 + 2\sqrt{2(k+1)})^2$. Thus D_q does not have property $P(3,k)$ for $q < (-1 + 2\sqrt{2(k+1)})^2$. We suspect that this is true for all $q \leq (1 + 2\sqrt{2k})^2$.

We can extend the property $P(n,k)$ as follows. We say a tournament T_q of order q has property $P(m,n,k)$ if for any set of $m+n$ distinct vertices of T_q there exists at least k other vertices each of which dominates the first m vertices and is dominated by each of the latter n vertices. We have the following result.

Theorem 3.2. *Let $q \equiv 3(\mod 4)$ be a prime power and k a positive integer. If*

$$q > \{(t-3)2^{t-1}+2\}\sqrt{q}+(t+2k-1)2^{t-1}-1, \tag{3.3}$$

then D_q has property $P(m,n,k)$, where $t=m+n$.

Proof: Let A and B be disjoint subsets of vertices of D_q with $|A|=m$ and $|B|=n$. Then there are at least k other vertices, each of which dominates every vertex of A but is dominated by every vertex of B if any only if

$$h=\sum_{\substack{x\in F_q\\ x\notin A\cup B}}\prod_{a\in A}\{1+\eta(x-a)\}\prod_{b\in B}\{1-\eta(x-b)\}>(k-1)2^t.$$

Let

$$g=\sum_{x\in \mathbf{F}_q}\prod_{a\in A}\{1+\eta(x-a)\}\prod_{b\in B}\{1-\eta(x-b)\}.$$

Using Lemma 2.3(a) we have

$$g\geq q-\{(t-3)^{t-1}+2\}\sqrt{q}-\{2^{t-1}-1\}.$$

Then

$$\begin{aligned}g-h&=\sum_{x\in A\cup B}\prod_{a\in A}\{1+\eta(x-a)\}\prod_{b\in B}\{1-\eta(x-b)\}\\&\leq t2^{t-1},\end{aligned}$$

since, in each product, each factor is at most 2 and one factor is 1, so each of these terms is at most 2^{t-1}. Therefore

$$\begin{aligned}h&\geq g-t2^{t-1}\\&\geq q-\{(t-3)2^{t-1}+2\}\sqrt{q}-\{(t+1)2^{t-1}-1\}.\end{aligned}$$

Now if inequality (3.3) holds, then $h>(k-1)2^t$ as required. Since A and B are arbitrary this completes the proof. ∎

For $m=n$ we have the following sharper result.

Theorem 3.3. *Let $q \equiv 3(\mod 4)$ be a prime power and k a positive integer. If*

$$q > \{(2n-3)2^{2n-1}+2\}\sqrt{q}+(n+2k)2^{2n-1}-2n^2-1, \tag{3.4}$$

then D_q has property $P(n,n,k)$.

Proof: Let A and B be disjoint subsets of vertices of D_q with $|A| = |B| = n$. Then there are at least k other vertices each of which dominates A and is dominated by B if any only if

$$h = \sum_{\substack{x\in F_q \\ x\notin A\cup B}} \prod_{a\in A}\{1+\eta(x-a)\}\prod_{b\in B}\{1-\eta(x-b)\} > (k-1)2^{2n}.$$

Let

$$g = \sum_{x\in \mathbf{F}_q}\prod_{a\in A}\{1+\eta(x-a)\}\prod_{b\in B}\{1-\eta(x-b)\}.$$

Using Lemma 2.3 (b) we have

$$g \geq q - \{(2n-3)2^{2n-1}+2\}\sqrt{q} - \{2^{2n-1}-2n^2-1\}.$$

Consider

$$g-h = \sum_{x\in A\cup B}\prod_{i=1}^{n}\{1+\eta(x-a_i)\}\{1-\eta(x-b_i)\}, \tag{3.5}$$

where $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$.

If $g-h \neq 0$, then for some x the product

$$\prod_{i=1}^{n}\{1+\eta(x-a_i)\}\{1-\eta(x-b_i)\} \neq 0. \tag{3.6}$$

Without any loss of generality suppose $x = a_k$. For (3.6) to hold we must have $\eta(a_k - a_i) \neq -1$ and $\eta(a_k - b_i) \neq 1$ for all i. This means that $\eta(a_k - a_i) = 1$ for $i \neq k$ and $\eta(a_k - b_i) = -1$ for all i. Hence the term in (3.5) with $x = a_i$ for $i \neq k$ contributes zero to the sum. Hence we can write (3.5) as

$$\begin{aligned} g-h &= \sum_{x\in\{a_k\}\cup B}\prod_{i=1}^{n}\{1+\eta(x-a_i)\}\{1-\eta(x-b_i)\} \\ &\leq (n+1)2^{2n-1}, \end{aligned}$$

since, in each product, each factor is at most 2 and at least one factor is 1. Hence

$$\begin{aligned} h &\geq g-(n+1)2^{2n-1} \\ &\geq q - \{(2n-3)2^{2n-1}+2\}\sqrt{q} - \{(n+2)2^{2n-1}-2n^2-1\}. \end{aligned}$$

Now if inequality (3.4) holds, then $h > (k-1)2^{2n}$ as required. Since A and B are arbitrary, this completes the proof of the theorem. ∎

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