Elegant labelings and Edge-colorings
A proof of two conjectures of Hartman and Chang, Hsu, Rogers.

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Abstract. A graph $G = (V, E)$ is said to be elegant if it is possible to label its vertices by an injective mapping g into $\{0, 1, \dots, |E|\}$ such that the induced labeling h on the edges defined for edge x, y by $h(x, y) \equiv g(x) + g(y) (\bmod |E| + 1)$ takes all the values in $\{1, \dots, |E|\}$. In the first part of this paper, we prove the existence of a coloring of K_n with a omnicolored path on n vertices as subgraph, which had been conjectured by Hartman [2].

In the second part we prove that the cycle on n vertices is elegant if and only if $n \not\equiv 1 (\bmod 4)$ and we give a new constuction of an elegant labeling of the path $P_n, n \not\equiv 4$.

1. Introduction.

Two additive versions of the well known notion of *graceful graphs* ([3], [4], [5], [6]) have been proposed.

A connected graph with m edges is called *harmonious* if it is possible to label its vertices with distinct numbers $(\bmod m)$ in such a way that the values on the edges obtained by sums $(\bmod m)$ of their endpoints labelings are also distinct (R.L. Graham and N.J.A. Sloane [7]).

A connected graph with m edges is called *elegant* if it is possible to label its vertices with distinct numbers $(\bmod m + 1)$ in such a way that the values on the edges obtained by sums $(\bmod m + 1)$ of their endpoints labelings are distinct and non zero (G.J. Chang, D.F. Hsu and D.G. Rogers [1]).

Very few graphs are known to be elegant. In [1], the cycles C_{4p} and C_{4p+3} and the paths P_{4p+1}, P_{4p+2} and P_{4p+3} were proved to be elegant. Cahit [8] proved that P_{4p} is elegant if $p > 1$. We shall prove that C_{4p+2} is elegant, and give a simpler and more geometric proof of Cahit's result.

The concept of graph labeling can be applied to problems on edge-coloring. An edge-coloring of a graph is said to be optimal if no two edges incident with the same vertex have the same color, and the minimum number of colors is used.

Hartman [2] posed the problem of characterizing the family M_n of graphs G such that there exists an optimal edge-coloring of the complete graph K_n in which G appears as a omnicolored subgraph, i.e., every edge of G has a different color.

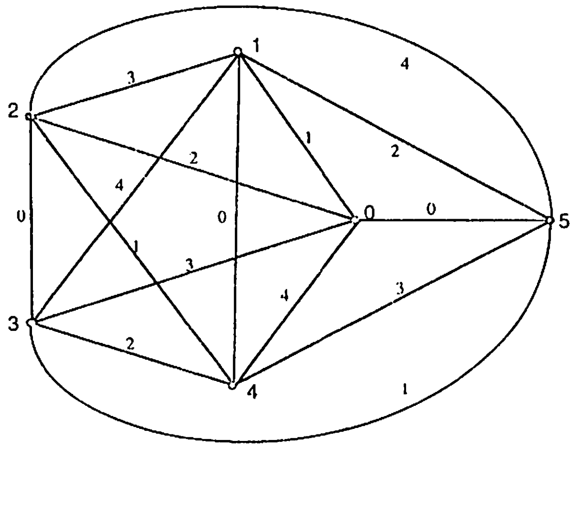
Note that an harmonious graph with $2p + 1$ edges belongs to M_{2p+1}. In [2], it was stated that P_n is in M_n for $n \not\equiv 6 (\bmod 8)$, and it was conjectured that P_n belongs to M_n for $n \not\equiv 4, 6$. We shall settle this conjecture in the affirmative.

2. An Edge-coloring problem.

Theorem 1. *The paths on n vertices belongs to M_n if and only if $n \neq 4, 6$.*

Proof: In the classic coloring of the edges of K_{2p+1} with $2p+1$ colors, we label the vertices $0, 1, \ldots, 2p$, and then color the edges according to the sum (mod $2p+1$) of the labels of their endpoints. If the vertices form a regular polygon $0, 1, \ldots, 2p$, then a color class is a set of parallel edges. The boundary is a omnicolored cycle. By deleting one edge we obtain a omnicolored path on $2p+1$ vertices.

In every optimal edge-coloring of K_{2p+1} exactly one color is missing at each vertex and all these missing colors are different. Thus we obtain a coloring of K_{2p+2} with $2p+1$ colors by coloring the edges incident to the new vertex with these missing colors.

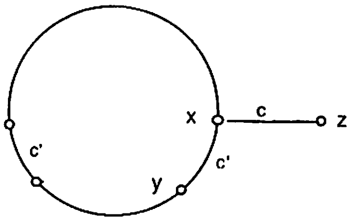


Assume we are able to color K_{2p+1} in such a way that:

(a) in the boundary every color appears exactly once except one color c which does not appear and one color c' which appears twice.

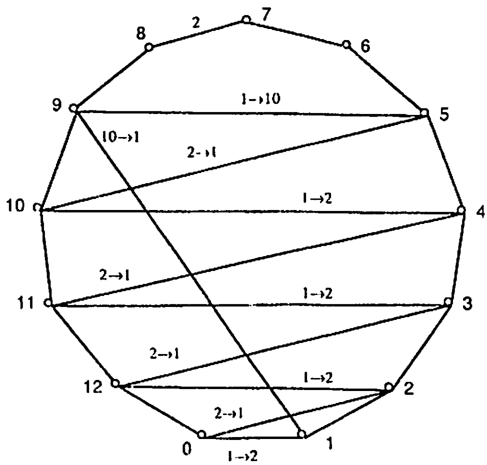
(b) the vertex x where c is missing is incident to an edge $\{x, y\}$ of the boundary colored with c'.

Extend this coloring to K_{2p+2}. Let z be the new vertex then $\{z, x\}$ is colored with c, and by deleting the edge $\{x, y\}$ from the boundary and adding the edge $\{x, z\}$ we obtain a omnicolored path $(y, \ldots, x, z)$.



We are now going to exhibit such a coloring of K_{2p+1} by some changes on the classical coloring defined above.

Case 1. $p \equiv 0 \pmod 3$.

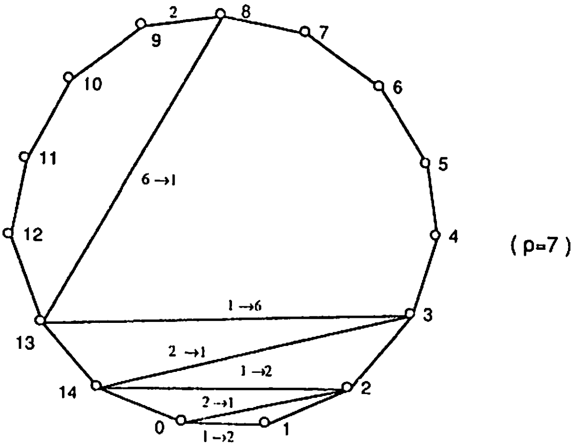


(p=6)

The cycle defined by the vertices $1, 0, 2, 2p, 3, 2p-1, \ldots, 2p/3+1, 4p/3+1, 1$ has its edges colored $1,2,\ 1,2, \ldots, 2,\ 1,\ 4p/3+2$. We change this coloring to $2, 1, 2, 1, \ldots, 1, 4p/3+2, 1$. We have a required coloring of K_{2p+1} with $c = 1$, $c' = 2$, $x = p+1$ and $y = p+2$.

In the new coloring: the color missing at vertex 1 is now $4p/3+2$ instead of 2, the color missing at vertex $2p/3+1$ is now 2 instead of $4p/3+2$.

Case 2. $p \equiv 1 \pmod 3$.



The path $1, 0, 2, 2p, 3, 2p-1, \ldots, (p+2)/3, (5p+4)/3, p+1$ is colored 1, 2, 1, 2, ..., 2, 1, $(2p+4)/3$ and the color 1 is missing at $p+1$. We change this coloring to 2, 1, 2, 1, ..., 1, $(2p+4)/3$, 1. Now the color 1 is missing in 1 so that we have a required coloring of K_{2p+1} with $c = 1, c' = 2, x = 1$ and $y = 0$.

In the new coloring: the color missing at vertex 1 is now 1 instead of 2,

the color missing at vertex $p+1$ is now $(2p+4)/3$ instead of 1,

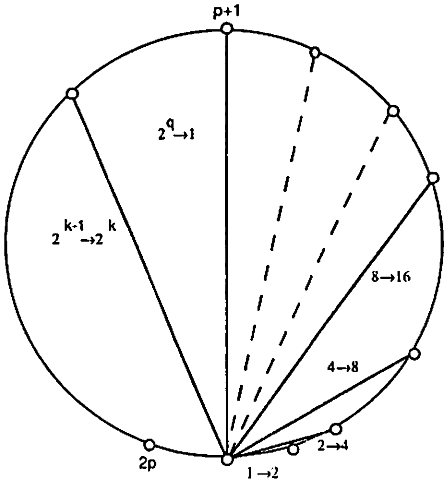
the color missing at vertex $(p+2)/3$ is now 2 instead of $(2p+4)/3$.

Case 3. $p \equiv 2 \pmod 3$.

Consider the sequence of vertices $1, 2, 4, \ldots, 2^k, \ldots$ (taken $\pmod{2p+1}$). All these vertices are different until 1 is met again (because $2p+1$ is odd).

Case 3.1.

Assume first that $2p$ does not belong to the sequence $1, 2, 4, \ldots, 2^k, \ldots, 2^q = p+1, 1$. Then all the edges between the vertex 0 and $1, 2, 4, \ldots, 2^k, \ldots, p+1$ except the first one are not in the boundary. In the classical coloring for each k the edge $\{0, 2^k\}$ is colored with 2^k and the color 2^{k+1} is missing at the vertex 2^k. Color this edge with 2^{k+1}. At the vertex 0 we did nothing else than a circular permutation on the colors used for all these adjacent edges. Thus we still have a coloring. This is a required coloring with $c = 1, c' = 2, x = 1$ and $y = 0$.

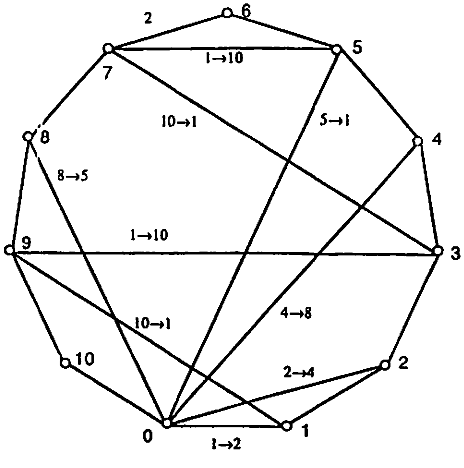


Case 3.2.

If we are not in the above case, we have a sequence of different vertices 1,2,4, $\ldots, 2^k, \ldots, p, 2p$. In this case we cannot use the same change because the edge $\{0, 2p\}$ is in the boundary.

Case 3.2.1. $\boldsymbol{p \equiv 5 \pmod 6}$.

There is a path $(p, \boldsymbol{p+2}, p-2, \boldsymbol{p+4}, p-4, \ldots, 3, \boldsymbol{2p-1}, 1)$, the edges of which are colored $1, 2p, 1, 2p, \ldots, 1, 2p$.



(p=5)

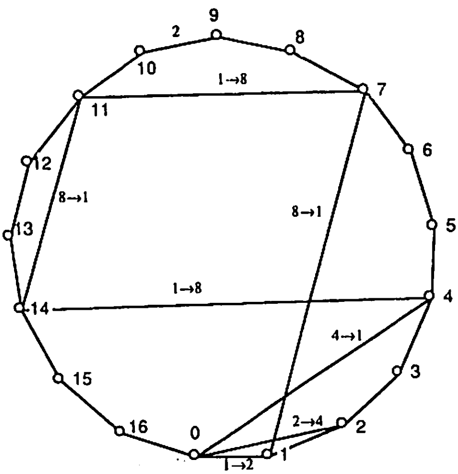
Consider now the sequence of edges $\{0,1\},\{0,2\},\{0,4\},\dots,\{0,2^k\},\dots,$ $\{0,p\},\{p,p+2\},\{p+2,p-2\},\{p-2,p+4\},\dots,\{3,2p-1\},\{2p-1,1\}$ colored $1,2,4,\dots,2^k,\dots,p,1,2p,1,2p,\dots,1,2p$. We change these colors to $2,4,8,\dots,2^{k+1},\dots,1,2p,1,2p,\dots,1,2p,1$. This yields a required coloring with $c=1, c'=2, x=p+1$ and $y=p+2$.

Case 3.2.2 $p\equiv 2 \pmod 6$.

We have $p/2 \equiv 1 \pmod 3$. Consider the path $P = u_0,u_1,\dots,u_i,\dots,u_{(2p-4)/3}$ with

$$
\begin{aligned}
u_i &= p/2 - 3i/4 && \text{if } i\equiv 0 \pmod 4,\\
u_i &= 3p/2+2+3(i-1)/4 && \text{if } i\equiv 1 \pmod 4,\\
u_i &= 3p/2-1-3(i-2)/4 && \text{if } i\equiv 2 \pmod 4,\\
u_i &= p/2+3+3(i-3)/4 && \text{if } i\equiv 3 \pmod 4.
\end{aligned}
$$

In other words $P = (p/2, 3p/2+2, 3p/2-1, p/2+3, p/2-3, 3p/2+5, 3p/2-4, p/2+6, p/2-6,\dots,4,2p-2,p+3,p-1,1)$.



(p=8)

The edges of P are colored $1,p,1,p,\dots,1,p$. Consider now the sequence of edges $\{0,1\},\{0,2\},\{0,4\},\dots,\{0,p/2\},\{p/2,3p/2+2\},\{3p/2+2,3p/2-1\},\dots,\{p-1,1\}$ colored $1,2,4,\dots,2^k,\dots,p/2,1,p,1,p,\dots,1,p$. We change these colors to $2,4,\dots,2^k,\dots p/2,1,p,1,p,\dots,1,p,1$. We have a required coloring with $c=1, c'=2, x=p+1$ and $y=p+2$.

3. Elegant graphs.

Theorem 2. *The cycles on n vertices are elegant if and only if $n \not\equiv 1 \pmod 4$.*

Notice first that in an elegant labeling of the cycle C_n the sum of labels of the edges, $1+2+,\dots,+n$, is twice the sum of labels of the vertices $\pmod{n+1}$. This remark gives the necessary condition $n\not\equiv 1 \pmod 4$.

The two cases $n \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$ have been previously solved in [1]. We repeat these short arguments to make our proof self-contained
If $n = 4p$ an elegant labeling of C_{4p} is given by:

$$1,2,\ldots,2p-1,2p,$$
$$2p+2,\mathbf{2p+1},2p+4,\mathbf{2p+3},\ldots,4p,\mathbf{4p-1}$$

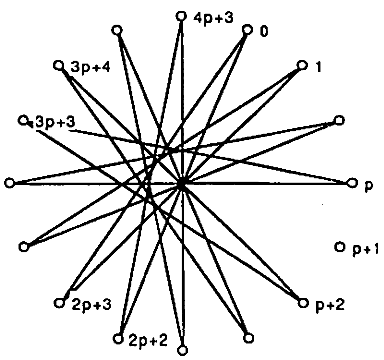
We point out that the edges with even labels form a path, as do the edges with odd labels.
If $n = 4p+3$, an elegant labeling of C_{4p+3} is given by:

$$0,\mathbf{2p+3},1,\mathbf{2p+4},\ldots,\mathbf{3p+2},p,\mathbf{3p+3}$$
$$p+2,\mathbf{3p+4},p+3,\mathbf{3p+5},\ldots,2p+1,\mathbf{4p+3},2p+2$$

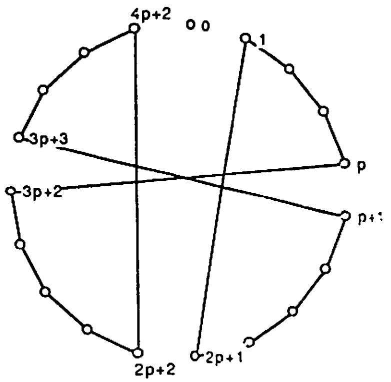
If $n = 4p+2$, an elegant labeling of C_{4p+2} is given by:

$$1,2,\ldots,p-1,p,$$
$$3p+2,3p+1,\ldots,2p+3,2p+2,$$
$$4p+2,4p+1,\ldots,3p+4,3p+3,$$
$$p+1,p+2,\ldots,2p,2p+1$$

It is easy to verify that the induced values on the edges are $\{1,\ldots,4p+2\}$. In the two later cases the geometric representation is more interpretable:



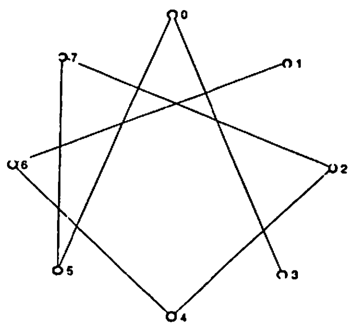
case $4p+3$



case $4p+2$

Theorem 3. *The paths on n vertices are elegant if and only if $n \neq 4$.*

The cases $n \not\equiv 0 \pmod 4$ have been proved in [1]. The labels are obtained from the perfect shuffle of two arithmetic progressions.
If $n = 4p+1$, we have:
$3p+1, \mathbf{p+1}, 3p+2, \mathbf{p+2}, \ldots, p-1, \mathbf{3p}, p$.
If $n = 4p+2$, we have:
$p+1, \mathbf{3p+2}, p+2, \mathbf{3p+3}, \ldots, 3p+1, \mathbf{p}$.
If $n = 4p+3$, we have:
$p+1, \mathbf{3p+3}, p+2, \mathbf{3p+4}, .., 3p+1, \mathbf{p}, 3p+2$.
The remaining case $n = 4p$ has been solved in [8]. It requires a more complicated labeling. We are going to construct another elegant labeling of P_{4p}, which is simpler and more geometric.
First notice that P_4 is not elegant and an elegant labeling of P_8 is given in the following figure.



An elegant labeling of P_8.

Assume now that $n = 4p$ with $p \geq 3$.

If *p is odd*, an elegant labeling of P_{4p} is given by:

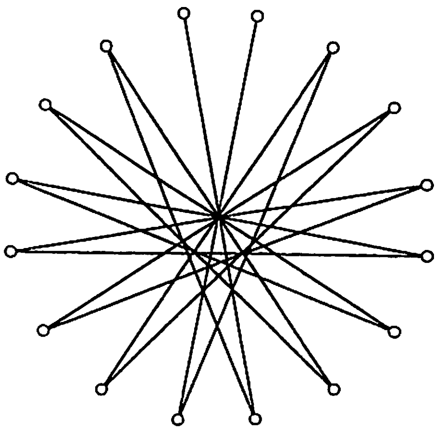
$1,$
$2p+1, 2p+3, \ldots, 3p-2, 3,$
$3p+3, 3p+5, \ldots, 4p-2, 0,$
$2p, \mathbf{4p-1}, 2p-2, \mathbf{4p-3}, \ldots, \mathbf{3p+4}, p+3, \mathbf{3p+2}, p+1,$
$\mathbf{3p+1}, p, \mathbf{3p-1}, p-2, \ldots, 5, \mathbf{2p+4}, 3, \mathbf{2p+2},$
$2, 4, \ldots, p-3, p-1,$
$p+2, p+4, \ldots, 2p-3, 2p-1.$

If *p is even*, an elegant labeling is given by:

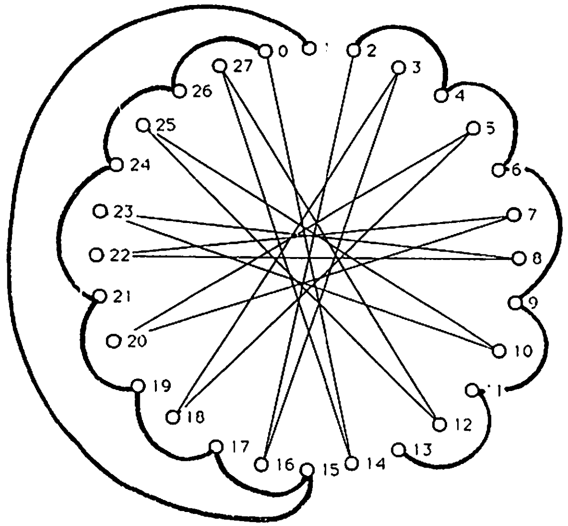
$1,$
$2p+1, 2p+3, \ldots, 3p-3, 3p-1,$
$3p+2, 3p+4, \ldots, 4p-2, 0,$
$2p, \mathbf{4p-1}, 2p-2, \mathbf{4p-3}, \ldots, p+4, \mathbf{3p+3}, p+2, \mathbf{3p+1},$
$p+1, \mathbf{3p}, p-1, \mathbf{3p-2}, p-3, \ldots, 5, \mathbf{2p+4}, 3, \mathbf{2p+2},$
$2, 4, \ldots, p-2, p,$
$p+3, p+5, \ldots, 2p-3, 2p-1.$

In each of the two cases the given labeling is obtained by a common geometric construction. On a regular polygon with vertices labelled $0, 1, \ldots n-1$, edges are parallel if and only if the sum of the labels of their endpoints are equal.

Consider the following path on $2p+1$ vertices.



Insert now the $2p-1$ remaining vertices joined by two disjoint paths as follows.



We obtain now a path on the $4p$ vertices such that edges are not pairwise "parallel" and none is "parallel" to $[1, n-1]$. It follows that the labelling is elegant.

References

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