Bandwidth for the Sum of k Graphs

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Abstract. The sum of a set of graphs $G_1, G_2, \ldots, G_k$, denoted $\sum_{k=1}^{k} G_i$, is defined to be the graph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) \cup \{uw: u \in V(G_i), w \in V(G_j) \text{ for } i \neq j\}$. In this paper, the bandwidth $B(\sum_{k=1}^{k} G_i)$ for $|V(G_i)| = n_i \geq n_{i+1} = |v(G_{i+1}|, (1 \leq i < k)$ with $B(G_1) \leq \lceil n_1/2 \rceil$ is established. Also, tight bounds are given for $B(\sum_{k=1}^{k} G_i)$ in other cases. As consequences, the bandwidths for the sum of a set of cycles, a set of paths, and a set of trees ae obtained.

1. Introduction

Bandwidth on graphs, and the analogous problem of bandwidth on matrices, has been studied since the early 1950s (see [1].) Following the notation of [1] and [9], we may define bandwidth as follows. Let $G = (V, E)$ be a graph on n vertices. A $1-1$ mapping $f : V \rightarrow \{1, 2, \ldots, n\}$ will be called a *proper numbering* of G. The *bandwidth of a proper numbering f of G*, denoted $B_f(G)$, is the number

$$\max\{|f(u) - f(v)| : \text{ edge } uv \in E(G)\},$$

and the *bandwidth of G*, denoted $B(G)$, is the number

$$\min\{B_f(G) : f \text{ is a numbering of } G\}.$$

The decision problem corresponding to finding the bandwidth of an arbitrary graph was shown to be NP-complete in [11]. In [7] it was shown that the problem is NP-complete even for trees of maximum degree 3. However, the bandwidth problem has been solved for many special types of graphs. It is easy to find the bandwidth for graphs such as K_n, P_n, C_n and $K_{1,n}$ and others. [8] has established the bandwidth for hypercubes (or n-cubes). [3] found the bandwidth for $P_n \times C_m$ and for a number of other special graphs. [2] and [1] contain a number of survey results pertaining to solved problems. [1] also surveys a number of bounds on bandwidth. [6] provides results relating to the relationship between bandwidth and bandsize, and [10] provides some insight on the relationship between bandwidth and VLSI layout width.

[9] gives the bandwidth $B(G_1 + G_2)$ for $|V(G_1)| = n_1 \geq n_2 = |V(G_2)|$ with $B(G_1) \leq \lceil n_1/2 \rceil$, and also provides bounds for $B(G_1 + G_2)$ in other cases.

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2. Bandwidth on finite graph sums

Theorem 1. *Let* $G = G_1 + G_2 + \cdots + G_k$ *with* $n_i = |V(G_i)|$, *and* $n_1 \geq n_2 \geq \cdots \geq n_k$. *If* $B(G_1) < \lceil n_1/2 \rceil$, *then* $B(G) = \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil$.

Proof: For $n_1 \leq 2$ it is trivial to check the result. We assume $n_1 \geq 3$. First let $y_j = j$ for $1 \leq j \leq \lfloor n_1/2 \rfloor$, $y_j = \sum_{i=2}^{k} n_i + j$ for $\lfloor n_1/2 \rfloor + 1 \leq j \leq n_1$. Let $f_1 : V(G_1) \to \{1,2,\ldots,n_1\}$ be a proper numbering such that $B_{f_1}(G_1) = B(G_1)$ and $f_i : V(G_i) \to \{1,2,\ldots,n_i\}$ be a proper numbering such that $B_{f_i}(G_i) = B(G_i)$ for $1 < i \leq k$. Now, we construct a proper numbering $f : V(\sum_{i=1}^{k} G_i) \to \{1,2,\ldots,\sum_{i=1}^{k} n_i\}$ as follows: $f(v_i) = y_j$ if and only if $f_1(v_i) = j$; $f(w_i) = \lfloor n_1/2 \rfloor + \sum_{p=2}^{q-1} n_p + j$ if and only if $f_q(w_i) = j$. Then,

$$\max_{v_iv_j \in E(G_1)} |f(v_i) - f(v_j)| \leq B(G_1) + \sum_{i=2}^{k} n_i \leq \lceil n_1/2 \rceil + \sum_{i=2}^{k} n_i - 1$$

$$= \sum_{i=2}^{k} n_i + n_1 - \lceil (n_1+1)/2 \rceil = \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil,$$

$$\max_{w_iw_j \in E(\sum_{i=2}^{k} G_i)} |f(w_i) - f(w_j)| \leq \sum_{i=2}^{k} n_i - 1 < \sum_{i=2}^{k} n_i + \lceil n_1/2 \rceil - 1$$

$$= \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil,$$

and for

$v_i \in V(G_1)$ and $w_j \in V(G_l)$ with $l \neq 1$,

$$\max_{v_iw_j} |f(v_i) - f(w_j)| = \max\left\{ \sum_{i=1}^{k} n_i - (\lfloor n_1/2 \rfloor + 1), \sum_{i=2}^{k} n_i + \lfloor n_1/2 \rfloor - 1 \right\}$$

$$= \sum_{i=2}^{k} n_i + \lceil n_1/2 \rceil - 1 = \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil.$$

Therefore,

$$\max_{x_ix_j \in E(G)} |f(x_i) - f(x_j)| = \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil$$

which implies that $B(G) \leq \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil$.

To see that $B(G) \geq \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil$, suppose that $l : V(G) \to \{1,2,\ldots,\sum_{i=1}^{k} n_i\}$ is a minimal proper numbering. Let $l(v) = 1$ and $l(w) = n_i$, then either v and w are in $V(G_1)$ or they are in $V(G_i), i \neq 1$, since $B(G) \leq \sum_{i=1}^{j} n_i - \lceil (n_1+1)/2 \rceil$. If v and w are in $V(G_1)$ then all labels $1, 2, \ldots, \lceil (n_1 + 1)/2 \rceil - 1$ are in $V(G_1)$, as are all labels $\sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil + 2, \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil + 3, \ldots, \sum_{i=1}^{k} n_i$, since $B(G) \leq \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$. This implies $B(G) \geq \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$ since at least one of the vertices with labels $\lceil (n_1 + 1)/2 \rceil$ and $\sum_{i=1}^{k} ni - \lceil (n_1 + 1)/2 \rceil + 1$ is not in $V(G_1)$. If v and w are in $V(G_i), i \neq 1$, then, similarly, $1, 2, \ldots, \lceil (n_1+1)/2 \rceil - 1, \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil + 2, \sum_{i=1}^{k} n_i - \lceil (n_1 + 1) \rceil + 3, \ldots, \sum_{i=1}^{k} n_i$ must be labels of the vertices of G_i. But then we either get a contradiction or obtain $B(G) \geq \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$ since $n_i \leq n_1$. Therefore $B(G) \geq \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$. Combining the inequalities, we have $B(G) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$. ∎

Alternatively, the second part of the proof in Theorem 1 follows from Theorem 4.1.2 in [1] which appeared initially in [5].

Corollary 1. *If $T_1, T_2, \ldots, T_k$ are each trees such that $|V(T_i)| = n_i \geq n_{i+1} = |V(T_{i+1})|$, $1 \leq i < k$, and T_1 is not a star of even order, then $B(\sum_{i=1}^{k} T_i) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$.*

Proof: By a result in [4], we know $B(T) < n/2$ if T is a tree of order n and T is not a star of even order. So we must have $B(T_1) < \lceil n_1/2 \rceil$. The result, then, follows immediately from Theorem 1. ∎

Corollary 2. *If $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ are each paths with $n_i \geq n_{i+1}$, $1 \leq i < k$, and $n_1 \geq 3$, then $B(\sum_{i=1}^{k} P_{n_i}) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$.*

Since a path is a special tree, this corollary follows immediately from Corollary 1.

Corollary 3. *If $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$, are each cycles with $n_i \geq n_{i+1}$, $1 \leq i < k$, and $n_1 \geq 5$, then $B(\sum_{i=1}^{k} C_{n_i}) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$.*

Proof: Since $B(C_{n_1}) = 2$, $B(C_{n_1}) < \lceil n_1/2 \rceil$ if $n_1 \geq 5$. Thus it follows from Theorem 1 that $B(\sum_{i=1}^{k} C_{n_i}) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil$ for $n_1 \geq 5$. ∎

Theorem 2. *Let $G_1, G_2, \ldots, G_k$ be a set of graphs such that $n_1 = |V(G_1)| = c$ with $B(G_1) = c/2$, $n_i = |V(G_i)| = c$ and $B(G_i) \geq c/2$ for $2 \leq i \leq t$, and $n_i = |V(G_i)| < c$ for $t + 1 \leq i \leq k$ where c is even and $1 \leq t \leq k$. Then $B(\sum_{i=1}^{k} G_i) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil + 1$.*

Proof: For $c = 2$ the result is trivial. So we assume $c \geq 4$. For convenience, let $G = \sum_{i=1}^{k} G_i$ and $n = \sum_{i=1}^{k} n_i$. Similarly to the first half of the proof of Theorem 1, we obtain $B(\sum_{i=1}^{k} G_i) \leq \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil + 1$. Let $f : V(G) \to \{1, 2, \ldots, n\}$ be a minimal proper numbering such that $B_f(G) = B(G)$ and let $f(u) = 1, f(v) = n$. Since $B(G) \leq n - \lceil (n_1 + 1)/2 \rceil + 1$ and $n_1 \geq 4$, there exists j such that $u, v \in V(G_j)$. Suppose $B(G) \neq n - \lceil (n_1+1)/2 \rceil + 1$, then $B(G) \leq n - \lceil (n_1+1)/2 \rceil$, which implies that all labels $1, 2, \ldots, c/2, n-c/2+1, n-c/2+2, \ldots, n$ are in G_j, and so $j \leq t$. Since $B(G) \leq n - \lceil (c+1)/2 \rceil$, $\max_{xy \in E(G_i)} |f(x) - f(y)| \leq n - \lceil (c+1)/2 \rceil$. It follows that $B_{f_1}(G_j) < c/2$ for the induced proper numbering f_1 of G_j from f such that $f_1(w) = f(w)$ if and only if $f(w) \leq c/2$ and $f_1(w) = f(w) - (n - n_j)$ if and only if $f(w) > c/2$ which contradicts $B(G_j) \geq c/2$. Therefore $B(\sum_{i=1}^{k} G_i) = \sum_{i=1}^{k} n_i - \lceil (n_1 + 1)/2 \rceil + 1$. ∎

Corollary 4. *If $T_1, T_2, \ldots, T_k$ are each trees such that T_i is a star of even order $n_i = c$ for $1 \leq i \leq t$, and T_i is of order $n_i < c$ for $t+1 \leq i \leq k$, then $B(\sum_{i=1}^{k} T_i) = \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil + 1$, where c is an even number and $1 \leq t \leq k$.*

This corollary follows immediately from Theorem 2 since $B(T_1) = n_1/2$.

Theorem 3. *Let $G_1, G_2, \ldots, G_k$ be a set of graphs with $n_i = |V(G_i)| \geq |V(G_{i+1})| = n_{i+1}$, n_1 is odd, and $B(G_1) = \lceil n_1/2 \rceil = (n_1+1)/2$, then $\sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil \leq B(\sum_{i=1}^{k} G_i) \leq \sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil + 1$.*

The proof of the upper bound is similar to the first half of the proof of Theorem 1, while the proof of the lower bound is similar to the second half of Theorem 1.

To see that the two values $\sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil$ and $\sum_{i=1}^{k} n_i - \lceil (n_1+1)/2 \rceil + 1$, as given in Theorem 3, may each be achieved we consider the following two examples.

Example 1: Let $G_1 = H_1 \cup H_2$, where $H_1 \cong H_2 \cong K_{k+2}$ and $|V(H_1) \cap V(H_2)| = 3$, $G_2 = D_1 \cup D_2$ with $D_1 \cong D_2 \cong K_{k+1}$ and $|V(D_1) \cap V(D_2)| = 2$, $G_3 = E_1 \cup E_2$ with $E_1 \cong E_2 \cong K_k$ and $|V(E_1) \cap V(E_2)| = 1$. Then, $n_1 = 2k+1, n_2 = 2k, n_3 = 2k-1$, and $B(G_1) = k+1 = (n_1+1)/2$. So G_1, G_2 and G_3 satisfy the conditions of Theorem 3. Thus, $B(G_1 + G_2 + G_3) \leq \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil + 1$. On the other hand, since $\delta(G_1 + G_2 + G_3) \geq 5k$, $B(G_1 + G_2 + G_3) \geq 5k = \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil + 1$. Therefore $B(G_1 + G_2 + G_3) = \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil + 1$.

Example 2: Let G_1 be as in Example 1. Let G_2 be the graph of order $2k$ with no edges and $G_3 = K_2$. Then G_1, G_2 and G_3 satisfy the conditions of Theorem 3. Thus $B(G_1 + G_2 + G_3) \geq \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil$, where $n_1 = 2k+1, n_2 = 2k$ and $n_3 = 2$. On the other hand, the proper numbering, f, shown in Figure 1 has $B_f(G_1 + G_2 + G_3) = \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil$. Therefore $B(G_1 + G_2 + G_3) = \sum_{i=1}^{3} n_i - \lceil (n_1+1)/2 \rceil$.

Corollary 5. *Let $G_1, G_2, \ldots, G_k$ be a set of graphs with $G_1 \cong G_2 \cong \cdots \cong G_k$ and $n = |V(G_i)|$ for all $1 \leq i \leq k$, n is odd and $B(G_1) = \lceil n/2 \rceil = (n+1)/2$ then $((2k-1)n - 1)/2 \leq B(\sum_{i=1}^{k} G_i) \leq ((2k-1)n + 1)/2$.*

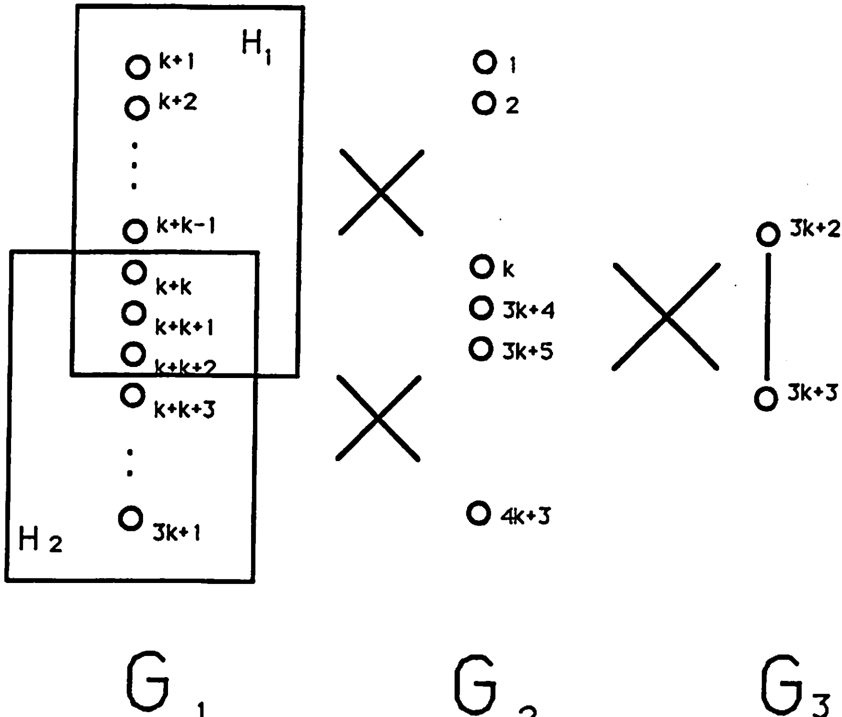

Figure 1: Proper numbering of $G_1 + G_2 + G_3$, as in Example 2

Corollary 5 is simply a special case of Theorem 3.

To this point, all results require that $G_1, G_2, \ldots, G_k$ satisfy conditions $n_i = |V(G_i)| \geq |V(G_{i+1})| = n_{i+1}$, and $B(G_1) \leq \lceil n_1/2 \rceil$. Next we consider the case where $B(G_1) > \lceil n_1/2 \rceil$. We provide the following lower and upper bounds in that case.

Theorem 4. *Let $G_1, G_2, \ldots, G_k$ be a set of graphs with $n_i = |V(G_i)|$, $n_i \geq n_{i+1}$, $1 \leq i < k$, and $B(G_1) > \lceil n_1/2 \rceil$.* $\sum_{i=2}^{k} n_i + n_1/2 - 1 \leq B(\sum_{i=1}^{k} G_i) \leq \min\{B(\sum_{i=1}^{k-1} G_i) + n_k, \max\{B(G_k) + \sum_{i=1}^{k-1} n_i, \lceil n_k/2 \rceil + \sum_{i=1}^{k-1} n_i - 1\}\}$.

Proof: The proof of the lower bound is similar to the second half of the proof of Theorem 1. To see that $B(\sum_{i=1}^{k} G_i) < \min\{B(\sum_{i=1}^{k-1} G_i) + n_k, \max\{B(G_k) + \sum_{i=1}^{k-1} n_i, \lceil n_k/2 \rceil + \sum_{i=1}^{k-1} n_i = 1\}\}$, let $k = 2$. $n_1/2 + n_2 - 1 \leq B(G_1 + G_2) \leq \min\{B(G_1) + n_2, \max\{B(G_2) + n_1, \lceil n_2/2 \rceil + n_1 - 1\}\}$ has been proved in [9]. Proceeding inductively we assume that $k = k'$ is true (that is $\sum_{i=1}^{k'} n_i + n_1/2 - 1 \leq B(\sum_{i=1}^{k'} G_i) \leq \min\{B(\sum_{i=1}^{k'-1} G_i) + n_{k'}, \max\{B(G_{k'}) + \sum_{i=1}^{k'-1} n_i,$

$\lceil n_k/2\rceil+\sum_{i=1}^{k'-1} n_i-1\}\})$. For $k = k'+1$, since $|V(\sum_{i=1}^{k'} G_i)| > |V(G_{k'+1})|$ and $B(\sum_{i=1}^{k'} G_i) \geq \sum_{i=2}^{k'} n_i + n_1/2 - 1 > \lceil\sum_{i=1}^{k'} n_i/2\rceil$ applying Theorem 5 in [9] we obtain $B(\sum_{i=1}^{k'} G_i + G_{k'+1}) = B(\sum_{i=1}^{k'+1} G_i) \leq \min\{B(\sum_{i=1}^{k'} G_i) + n_{k'+1}, \max\{B(G_{k'+1} + \sum_{i=1}^{k'} n_i, \lceil n_{k'+1}/2\rceil + \sum_{i=1}^{k'} n_i - 1\}\}$. Then by mathematical induction, we obtain $B(\sum_{i=1}^{k} G_i) \leq \min\{B(\sum_{i=1}^{k-1} G_i) + n_k, \max\{B(G_k) + \sum_{i=1}^{k-1} n_i, \lceil n_k/2\rceil + \sum_{i=1}^{k-1} n_i - 1\}\}$. ∎

To see that the upper bound and lower bound as given in Theorem 4 may be achieved, we consider the following examples.

Example 3: Let $G_1 = P_5 + P_5, G_2 \cong G_3 \cdots \cong G_{k-1} = P_4, G_k = P_3$. Then $n_1 = 10, n_i = 4$ for $2 \leq i \leq k-1$ and $n_k = 3$. Since $B(G_1) = 7 > \lceil 10/2\rceil$, $G_1, G_2, \ldots, G_k$ satisfy the conditions in Theorem 4. By Corollary 1, $B(\sum_{i=1}^{k} G_i) = B(P_5 + P_5 + P_4 + \cdots + P_4 + P_3) = 5+5+4(k-2)+3-3 = 10+4(k-2)$ and $B(\sum_{i=1}^{k-1} G_i) = 5+5+4(k-2)-3 = 7+4(k-2)$. It follows that $\min\{B(\sum_{i=1}^{k-1} G_i+n_k, \max\{B(G_k)+\sum_{i=1}^{k-1} n_i, \lceil n_k/2\rceil+\sum_{i=1}^{k-1} n_i-1\}\} = \min\{7+4(k-2)+3, \max\{1+10+4(k-2), 2+10+4(k-2)-1\}\} = 10+4(k-2)$.

Example 4: Let $G_i = K_{n_i}$ for $1 \leq i \leq k-1$ with $n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq 4$ and $G_k = C_4$. Since $B(G_1) = n_1 - 1 > \lceil n_1/2\rceil$ $G_1, G_2, \ldots, G_k$ satisfy the conditions in Theorem 4. Since $\sum_{i=1}^{k} G_i = K_{n_1} + K_{n_2} + \cdots + K_{n_{k-1}} + C_4 = C_4 + K_1 + K_1 + \cdots + K_1$ (with $\sum_{i=1}^{k-1} n_i$ copies of K_1), it follows from Theorem 2 that $B(\sum_{i=1}^{k-1} G_i) = \sum_{i=1}^{k-1} n_i + 4 - \lceil(4+1)/2\rceil + 1 = \sum_{i=1}^{k-1} n_i + 2$. Note that $B(\sum_{i=1}^{k-1} G_i) = \sum_{i=1}^{k-1} n_i - 1$ and $B(C_4) = 2$. From Theorem 4 we have $B(\sum_{i=1}^{k} G_i) = \min\{\sum_{i=1}^{k-1} n_i + 3, \max\{\sum_{i=1}^{k-1} n_i + 2, \sum_{i=1}^{k-1} n_i + 1\}\} = \sum_{i=1}^{k-1} n_i + 2$.

Example 5: Let $G_1 = C_4 + C_4, G_2 \cong G_3 \cong \cdots \cong G_{k-1} = C_4$, and let G_k be the graph of order 4 with no edges. Since $B(G_1) = 6 > \lceil 8/2\rceil, G_1, G_2, \ldots, G_k$ satisfy the conditions in Theorem 4. By Theorem 1, $B(\sum_{i=1}^{k} G_i) = B(G_3 + B(C_4 + \cdots + C_4) = 4(k+1) - 3 = 4k+1$ $B(\sum_{i=1}^{k} G_i) = B(C_4 + \cdots + C_4) = 4k-3+1 = 4k-2$, and by Theorem 2, $B(\sum_{i=1}^{k} G_i) = B(C_4+\cdots+C_4) = 4k-3+1 = 4k-2$. It follows that $\min\{B(\sum_{i=1}^{k-1} G_i) + n_k, \max\{B(G_k) + \sum_{i=1}^{k-1} n_i, \lceil n_k/2\rceil + \sum_{i=1}^{k-1} n_i - 1\}\} = \min\{4k-2+4, \max\{4k, 2+4k-1\}\} = 4k+1$.

Example 6: Let $G_1, G_2, \ldots, G_k$ be a set of graphs with $n_i = |V(G_i)|$ such that $n_1 = n_2 \geq n_3 \geq \cdots \geq n_k, B(G_1) > \lceil n_1/2\rceil$ and $B(G_2) < \lceil n_2/2\rceil = \lceil n_1/2\rceil$. Then $G_1, G_2 \ldots G_k$ satisfy the conditions in Theorem 4. However, by reversing the positions of G_1 and G_2 Theorem 1 gives $B(\sum_{i=1}^{k} G_i) = \sum_{i=1}^{k} n_i - \lceil(n_1 + 1)/2\rceil = \sum_{i=2}^{k} n_i + \lceil n_1/2\rceil - 1$.

3. Conclusions

Giving the sum of graphs $G = G_1 + G_2 + \cdots + G_k$ the bandwidth problem has been solved or bounded for all cases and all bounds have been shown to be tight.

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