On the Existence of Perfect Mendelsohn Designs without Repeated Blocks

D.F. Hsu
Department of Computer and Information Science
Fordham University
Bronx, New York
U.S.A. 10458-5198

H. Shen
Department of Applied Mathematics
Shanghai Jiao Tong University
Shanghai 200030
China

Abstract. In this paper we study the existence of perfect Mendelsohn designs without repeated blocks and give several general constructions. We prove that for $k = 3$ and any λ, and $(k,\lambda) = (4,2),(4,3)$ and $(4,4)$, the necessary conditions are also sufficient for the existence of a simple (v,k,λ)-PMD, with the exceptions $(k,\lambda) = (6,1)$ and $(6,3)$

Introduction

Let v and k be given positive integers. Let X be a finite set containing v elements and $B = (x_0, x_1, \dots, x_{k-1})$ be a cyclically ordered k–subset of X consisting of k ordered pairs $(x_0,x_1),(x_1,x_2),\dots,(x_{k-2},x_{k-1})$ and (x_{k-1},x_0). Let $t = 1,2,\dots,k-1$, two elements x_i and x_{i+t} are said to be t-apart in B where $i+t$ is taken modulo k. Two cyclically ordered subsets are considered to be the same if one can be obtained from the other by a cyclic permutation.

Let v, k and λ be given positive integers. A (v,k,λ)–*Mendelsohn design* (or briefly written as (v,k,λ)–MD) is a pair $(X,\mathbf{B})$ where X is a v-set and $\mathbf{B}$ is a collection of cyclically ordered k–subsets (called blocks) of X such that each ordered pair of distinct elements of X is contained in exactly λ blocks of $\mathbf{B}$.

Let $(X,\mathbf{B})$ be a (v,k,λ)–MD and r be an integer, $1 \le r \le k-1$. If for all $t = 1,2,\dots,r$, each ordered pair of distinct elements of X appears t-apart in exactly λ blocks, then $(X,\mathbf{B})$ is called *r-fold perfect*. A $(k-1)$-fold perfect (v,k,λ)-MD is called *perfect* and denoted (v,k,λ)-PMD.

The concept of PMDs was introduced in the case $k = 3$ by N.S. Mendelsohn [13] under the name of perfect cyclic designs. The terminology of Mendelsohn designs was first used in its general form by Hsu and Keedwell [9]. The existence and construction of perfect Mendelsohn designs, its relationship with various mathematical structures has been widely studied and many new results obtained in recent years. The interested reader may refer to the survey paper [17].

It is not difficult to verify that any (v, k, λ)-PMD contains $\lambda v(v-1)/k$ blocks and hence the following is a necessary condition for the existence of a (v, k, λ)-PMD:

$$\lambda v(v-1) \equiv 0 \pmod{k}. \tag{1}$$

It was proved [1, 11] that (1) is also sufficient for the existence of a $(v, 3, \lambda)$-PMD, except $(v, \lambda) = (6, 1)$. It is proved [6,16] that for $k = 4$, (1) is also sufficient for the existence of a $(v, 4, \lambda)$-PMD except $v = 4$ and λ odd, $(v, \lambda) = (8, 1)$ and possibly except $(v, \lambda) = (12, 1)$. For $k = 5$, an almost complete solution can be found in [3,5]. But in their constructions, repeated blocks are permitted. In many occasions, naturally, we are more interested in perfect Mendelsohn designs without repeated blocks, such designs will be called *simple*.

A *balanced incomplete block design* (v, k, λ)-BIBD is a pair $(X, \mathbf{B})$ where X is a v-set and $\mathbf{B}$ is a collection of k-subsets (called blocks) of X such that each unordered pair of distinct elements of X is contained in exactly λ blocks.

If we ignore the cyclic order in the blocks of a (v, k, λ)-PMD, then we obtain a $(v, k, \lambda(k-1))$-BIBD and this BIBD is called the underlying BIBD of the PMD. A (v, k, λ)-PMD is called *pure* according to [2] or called *tight* according to [9] if the underlying $(v, k, \lambda(k-1))$-BIBD contains no repeated blocks. Obviously, a pure (v, k, λ)-PMD must be simple. We also note that for $\lambda = 1$, any $(v, k, 1)$-PMD must be simple but not necessarily pure.

The main purpose of this paper is to study the existence of simple or pure perfect Mendelsohn designs. Constructions of simple perfect Mendelsohn designs with additional properties are also discussed.

General Constructions

Let v be a positive integer and K be a set of positive integers. A *pairwise balanced design* $(v, K, 1)$-PBD is a pair $(X, \mathbf{B})$ where X is a v-set and $\mathbf{B}$ is a set of subsets (called blocks) of X such that $|B| \in K$ for each $B \in \mathbf{B}$ and each pair of distinct elements of X is contained in a unique block of $\mathbf{B}$.

Let

$$B(K) = \{v \mid \text{there exists a } (v, K, 1)-\text{PBD}\}.$$

If $B(K) = K$, then K is called a PBD-closed set.

Now let

$$NPM(k, \lambda) = \{v \mid \text{there exists a simple } (v, k, \lambda)-\text{PMD}\},$$

and

$$PPM(k, \lambda) = \{v \mid \text{there exists a pure } (v, k, \lambda)-\text{PMD}\}.$$

We prove the following theorem which provides a recursive construction for simple or pure perfect Mendelsohn designs.

Theorem 2.1. *For given positive integers k and λ, $NPM(k,\lambda)$ and $PPM(k,\lambda)$ are PBD-closed sets.*

Proof: Let $(X,\mathbf{B})$ be a $(v,K,1)$-PBD such that $m \in NPM(k,\lambda)$ for each $m \in K$. For each $B \in \mathbf{B}$, $|B| = m$, we form a simple (m,k,λ)-PMD on B and denote the block set by $\mathbf{A}_B$. Now let

$$\mathbf{A} = \cup_{B\in\mathbf{B}} \mathbf{A}_B,$$

then $(X,\mathbf{A})$ is a simple (v,k,λ)-PMD. This proves that $NMP(k,\lambda)$ is PBD-closed. Similarly, we can prove that $PPM(k,\lambda)$ is a PBD-closed set. This completes the proof. ∎

Let $(X,\mathbf{A})$ be a (v,k,λ)-PMD and $(Y,\mathbf{B})$ be a (w,k,λ)-PMD. If $Y \subset X$ and $\mathbf{B}$ is a subcollection of $\mathbf{A}$, then $(Y,\mathbf{B})$ is called a subdesign of $(X,\mathbf{A})$ or $(Y,\mathbf{B})$ is said to be embedded in $(X,\mathbf{A})$. More generally, let X be a v-set and Y be a w-subset of X, and let $\mathbf{A}$ be a collection of cyclically ordered k-subsets (called blocks) of X such that $|B \cap Y| \leq 1$ for each $B \in \mathbf{A}$ and each ordered pair of distinct elements of X, not both in Y, appears t-apart in exactly λ blocks of $\mathbf{A}$, for each $t = 1,2,\ldots,k-1$, then $(X,Y,\mathbf{A})$ is called a $(v,w;k,\lambda)$-IPMD *incomplete perfect Mendelsohn design.* The concept of incomplete block design $(v,w;k,\lambda)$-IPBD can be defined in a similar way (see for example [13]). The following theorem for the construction of $(v,w;k,\lambda)$-IPBD was proved in [14].

Theorem 2.2. *If there exists a simple $(v,w;k,\lambda_1)$-IPBD, a simple $(v,w;k,\lambda_2)$-IPBD and*

$$\lambda_1\lambda_2(k-2)!(v-w)\{kw(v-w-k+1)+(v-(k-1)w-1)^2\}\cdot$$

$$(v-w-k)! < k(k-1)(v-w-1)! \qquad (2)$$

Then there exists a simple $(v,w;k,\lambda_1+\lambda_2)$-IPBD.

We prove the following analogue for simple incomplete perfect Mendelsohn designs.

Theorem 2.3. *If there exists a simple $(v,w;k,\lambda_1)$-IPMD, a simple $(v,w;k,\lambda_2)$-IPMD and*

$$\lambda_1\lambda_2(v-w)(v-w-k)!\{kw(v-w-k+1)+(v-1-(k-1)w)^2\} < k(v-w-1)!$$

Then there exists a simple $(v,w;k,\lambda_1+\lambda_2)$-IPMD.

Proof: Let $(X,Y,\mathbf{A})$ be a $(v,w;k,\lambda)$-IPMD and let

$$\mathbf{A}_i = \{B \in \mathbf{A} \mid |B \cap Y| = i\}, i = 0,1.$$

For any $y \in Y$ and $x \in X \backslash Y$, the ordered pair (x, y) is contained in exactly λ blocks. As for any block $B, |B \cap Y| \leq 1$, so y is contained in precisely $\lambda(v-w)$ blocks and therefore it is easy to check that $|\mathbf{A}_1| = \lambda w(v-w)$. Since the number of blocks contained in a $(v, w; k, \lambda)$-IPMD is

$$|\mathbf{A}| = \frac{\lambda v(v-1) - \lambda w(w-1)}{k},$$

then we have

$$\begin{aligned} |\mathbf{A}_0| &= |\mathbf{A}| - |\mathbf{A}_1| \cdot \frac{\lambda v(v-1) - \lambda w(w-1)}{k} - \lambda w(v-w) \\ &= \frac{\lambda(v-w)}{k}\{v - 1 - (k-1)w\}. \end{aligned}$$

Now let S be the symmetric group on X and $\pi \in S$ be a permutation. For each cyclically ordered set $M = (x_1, x_2, \ldots, x_m)$ of X, let

$$\pi(M) = (\pi(X_1), \pi(X_2), \ldots, \pi(X_m)),$$
$$\pi(A) = \{\pi(B) \mid B \in \mathbf{A}\}.$$

Let G be the subgroup of S fixing Y, then

$$|G| = w!(v-w)!$$

and for any $\pi \in G$, $(X, Y, \pi(\mathbf{A}))$ is a simple $(v, w; k, \lambda)$-IPMD.

Now let $(X, Y, \mathbf{A})$ and $(X, Y, \mathbf{B})$ be a simple $(v, w; k, \lambda_1)$-IPMD and a $(v, w; k, \lambda_2)$-IPMD, respectively. For two given blocks $B_1 \in \mathbf{A}$ and $B_2 \in \mathbf{B}$, if $|B_1 \cap Y| \neq |B_2 \cap Y|$, then there doesn't exist $\pi \in G$ such that $\pi(B_1) = B_2$. If $|B_1 \cap Y| = |B_2 \cap Y| = 0$, then the number of permutations $\pi \in G$ such that $\pi(B_1) = B_2$ is $w!k(v-w-k)!$. If $|B_1 \cap Y| = |B_2 \cap Y| = 1$, then the number of such permutations is

$$(w-1)!\,(v-w-k+1)!.$$

Let n be the number of permutations $\pi \in G$ such that

$$|\pi(\mathbf{A}) \cap \mathbf{B}| \geq 1,$$

then we have

$$\begin{aligned} n \leq\, & \lambda_1\lambda_2 w^2(v-w)^2(w-1)!\,(v-w-k+1)! \\ &+ \frac{\lambda_1\lambda_2}{k^2}(v-w)^2\{v-1-(k-1)w\}^2 w!\,k(v-w-k)! \\ =\, & \frac{\lambda_1\lambda_2}{k}(v-w)^2 w!\,(v-w-k)!\,\{kw(v-w-k+1) \\ &+ (v-1-(k-1)w)^2\} \\ <\, & w!\,(v-w)!. \end{aligned}$$

Thus there exists a permutation $\pi \in G$ such that $\pi(\mathbf{A}), \mathbf{B}$ have no blocks in common and therefore $(X, Y, \pi(\mathbf{A}) \cup \mathbf{B})$ is a simple $(v, w; k, \lambda_1 + \lambda_2)$-IPMD.
∎

As corollaries of the above theorem, we have the following constructions for simple Mendelsohn designs.

Theorem 2.4. *If there exist a simple $(v, w; k, \lambda_1)$-IPMD, a simple $(v, w; k, \lambda_2)$-IPMD and a simple $(w, k, \lambda_1 + \lambda_2)$-PMD such that*

$$\lambda_1\lambda_2(v-w)(v-w-k)! \cdot$$
$$\{kw(v-w-k+1) + (v-1-(k-1)w)^2\} < k(v-w-1)!$$

Then there exists a simple $(v, k, \lambda_1 + \lambda_2)$-PMD containing a simple $(w, k, \lambda_1 + \lambda_2)$-PMD as a subdesign.

Proof: Let $(X, Y, \mathbf{B}_1)$ be a simple $(v, w; k, \lambda_1)$-IPMD, $(X, Y, \mathbf{B}_2)$ be a simple $(v, w; k, \lambda_2)$-IPMD and $(Y, \mathbf{B}_0)$ be a simple $(w, k, \lambda_1 + \lambda_2)$-PMD. By Theorem 2.3, there is a permutation $\pi \in G$ such that $\pi(\mathbf{B}_1) \cap \mathbf{B}_2 = \phi$. Now let $\mathbf{A} = \mathbf{B}_0 \cup \{\cup_{i=1}^{2} \pi(B_i)\}$, then $(X, \mathbf{A})$ is a simple $(v, k, \lambda_1 + \lambda_2)$-PMD containing $(Y, \mathbf{B}_0)$ as a subdesign. ∎

Theorem 2.5. *If there exists a simple (v, k, λ_1)-PMD containing a subdesign (w, k, λ_1)-PMD, a simple (v, k, λ_2)-PMD containing a subdesign (w, k, λ_2)-PMD and*

$$\lambda_1\lambda_2(v-w)(v-w-k)!\{kw(v-w-k+1)(v-1-(k-1)w)^2\}$$
$$< k(v-w-1)!$$
$$\lambda_1\lambda_2 w(w-k)!(w-1) < k(w-2)!$$

Then there exists a simple $(v, k, \lambda_1 + \lambda_2)$-PMD containing a subdesign $(w, k, \lambda_1 + \lambda_2)$-PMD.

Proof: Let $(X, \mathbf{A})$ be a simple (v, k, λ)-PMD which contains $(Y, \mathbf{B})$ as a subdesign (w, k, λ)-PMD, then $(X, Y, \mathbf{A}, \mathbf{B})$ is a simple $(v, w; k, \lambda)$-IPMD. We also note that a $(w, 0; k, \lambda)$-IPMD is in fact a (w, k, λ)-PMD. Now the conclusion follows from Theorem 2.4. ∎

Existence of Simple Perfect Mendelsohn Designs

We know that any (v, k, λ)-PMD contains $\lambda v(v-1)/k$ blocks and a v-set contains $\binom{v}{k}(k-1)!$ cyclically ordered k-subsets. So we have the following necessary conditions for the existence of a simple (v, k, λ)-PMD:

$$\lambda v(v-1) \equiv 0 \pmod{k},$$
$$\lambda \leq (v-2)(v-3)\ldots(v-k+1). \tag{3}$$

The main purpose of this section is to prove that the necessary conditions (3) are also sufficient for the existence of a simple (v, k, λ)-PMD with (i) $k = 3$ and λ even and (ii) $k = 4, \lambda = 2, 3, \text{or}, 4$. Partial results for $k \geq 5$ are also obtained.

Lemma 3.1 [15]. *If $v \equiv 0 \pmod 2$, then there exists a pure $(v, 3, (v-2)/2)$-PMD.*

Theorem. *For any positive integer λ, there exists a simple $(v, 3, \lambda)$-PMD if and only if*

$$\begin{aligned} &\lambda v(v-1) \equiv 0 \pmod 3, \quad \lambda \leq v-2, \\ &(v, \lambda) \neq (6,1), (6,3). \end{aligned} \tag{4}$$

Proof: Let V be a v-set and A be the set of all the $v(v-1)(v-2)/3$ cyclicaly ordered 3-subsets of V, then $(V, \mathbf{A})$ is a simple $(v, 3, v-2)$-PMD. Further, if $(V, \mathbf{B})$ is a simple $(v, 3, \lambda)$-PMD, then $(V, \mathbf{A} \backslash B)$ is obviously a simple $(v, 3, v-2-\lambda)$-PMD. So the non-existence of a simple (6,3,3)-PMD follows from the non-existence of a (6,3,1)-PMD. By (3), if there exists a simple $(v, 3, \lambda)$-PMD, then $\lambda v(v-1) \equiv 0 \pmod 3$ and $\lambda \leq v-2$. So (4) is necessary for the existence of a simple $(v, 3, \lambda)$-PMD. To prove the sufficiency, by Lemma 3.1, it is sufficient to consider the problem for $\lambda < (v-2)/2$.

If $v \equiv 0$ or $1 \pmod 3$, we may suppose $v \neq 6$ and write $v = 3t$ or $3t+1$. It is proved in [10] that there exist $2t$ pairwise disjoint $(v, 3, 1)$-PMDs for each $v = 3t$ or $3t+1$, $v \neq 6$. Let $(V, \mathbf{B}_i)$, $1 \leq i \leq 2t$, be these $(v, 3, 1)$-PMDs. For $1 \leq \lambda \leq 2t$, let $\mathbf{B} = \cup_{1 \leq i \leq \lambda} \mathbf{B}_i$, then $(V, \mathbf{B})$ is a simple $(v, 3, \lambda)$-PMD.

If $v \equiv 5 \pmod 6$ and a $(v, 3, \lambda)$-PMD exists, then $\lambda \equiv 0 \pmod 3$. It is well known that there exists a simple $(v, 3, \lambda)$-BIBD for $v \equiv 5 \pmod 6$ and $\lambda \equiv 0 \pmod 3$, $\lambda \leq v-2$ (see [7]). Let $(V, \mathbf{B})$ be such a simple $(v, 3, \lambda)$-BIBD. We replace each (unordered) triple $\{a, b, c\} \in \mathbf{B}$ by two cyclically ordered triples (a, b, c) and (a, c, b) and let

$$\mathbf{A} = \{(a, b, c), (a, c, b) \mid \{a, b, c\} \in \mathbf{B}\},$$

then $(V, \mathbf{A})$ is a simple $(v, 3, \lambda)$-PMD.

Now the only case remained to be dealt with is $v \equiv 2 \pmod{16}$. If $v = 14$, we form a simple (14,3,3)-PMD as follows:

$$\begin{aligned} &\mathbf{X} = \mathbf{Z}_{13} \cup \{\infty\}. \\ &\mathbf{B} : \{(\infty, 0, 4), (\infty, 0, 10), (\infty, 0, 12), (0, 1, 4), (0, 2, 8), \\ &\quad (0, 3, 12), (0, 4, 3), (0, 5, 7), (0, 6, 11), (0, 7, 2), \\ &\quad (0, 8, 6), (0, 9, 10), (0, 10, 1), (0, 11, 5)\} \pmod{13}. \end{aligned}$$

If $v \neq 14$, write $v = 6t+2$, then $3t \neq 6$ and there exists a simple $(3t, 3, \lambda)$-PMD for each $\lambda \equiv 0 \pmod 3$, $\lambda \leq 3t-3$, as shown above. We form a simple $(6t+2, 3, \lambda)$-PMD for each $\lambda \equiv 0 \pmod 3$, $\lambda \leq 3t-3$, as follows:

Let $X = \{\infty_1, \infty_2, \ldots, \infty_{3t}\}$ and $(X, \mathbf{A})$ be a simple $(3t, 3, \lambda)$-PMD. Let $D(3t+2, \lambda)$ be the multiset containing each $i \in Z_{3t+2} \backslash \{0\}$ λ times. The elements of $D(3t+2, \lambda)$ are called differences. As $\lambda \leq 3t-3$, we can always partition $D(3t+2, \lambda)$ into two parts C and D, where C contains λ differences which can be divided into $\lambda/3$ pairwise distinct groups $C_i = \{c_{i_1}, c_{i_2}, c_{i_3}\}$, $1 \le i \le \lambda/3$, such that $c_{i_1} + c_{i_2} + c_{i_3} \equiv 0 \pmod{(3t+2)}$ for each i, and D contains the remaining $3\lambda t$ differences which can be divided into $3t$ groups D_j, $1 \le j \le 3t$, each D_j contains λ distinct differences: $D_j = \{d_{j_1}, d_{j_2}, \ldots, d_{j_\lambda}\}$. Now let

$$Y = X \cup Z_{3t+2}, \quad \mathbf{B} = \mathbf{A} \cup \mathbf{A}_1,$$

where $\mathbf{A}_1$ consists of the following cyclically ordered triples:

$$(0, c_{i_1}, c_{i_1} + c_{i_2}), \quad (\text{mod } (3t+2)), \quad 1 \le i \le \lambda/3;$$

$$(\infty_j, 0, d_{j_1}), (\infty_j, 0, d_{j_2}), \ldots, (\infty_j, 0, d_{j_\lambda}) \quad (\text{mod } (3t+2))), 1 \le j \le 3t.$$

Then $(Y, \mathbf{B})$ is a simple $(6t+2, 3, \lambda)$-PMD.

This completes the proof. ∎

Let $w = 0$ in Theorem 2.3. We have the following construction for simple perfect Mendelsohn designs.

Theorem 3.2. *If there exist a simple (v, k, λ_1)-PMD and a simple (v, k, λ_2)-PMD and*

$$\lambda_1 \lambda_2 v(v-1)(v-k)! < k(v-2)! \qquad (6)$$

then there exists a simple $(v, k, \lambda_1 + \lambda_2)$-PMD.

Theorem 3.3. *There exists a simple $(v, 4, 2)$-PMD if $v \ge 4$.*

Proof: It is proved that for each $v \equiv 0$ or $1 \pmod 6$, $v > 4$, $v \ne 8, 12$, there exists a $(v, 4, 1)$-PMD. For $v \equiv 0$ or $1 \pmod 4$, $v \ge 5$, $v \ne 8, 12$, let $\lambda_1 = \lambda_2 = 1$ in Theorem 3.2, we obtain a simple $(v, 4, 2)$-PMD. For $v = 4, 8$ and 12, the following designs were constructed in [6]:

$(4,4,2) - PMD: \quad X = Z_3 \cup \{\infty\}$

$\mathbf{B} : \{(\infty, 0, 1, 2,), (\infty, 0, 2, 1)\} \pmod 3$

$(8,4,2) - PMD: \quad X = Z_7 \cup \{\infty\}$

$\mathbf{B} : \{(\infty, 0, 1, 3), (\infty, 0, -1, -3), (0, 1, -2, 2),$
$(0, -1, 2, -2)\} \pmod 7$

$(12,4,2) - PMD: \quad X = Z_{11} \cup \{\infty\}$

$\mathbf{B} : \{(\infty, 0, 1, 5), (\infty, 0, -1, -5), (0, 1, 3, 8),$
$(0, -1, -3, -8), (0, -4, -2, -5), (0, 3, 1, 5)\} \pmod{11}$

It can be readily checked that these designs are simple.

For $v \in \{6, 7, 10, 11, 14, 15, 18, 19, 23\}$, $(v, 4, 2)$-PMDs can also be found in [6] and all of these designs are simple.

Let

$$K_4 = \{4,5,\ldots,12,14,15,18,19,23\}$$

It is proved [8] that for every integer $v \geq 4$, we have $v \in B(K_4)$. Since there exists a simple $(v,4,2)$-PMD for each $v \in K_4$, then as $NPM(4,2)$ is a PBD-closed set, we have proved that there exists a simple $(v,4,2)$-PMD for every $v \geq 4$. ∎

Theorem 3.4. *There exists a simple $(v,4,3)$-PMD if and only if*

$$v \equiv 0 \text{ or } 1 \pmod 4, \quad v \geq 5 \tag{7}$$

Proof: To prove the theorem, we only need to prove the sufficiency. For each $v \equiv 0$ or $1 \pmod 4$, $v > 8$, $v \neq 12$, let $\lambda_1 = 1$ and $\lambda_2 = 2$ in Theorem 3.2. We obtain a simple $(v,4,3)$-PMD from a $(v,4,1)$-PMD and a simple $(v,4,2)$-PMD. For $v = 8$ or 12, the $(v,4,3)$-PMD's constructed in [6] are simple. To complete the proof of the theorem, we construct a simple (5,4,3)-PMD as follows:

$$X = Z_5$$

$$\begin{aligned}\mathbf{B}:\{&(0,1,2,3),(0,2,4,1),(0,3,1,4),(0,4,3,2),(1,3,4,2);\\&(1,2,3,4),(1,3,0,2),(1,4,2,0),(1,0,4,3),(2,4,0,3);\\&(2,3,4,0),(2,4,1,3),(2,0,3,1),(2,1,0,4),(3,0,1,4)\}.\end{aligned}$$

Now we consider the existence of simple $(v,4,4)$-PMDs.

Lemma 3.2. *If $p \geq 5$ is a prime, then there exists a simple $(p,4,4)$-PMD and a simple $(p+1,4,4)$-PMD. If $p > 7$ is a prime, then there exists a pure $(p,4,4)$-PMD and a pure $(p+1,4,4)$-PMD.*

Proof: Let $X = Z_p$ and $B = (0,1,2,4)$. For each $t \in Z_p \backslash \{0\}$, let $tB = t \cdot (0,1,2,4) = (0,t,2t,4t)$. Let **B** be obtained by developing the $p-1$ base blocks $B, 2B, \ldots, (p-1)B$. Then $(Z_p, \mathbf{B})$ is a simple $(p,4,4)$-PMD if $p \geq 5$ and it is pure if $p > 7$.

Now in the above $(p,4,4)$-PMD, let the base blocks B and $(p-1)B$ be replaced by the following four base blocks:

$$(0,1,2,\infty),(0,2,-2,\infty),(0,-1,-3,\infty),(0,-1,3,\infty).$$

Then we obtain a simple $(p+1,4,4)$-PMD on $Z_p \cup \{\infty\}$ if $p \geq 5$ and it is pure if $p > 7$. ∎

Lemma 3.3. *If there exists a simple (or pure)* $(v,4,4)$*-PMD and* $p \geq 2v+1$ *is a prime, then there exists a simple (or pure, respectively)* $(v+p,4,4)$*-PMD containing the* $(v,4,4)$*-PMD as a subdesign.*

Proof: Since there is a simple $(v,4,4)$-PMD, then $v \geq 5$ and so $p \geq 2v+1 \geq 11$. Let $Y = \{\infty_1, \infty_2, \ldots, \infty_v\}$ and $(Y, \mathbf{B}_0)$ be a simple (or pure) $(v,4,4)$-PMD. For $t \in Z_p \backslash \{0\}$, let the $p-1$ base blocks tB be defined as in Lemma 3.2. Now for $t = 1,2,\ldots,v$, let tB and $(-t)B$ be replaced by the following base blocks:

$$t \cdot (0,1,2,\infty_t), t \cdot (0,2,-2,\infty_t), -t \cdot (0,1,3,\infty_t), -t \cdot (0,1,-3,\infty_t).$$

Let $\mathbf{B}_1$ be the block set obtained by developing the above $4v$ base blocks and the $p-2v-1$ base blocks $t \cdot B$, $t = \pm(v+1), \ldots, \pm(p-1)/2$. Then $(Z_p \cup Y, \mathbf{B}_0 \cup \mathbf{B}_1)$ is a simple (or pure, repectively) $(v+p,4,4)$-PMD containing $(Y, \mathbf{B}_0)$ as a subdesign. ∎

Lemma 3.4. *There exists a pure* $(v,4,4)$*-PMD for* $v = 9$*, 10 or* 15.

Proof: For $v = 9$, let $X = GF(9)$ and x be a fixed primitive element. Let $\mathbf{B}$ be obtained by developing the 8 base blocks $x^i \cdot (0,1,x,x^2)$, $0 \leq i \leq 7$. Then $(GF(9), \mathbf{B})$ is a pure $(9,4,4)$-PMD. When $v = 10$, take $X = GF(9) \cup \{\infty\}$. Let $x^0 \cdot (0,1,x,x^2) = (0,1,x,x^2)$ and $x^4 \cdot (0,1,x,x^2) = (0,-1,-x,-x^2)$ be replaced by the following 4 base blocks:

$$(\infty,0,1,x), (\infty,0,x^2-x,-x),$$

$$(\infty,0,-(x-1),-(x^2-1)), (\infty,0,-1,x^2-1).$$

Let $\mathbf{A}$ be obtained by developing the above 4 base blocks and the 6 base blocks $x^i \cdot (0,1,x,x^2)$, $1 \leq i \leq 7$, $i \neq 4$. Then $(X, \mathbf{A})$ is a pure (10,4,4)-PMD. For $v = 15$, $X = Z_{14} \cup \{\infty\}$. Base blocks of $\mathbf{B}$ (mod 14):

$$\begin{array}{llll}
(0,5,3,8), & (0,6,13,5), & (0,3,12,8), & (0,11,9,2), \\
(0,8,10,6), & (0,13,3,7), & (0,11,10,3), & (0,1,2,7), \\
(0,12,1,5), & (0,13,9,3), & (0,1,2,6), & \\
(\infty,0,2,4), & (\infty,0,3,6), & (\infty,0,2,1), & (\infty,0,5,3).
\end{array}$$

∎

Lemma 3.5. *There exists a simple* $(v,4,4)$*-PMD if* $5 \leq v \leq 21$.

Proof: If $v = 5,6,7,8,11,12,13,14,17,18,19,20$, then there exists a simple $(v,4,4)$-PMD by Lemma 3.2. For $v = 9, 10$ or 15, the existence of a simple $(v,4,4)$-PMD has been proved in Lemma 3.4. Let $v = 5$ and $p = 11$ in Lemma 3.3, then we have a simple $(16,4,4)$-PMD. Since $NPM(4,4)$ is a PBD-close set by Theorem 2.1 and there exists a (21,5,1)-BIBD, then we have a simple (21,4,4)-PMD. This completes the proof. ∎

Theorem 3.5. *There exists a simple $(v,4,4)$-PMD if and only if $v \geq 5$.*

Proof: For $5 \leq v \leq 21$, there exists a simple $(v,4,4)$-PMD by Lemma 3.5. Let $p \geq 17$ be a prime and $5 \leq v \leq (p-1)/2$, then by Lemma 3.3, there exists a simple $(v+p,4,4)$-PMD provided there exists a simple $(v,4,4)$-PMD. We prove the theorem by induction. Let $p = 17,19,23,29,37,47,61,83$ and 113 subsequently, this establishes the existence of a simple $(v,4,4)$-PMD for every v with $22 \leq v \leq 169$. Now let n be an integer such that $n \equiv 1 \pmod 2$, $n \not\equiv 0 \pmod 3$ and $n \geq 29$, then there exists a transversal design $TD(6,n)$. For $0 \leq s \leq n-5$, delete s points from a fixed group of the $TD(6,n)$, this gives a $(v,K,1)$-PBD with $v = 6n-s$ and $k = \{5,6,n-s,n\}$. Since $s \leq n-5$, then $n-s \geq 5$, so each block size of the $(v,K,1)$-PBD is at least 5. It can be easily checked that for each $v \geq 169$, there exist n and s satisfying the above conditions. As $NPM(4,4)$ is a PBD-closed set, this proves the existience of a simple $(v,4,4)$-PMD for every $v \geq 169$. ∎

Asymptotic Existence of Simple Perfect Mendelsohn Designs for $k \geq 5$

If $k \geq 5$, then for any fixed λ_1 and λ_2, there exists a $v_0 = v_0(k,\lambda_1,\lambda_2)$ such that for any $v > v_0$, we have

$$\lambda_1\lambda_2 v(v-1)(v-k)! < k(v-2)!$$

Thus, if $k \geq 5$ and there exists a simple (v,k,λ_0)-PMD and $\lambda \equiv 0 \pmod{\lambda_0}$, $v > v_0$, then by Theorem 3.2 there exists a simple (v,k,λ)-PMD. This proves the following theorem:

Theorem 4.1. *If $k \geq 5$ and there exists a simple (v,k,λ_0)-PMD and $m\lambda_0^2 v(v-1)(v-k)! < k(v-2)!$, then there exists a simple $(v,k,m\lambda_0)$-PMD.*

As it is proved in [3] that for $(k,\lambda) = (5,1)$, there exists a $(v,5,1)$-PMD if and only if

$$v \equiv 0 \text{ or } 1 \pmod 5$$

and

$$V \notin E = \{6,10,15,20,26,30,36,46,50,56,66,86,90, 110,126,130,140,146,186,206,246,286\}.$$

Thus we have the following theorem.

Theorem 4.2. *If $\lambda \not\equiv 0 \pmod 5$ and $\lambda < 5(v-2)(v-3)(v-4)/(v(v-1))$, $v \notin E$, then there exists a simple $(v,5,\lambda)$-PMD if and only if $v \equiv 0$ or $1 \pmod 5$.*

References

1. F. E. Bennett, *Direct constructions for perfect 3 cyclic designs*, Ann. Discrete Math. **15** (1982), 63–68.
2. F. E. Bennett and N. S. Mendelsohn, *On pure cyclic triple systems and semisymmetric quasigroups*, Ars Combinatoria **5** (1978), 13–22.
3. F. E. Bennett, K. T. Phelps, C. A. Rodger and L. Zhu, *Constructions of perfect Mendelsohn designs*, preprint.
4. F. E. Bennett and H. Shen, *On indecomposable pure Mendelsohn triple systems*, Discrete Math. (to appear).
5. F. E. Bennett, J. Yin and L. Zhu, *Existence of perfect Mendelsohn designs with* $k = 5$ *and* $\lambda > 1$. preprint
6. F. E. Bennett, Xuebin Zhang and L. Zhu, *Perfect Mendelsohn designs with block size four*, Ars Combinatoria **29** (1990), 65–72.
7. M. Dehon, *On the existence of 2-designs* $S_\lambda(2,3,4)$ *without repeated blocks*, Discrete Math. **43** (1983), 155–171.
8. D. A. Drake and J. A. Larson, *Pairwise balanced designs whose line sizes do not divide six,*, J. of Combinatorial Theory (A) **34** (1983), 266–300.
9. D. F. Hsu and A. D. Keedwell, *Generalized complete mappings, neofields, sequenceable groups and block designs, II*, Pacific J. Math. **117** (1985), 291–312.
10. C. C. Lindner, *On the number of disjoint Mendelsohn triple systems*, J. of Combinatorial Theory (A) **30** (1981), 326–330.
11. N. S. Mendelsohn, *A natural generalization of Steiner triple systems* in "Computers in Number Theory", (edited by A. O. L. Atkin and B. J. Birch), Academic Press, New York, 1971, pp. 323–338.
12. N. S. Mendelsohn, *Perfect cyclic designs*, Discrete Math. **20** (1971), 63–68.
13. R. Rees and D. R. Stinson, *On the existence of incomplete designs of block size four having one hole*, Utilitas Math. **38** (1988), 119–152.
14. H. Shen, *On the existence of simple and indecomposable block designs with block size 4*, J. Combinatorial Math. and Combinatorial Computing **8** (1990), 31–38.
15. H. Shen, *On the existence of pure Mendelsohn triple systems*, preprint.
16. Xuelin Zhang, *On the existence of* $(v, 4, 1)$*-PMD*, Ars Combinatoria **29** (1990), 3–12.
17. L. Zhu, *Perfect Mendelsohn designs*, J. Combinatorial Math. and Combinatorial Computing **5** (1989), 43–54.