A Pair of Mutually Polar Translation Planes

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1. Introduction

In this note we prove that the two flag-transitive translation planes of order 5^2 which were defined by D.A. Foulser in [7] are a mutually polar pair. Although these translation planes are well known it does not appear to have been noticed previously that they have this property.

We establish this result using Conway's isomorphism invariant of projective planes. This invariant, as well as another isomorphism invariant of translation planes are described in [4] and [5]. We have also determined the translation complements of both Foulser planes using this invariant. (This was also done in [9], [10], with standard arguments.) However our approach requires minimal background and the automorphism groups of other translation planes can also be determined by this method, thus we include here a fairly complete description of this method.

We conclude with an observation about a possible extension of our result to a family of two dimensional flag-transitive planes.

2. The Quadratic matrix

In the next two sections we shall prove that the two flag-transitive planes of order 5^2 π and π' defined by Foulser in [7] are a mutually polar pair. This concept will be defined in the course of our discussion; we refer the reader to [5] for an explanation of any undefined concepts.

We can use either the original description of π and π' given in [7], or an equivalent description found in [6]. From either description of π and π' we obtain two spreads S_1 and S_2 in the vector space $V(4,5)$; these spreads define the translation planes. (We can also work in the corresponding projective space $PG(3,5)$.) Next we calculate the *quadratic matrix* of each plane using the procedures which are described in [4].

We find that both π and π' have the same quadratic matrix. (See Fig. 1.) We have suppressed the signs.

Examination of this matrix shows that we can associate with this matrix a graph Γ as follows. The vertices of Γ are in one-one correspondence with the vectors of the quadratic matrix (Since the groups of these planes act flag-transitively the vectors are only of one type.) If we join two vertices whenever the corresponding entry of this matrix is ± 8, we obtain the graph in Figure 2.

25	0	8	4	4	4	4	4	4	4	4	8	0	0	4	8	8	4	4	4	4	4	4	8	8	4
0	25	0	8	4	4	4	4	4	4	4	4	8	4	0	4	8	8	4	4	4	4	4	4	8	8
8	0	25	0	8	4	4	4	4	4	4	4	4	8	4	0	4	8	8	4	4	4	4	4	4	8
4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4	4	4	4	4	4
4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4	4	4	4	4
4	4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4	4	4	4
4	4	4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4	4	4
4	4	4	4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4	4
4	4	4	4	4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8	4
4	4	4	4	4	4	4	8	0	25	0	8	4	4	4	4	4	4	4	8	8	4	0	4	8	8
4	4	4	4	4	4	4	4	8	0	25	0	8	8	4	4	4	4	4	4	8	8	4	0	4	8
8	4	4	4	4	4	4	4	4	8	0	25	0	8	8	4	4	4	4	4	4	8	8	4	0	4
0	8	4	4	4	4	4	4	4	4	8	0	25	4	8	8	4	4	4	4	4	4	8	8	4	0
0	4	8	8	4	4	4	4	4	4	8	8	4	25	4	4	8	4	0	4	4	0	4	8	4	4
4	0	4	8	8	4	4	4	4	4	4	8	8	4	25	4	4	8	4	0	4	4	0	4	8	4
8	4	0	4	8	8	4	4	4	4	4	4	8	4	4	25	4	4	8	4	0	4	4	0	4	8
8	8	4	0	4	8	8	4	4	4	4	4	4	8	4	4	25	4	4	8	4	0	4	4	0	4
4	8	8	4	0	4	8	8	4	4	4	4	4	4	8	4	4	25	4	4	8	4	0	4	4	0
4	4	8	8	4	0	4	8	8	4	4	4	4	0	4	8	4	4	25	4	4	8	4	0	4	4
4	4	4	8	8	4	0	4	8	8	4	4	4	4	0	4	8	4	4	25	4	4	8	4	0	4
4	4	4	4	8	8	4	0	4	8	8	4	4	4	4	0	4	8	4	4	25	4	4	8	4	0
4	4	4	4	4	8	8	4	0	4	8	8	4	0	4	4	0	4	8	4	4	25	4	4	8	4
4	4	4	4	4	4	8	8	4	0	4	8	8	4	0	4	4	0	4	8	4	4	25	4	4	8
8	4	4	4	4	4	4	8	8	4	0	4	8	8	4	0	4	4	0	4	8	4	4	25	4	4
8	8	4	4	4	4	4	4	8	8	4	0	4	4	8	4	0	4	4	0	4	8	4	4	25	4
4	8	8	4	4	4	4	4	4	8	8	4	0	4	4	8	4	0	4	4	0	4	8	4	4	25

Figure 1

The quadratic matrix for the Foulser planes, without signs.

We discuss the automorphism group of Γ. The vertices of the graph are labelled by their distance from 0. It is evident that there is an automorphism which interchanges the inner and outer 13-gons in Figure 2, so Aut(Γ) is transitive on the 26 vertices. The numbers on the figure show that there are just four points at distance 4 from the point 0 and that there is just one edge between two of these points.

As the initial point 0 varies, we obtain just the 26 edges of the outer and inner 13-gons in this way, so the partition of the vertices into these 2 polygons is characteristic. So Aut(Γ) is of order 52, the subgroup of index 2 fixing the outer polygon being obviously a dihedral group of order 26.

2.1 The Klein Correspondence

The Klein correspondence associates to a spread in $V(4,5)$ an ovoid in the quadratic space $\Omega^+(6,5)$. (An ovoid in $\Omega^+(6,q)$ is a set of q^2+1 isotropic vectors which has the property that inner product of any two distinct vectors is non-zero.) We can define $\Omega^+(6,5)$ to be the exterior square $V \wedge V$ together with

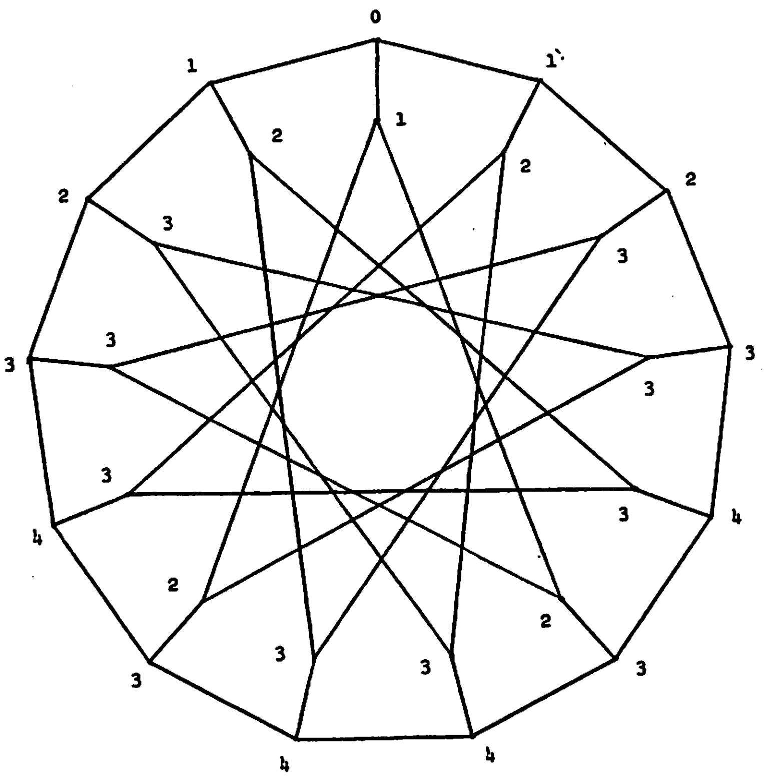

Figure 2

a quadratic form of Witt index 3.

Given the quadratic matrix of either Foulser plane in any ordering we can sort it into the order used in Figure 1 by the method of the previous section, and then display the Gram matrix of the ovoid vectors in the same order. We obtain the same matrix (Fig. 3) for each plane. This matrix was obtained as follows. S_1 and S_2 correspond to two ovoids in $\Omega^+(6,5)$; in this quadratic space, inner products are given by the bilinear form $B(x,y) = Q(x+y) - Q(x) - Q(y)$ which corresponds canonically to $Q(x) = x_1x_4 + x_2x_5 + x_3x_6$.

I found in this way that the 6-dimensional ovoids which correspond to the 2 planes are isomorphic (as ovoids), and concluded falsely[1] that these planes are isomorphic. The true situation is much more interesting.

Despite the fact that the ovoids are isomorphic, the corresponding 4-dimensional spreads are related by a *duality* of the 4-dimensional space rather that an isomor-

[1]This possibility was raised in my talk at the British Combinatorics Conference 1989.

0	3	3	4	3	2	1	1	2	3	4	3	3	4	4	3	3	2	4	2	2	4	2	3	3	4
3	0	3	3	4	3	2	1	1	2	3	4	3	4	4	4	3	3	2	4	2	2	4	2	3	3
3	3	0	3	3	4	3	2	1	1	2	3	4	3	4	4	4	3	3	2	4	2	2	4	2	3
4	3	3	0	3	3	4	3	2	1	1	2	3	3	3	4	4	4	3	3	2	4	2	2	4	2
3	4	3	3	0	3	3	4	3	2	1	1	2	2	3	3	4	4	4	3	3	2	4	2	2	4
2	3	4	3	3	0	3	3	4	3	2	1	1	4	2	3	3	4	4	4	3	3	2	4	2	2
1	2	3	4	3	3	0	3	3	4	3	2	1	2	4	2	3	3	4	4	4	3	3	2	4	2
1	1	2	3	4	3	3	0	3	3	4	3	2	2	2	4	2	3	3	4	4	4	3	3	2	4
2	1	1	2	3	4	3	3	0	3	3	4	3	4	2	2	4	2	3	3	4	4	4	3	3	2
3	2	1	1	2	3	4	3	3	0	3	3	4	2	4	2	2	4	2	3	3	4	4	4	3	3
4	3	2	1	1	2	3	4	3	3	0	3	3	3	2	4	2	2	4	2	3	3	4	4	4	3
3	4	3	2	1	1	2	3	4	3	3	0	3	3	3	2	4	2	2	4	2	3	3	4	4	4
3	3	4	3	2	1	1	2	3	4	3	3	0	4	3	3	2	4	2	2	4	2	3	3	4	4
4	4	3	3	2	4	2	2	4	2	3	3	4	0	4	3	1	2	1	1	1	1	2	1	3	4
4	4	4	3	3	2	4	2	2	4	2	3	3	4	0	4	3	1	2	1	1	1	1	2	1	3
3	4	4	4	3	3	2	4	2	2	4	2	3	3	4	0	4	3	1	2	1	1	1	1	2	1
3	3	4	4	4	3	3	2	4	2	2	4	2	1	3	4	0	4	3	1	2	1	1	1	1	2
2	3	3	4	4	4	3	3	2	4	2	2	4	2	1	3	4	0	4	3	1	2	1	1	1	1
4	2	3	3	4	4	4	3	3	2	4	2	2	1	2	1	3	4	0	4	3	1	2	1	1	1
2	4	2	3	3	4	4	4	3	3	2	4	2	1	1	2	1	3	4	0	4	3	1	2	1	1
2	2	4	2	3	3	4	4	4	3	3	2	4	1	1	1	2	1	3	4	0	4	3	1	2	1
4	2	2	4	2	3	3	4	4	4	3	3	2	1	1	1	1	2	1	3	4	0	4	3	1	2
2	4	2	2	4	2	3	3	4	4	4	3	3	2	1	1	1	1	2	1	3	4	0	4	3	1
3	2	4	2	2	4	2	3	3	4	4	4	3	1	2	1	1	1	1	2	1	3	4	0	4	3
3	3	2	4	2	2	4	2	3	3	4	4	4	3	1	2	1	1	1	1	2	1	3	4	0	4
4	3	3	2	4	2	2	4	2	3	3	4	4	4	3	1	2	1	1	1	1	2	1	3	4	0

Figure 3

The Gram matrix for the Foulser planes.

phism. A.A. Bruen has very recently informed me[2] that he called such planes *polar* in his 1972 paper [3] in which he found a complicated example of non isomorphic mutually polar translation planes. It does not appear to have been noticed before that the two Foulser planes are such a pair.

The following Lemma, suggested to me by S. P. Norton [11], provides us with a convenient criterion to cope with the polarity phenomenon. Before stating it we introduce some terminology. An automorphism of an ovoid in a 6-dimensional orthogonal geometry is an element M of the 6-dimensional orthogonal group such that $M'QM = \lambda Q$ for some $\lambda \in GF(q)$, where Q is the matrix of the quadratic form which defines the orthogonal geometry. We call λ the *scaling* factor of M.

2.1.1. Lemma. *An automorphism M of an 6-dimensional ovoid $\mathcal{O}$ corresponds (by the Klein correspondence), to an element of the collineation group of the corresponding translation plane if and only if the determinant of M equals λ^3.*

[2] After this work was nearly complete.

Proof: Since the translation planes we consider are defined over $GF(q)$ for q a prime, any automorphism (modulo the subgroup of translations), is an element of $GL_4(q)$. Let g be such an automorphism, then g can be put into the Jordan form

$$g = \begin{pmatrix} a & & & \\ \vdots & b & & \\ \vdots & \vdots & c & \\ \vdots & \vdots & \vdots & d \end{pmatrix}.$$

Now suppose that $A,\ B,\ C,\ D$ are the eigenvectors corresponding to the eigenvalues $a,\ b,\ c,\ d$ respectively, then: $A \wedge B, A \wedge C, A \wedge D, B \wedge C, D \wedge B, C \wedge D$ is a basis for the exterior square of $V(4,q)$ with itself, and with respect to this basis the corresponding element of the symmetry group of the ovoid has the Jordan form

$$G = \begin{pmatrix} ab & & & & & \\ \vdots & ac & & & & \\ \vdots & \vdots & ad & & & \\ \vdots & \vdots & \vdots & bc & & \\ \vdots & \vdots & \vdots & \vdots & bd & \\ \vdots & \vdots & \vdots & \vdots & \vdots & cd \end{pmatrix}.$$

The quadratic form of this quadratic space has the matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

then

$$G'QG = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & abcd \\ 0 & 0 & 0 & 0 & abcd & 0 \\ 0 & 0 & 0 & abcd & 0 & 0 \\ 0 & 0 & abcd & 0 & 0 & 0 \\ 0 & abcd & 0 & 0 & 0 & 0 \\ abcd & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that for $g \in GL_4(q)$, the determinant δ of the corresponding automorphism of the ovoid satisfies the relation $\delta = \lambda^3$, where λ is the scaling factor.

Now let U be a symmetry of the ovoid given by the matrix Q. The scaling factor of U is 1, but the determinant is -1. So the inverse image of U (under the Klein correspondence) is an element of $GL_4(q)\cdot 2$ which fails to satisfy the criterion. Since $GL_4(q)$ has index two in $GL_4(q)\cdot 2$, it follows that all the elements of this group which do not belong to $GL_4(q)$ have this property.

We summarise. Under the Klein correspondence an automorphism with $\delta = \lambda^3$ maps to a collineation of projective 3-space, while one with $\delta = -\lambda^3$ maps to a correlation (duality).

We can now prove our main result.

2.1.2. Theorem. *There are 2 mutually polar, but not isomorphic translation planes corresponding to the 6-dimensional ovoid described above.*

Remark. These are of course, the planes proved by Foulser in [7] to be the only flag-transitive planes of order 5^2.

Proof: The element of order 13 in the automorphism group cannot induce correlations so we need only consider the other elements of the group. Now the action of the element τ of order 4 is described by the following matrix

$$\tau = \begin{array}{c} \\ P_0 \\ P_5 \\ P_{-5} \\ Q_0 \\ Q_1 \\ Q_{-1} \end{array} \begin{array}{c} \begin{array}{cccccc} P_0 & P_5 & P_{-5} & Q_0 & Q_1 & Q_{-1} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & 4 & 4 & 4 \\ 2 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 0 & 4 & 2 & 2 \\ 4 & 4 & 4 & 0 & 4 & 4 \\ 4 & 2 & 2 & 4 & 0 & 3 \\ 4 & 2 & 2 & 4 & 3 & 0 \end{pmatrix} \end{array}.$$

But τ has the property that $\tau' Q \tau = Q$, where Q is the diagonal quadratic form. Hence the scaling factor corresponding to τ is 1. On the other hand the determinant of τ is 1, so by the previous lemma τ belongs to the subgroup of collineations of the translation plane corresponding to this ovoid. We conclude that every element of the automorphism group of the ovoid induces a collineation. In other words each ovoid automorphism takes each plane to itself, rather than to the polar plane.

We refer to the discussion of Chapter 2 in [4], as to why we get the same restricted sign matrices for a translation plane and the polar plane. Here we only mention that this is consistent with a conjecture regarding Conway's invariant which was discussed in [4].

It now follows from the above theorem (since a dual spread has the same abstract group in its contragradient representation), that the automorphism groups of π and π' have the same order. This was originally established by Foulser in [7] up to a factor of two, and the full automorphism group of both planes was determined by lengthy arguments in [9] and [10].

In view of the phenomenon described in Theorem 2.1.2, a 6-dimensional ovoid gives rise to either just one, or just two nonisomorphic translation planes, according as there is or is not an automorphism with $\delta = -\lambda^3$. This is often a delicate mater. The automorphism group of the plane, modulo the subgroup of kernel homologies, is of index at most 2 consisting just of those derived from the ovoid automorphisms with $\delta = \lambda^3$.

3. Polarity in other planes

Among the translation planes of order 7^2 that I have studied there is one such pair, and again it consists of two flag-transitive planes. These planes are an instance of a general construction for all flag-transitive translation planes of order q^2 given by Baker and Ebert in [1], [2] and [6]. In [6] Ebert was unable to decide whether the two planes of order 7^2 $[L_1] \cup [L'_1]$ and $[L_1] \cup [L''_1]$ were isomorphic. This is now a consequence of Kantor's recent work [8]. Prior to knowing about [8], I also have established this by our methods; I showed that these two planes are mutually polar.

This suggests that the automorphism groups of such planes acts in a special way. For instance, when $q = 7$ there are three nonisomorphic non-desarguesian flag-transitive planes planes of order 7^2. The automorphism group of two of these planes $[L_1] \cup [L'_1]$ and $[L_1] \cup [L''_1]$, coincides with the automorphism group of their ovoid. This is not the case for the plane $[L_3] \cup [L'_3]$ of [6]. For $q = 3$ there is only one non-desarguesian flag-transitive plane of order 3^2. Hence it is isomorphic to its polar plane.

We know now by Baker and Ebert's, and Kantor's work [8], that there are $(q-1)/2$ mutually nonisomorphic non-desarguesian flag-transitive translation planes of order q^2 if q is an odd prime. In view of our result and the previous remarks we ask is the following is true for all primes q, such that $q > 7$? If $q \equiv 3 \pmod 4$, the set of $(q-1)/2$ non-desarguesian flag-transitive planes is partitioned into one self polar plane and $(q-3)/4$ mutually polar planes. But when $q \equiv 1 \pmod 4$, this set is partitioned into $(q-1)/4$ pairs of mutually polar planes.

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