Simultaneous Decompositions Of Steiner Triple Systems

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1. Introduction

A *Steiner triple system* (STS) is a pair $(V, \mathbf{B})$ where V is a v-set of *elements*, and $\mathbf{B}$ is a collection of 3-subsets of V called *triples* or *lines* such that every 2-subset of V is contained in exactly one triple. The number $v = |V|$ is called the *order* of the STS. It is well known that an STS of order v (STS(v)) exists if and only if $v \equiv 1 \text{ or } 3 \pmod 6$; such values of v are called *admissible*. If in the definition of an STS "exactly" is replaced with "at most", we have a *partial Steiner triple system* (PTS). As in [GRR], [HR], we will use the term *configuration* to describe a PTS with a fixed small number of lines.

Given a configuration C, with k lines, let $A(C) = \{v\colon k \text{ divides } v(v-1)/6\}$. We can then ask the following questions:

a) Does there exist an STS(v) whose set of triples can be partitioned into copies of C?
b) What is the *spectrum* of C, $S(C) = \{v$: there exists an STS(v) whose set of triples can be partitioned into copies of $C\}$?
c) Is C v_0*-universal*? I.e., can the set of triples of every STS(v), v admissible, $v \geq v_0$, be partitioned into copies of C?
d) Is C *cyclic* v_0*-universal*? I.e, can the set of triples of every cyclic STS(v), v admissible, $v \geq v_0$, be partitioned into copies of C?

The papers [HR] and [GRR] deal with these questions for all configurations with at most four lines.

In this paper we take one further step in that we consider *simultaneous* decompositions of STSs. More precisely, given a set of configurations $\mathbf{C}$, we are interested in those values of v for which there exists an STS(v), say $(V, \mathbf{B})$, such that $\mathbf{B}$ can be decomposed into copies of C for all $C \in \mathbf{C}$.

In Sections 2 and 3 we consider simultaneous decompositions of STSs into three-line, and four-line configurations, respectively. In Section 4 we present a collection of configurations $\mathbf{C}^3$, the strongly 3-colourable configurations [RC] and prove that if $\mathbf{C}$ is any finite subset of $\mathbf{C}^3$, there exists a v_0 such that $S(\mathbf{C}) \cap \{v: v \geq v_0\} = A(\mathbf{C}) \cap \{v: v \geq v_0\}$. $\mathbf{C}^3$ contains all three-line configurations, all but one of the four-line configurations, all path-like, bipartite-like, tree-like, and cycle-like configurations.

For undefined design-theoretic terms, see [BJL].

2. Simultaneous decompositions

Let $\mathbf{C}$ be any set of configurations. The *admissible set* of $\mathbf{C}$, $A(\mathbf{C})$, is the set $A(\mathbf{C}) = \{v: v \equiv 1 \text{ or } 3 \pmod 6 \text{ and } |C| \text{ divides } v(v-1)/6 \text{ for all } C \in \mathbf{C}\}$. The *spectrum* of $\mathbf{C}$ is the set $S(\mathbf{C}) = \{v$: there exists an STS(v), $(V, \mathbf{B})$, such that $\mathbf{B}$ is decomposable into copies of C for all $C \in \mathbf{C}\}$.

In this paper, we will be primarily concerned with the case when all configurations in $\mathbf{C}$ contain the same number of lines. In what follows, let $\mathbf{C}_i$ be the set of *all* configurations with i lines.

We can easily dispose of the case of two lines. In this case, $\mathbf{C}_2$ consists just of two configurations, namely a pair of disjoint lines, and a pair of intersecting lines, respectively. It is straightforward to see that $S(\mathbf{C}_2) = \{v: v \equiv 1 \text{ or } 9 \pmod{12},\ v \geq 13\}$. In fact, both these configurations are universal as *every* STS(v) with $v \equiv 1 \text{ or } 9 \pmod{12}$ and $v \geq 13$ can be decomposed into copies of either of these two configurations [HR].

Next we deal with the case of three lines. Recall that $\mathbf{C}_3$ consists of 5 configurations shown in Figure 1.

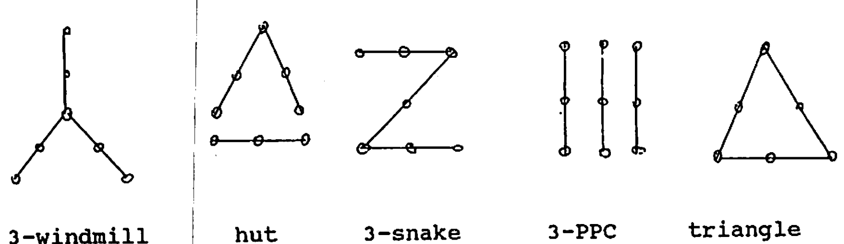

Figure 1

Clearly, we have $A(\mathbf{C}_3) = \{v: v \equiv 1 \text{ or } 9 \pmod{18}\}$. It is shown in [HR] that the 3-windmill is 9-universal, the 3-PPC is 27-universal, and the hut is 55-universal, and that the 3-snake is cyclic-universal. It is not known whether the triangle is universal or even cyclic universal. Thus in order to determine $S(C_3)$, it would suffice (at least for $v \geq 55$) to exhibit a cyclic STS(v) decomposable into triangles. In fact, when $v \equiv 1$ or $19 \pmod{72}$, such an STS(v) is already exhibited in [HR]. However, we pursue a different route here.

First we present some examples.

Lemma 2.1. $19 \in S(C_3)$.

Proof: Consider the following cyclic STS(19): its element-set is Z_{19} and the base triples are 0 1 4, 0 2 9, 0 5 11 (here and in what follows we omit set brackets for brevity). To see that this STS is decomposable into copies of any of the three-line configurations, it suffices to exhibit one such configuration with one line from each of the three orbits of triples. The remaining configurations are then obtained by the action of the cyclic group of order 19 given by $i \to i+1 \pmod{19}$. These "base configurations" are as follows:

3-PPC:	0 1 4, 2 7 13, 3 5 12
hut:	0 1 4, 0 2 9, 3 8 14
triangle:	0 1 4, 0 2 9, 4 9 15.

In addition, *any* STS(19) can be decomposed into copies of the 3-windmill, and *any cyclic* STS(19) can be decomposed into copies of the 3-snake (see [HR]). ∎

Lemma 2.2. $27 \in S(C_3)$.

Proof: Consider the following STS(27): $V = Z_9 \times \{1, 2, 3\}$, and the base triples are $0_1 3_1 0_3$, $0_1 1_1 5_2$, $0_1 2_1 3_2$, $0_1 4_1 6_2$, $0_2 1_2 4_2$, $0_2 2_2 4_3$, $0_3 1_3 2_2$, $0_3 3_3 3_2$, $0_3 4_3 5_1$, $0_3 2_3 4_1$, $0_1 0_2 1_3$, $0_1 8_2 2_3$, $0_1 7_2 3_3$ (instead of (a, i), we have written here for brevity a_i). Of the 13 orbits determined by these base triples, the first decomposes into 3 triangles. From the remaining 12 orbits we can choose 4 sets of "base" triangles as follows: $0_1 1_1 5_2$, $0_1 2_1 3_2$, $1_1 6_1 3_2$; $0_2 1_2 4_2$, $0_2 7_2 2_3$, $2_3 3_3 4_2$; $0_3 3_3 3_2$, $0_3 4_3 5_1$, $5_1 1_3 3_3$; $0_1 0_2 1_3$, $0_2 1_1 3_3$, $0_1 7_2 3_3$. Thus our STS(27) can be decomposed into triangles. Similarly, the orbit determined by the base triple $0_1 2_1 3_2$ decomposes into three 3-snakes, while from the remaining 12 orbits we can choose 4 sets of three base 3-snakes e.g. as follows: $0_1 3_1 0_3$, $0_1 1_1 5_2$, $1_1 5_1 7_2$; $0_2 1_2 4_2$, $0_2 2_2 4_3$, $2_2 0_3 1_3$; $0_3 3_3 3_2$, $0_3 4_3 5_1$, $4_3 6_3 8_1$; $0_1 0_2 1_3$, $0_1 8_2 2_3$, $1_1 8_2 4_3$. To show that our STS(27) can be decomposed into huts can be done in a similar manner but is even easier, and is thus omitted. Finally, note that any STS(27) can be decomposed into 3-windmills, or into 3-PPCs. ∎

Lemma 2.3. $37 \in S(C_3)$.

Proof: The proof is similar to that of Lemma 2.1 except that now the element-set is Z_{37}, and the base triples are 0 1 8, 0 2 13, 0 3 19, 0 4 14, 0 5 17, 0 6 15. The base configurations are:

hut:	0 1 8, 0 2 13, 3 6 22;	0 4 14, 0 5 17, 1 7 16
triangle:	0 1 8, 0 2 13, 8 13 25;	0 3 19, 0 9 31, 5 9 19.

By [HR], our cyclic STS(37) is decomposable into copies of the 3-windmill (or the 3-PPC, or the 3-snake, respectively). ∎

Lemma 2.4. $45 \in S(C_3)$.

Proof: The proof is similar to that of Lemma 2.2. We construct an STS(45) with the element-set $V = Z_{15} \times \{1,2,3\}$. Its base triples are $0_1 5_1 0_3$, $0_2 2_2 8_2$, $0_1 12_2 4_3$, $0_1 14_2 3_3$, $0_1 0_2 2_3$, $0_3 1_3 5_2$, $0_3 3_3 6_2$, $0_2 1_2 0_3$, $0_2 3_2 6_3$, $0_2 4_2 5_3$, $0_2 5_2 13_3$, $0_3 2_3 6_1$, $0_3 4_3 14_1$, $0_3 5_3 7_1$, $0_3 6_3 9_1$, $0_3 7_3 8_1$, $0_1 1_1 5_2$, $0_1 2_1 13_2$, $0_1 3_1 10_2$, $0_1 4_1 6_2$, $0_1 6_1 9_2$, $0_1 7_1 8_2$. The orbit determined by the first of these 22 base triples decomposes into 5 triangles. From the remaining 21 orbits we can choose 7 sets of "base" triangles as follows: $0_2 2_2 8_2$, $0_2 1_2 0_3$, $11_1 8_2 0_3$;

$$0_3 1_3 5_2,\ 0_3 3_3 6_2,\ 1_3 3_3 7_1;\quad 0_2 3_2 6_3,\ 0_2 4_2 5_3,\ 3_1 3_2 5_3;$$
$$0_3 2_2 7_2,\ 0_3 4_3 14_1,\ 14_1 6_1 7_2;\quad 0_3 5_3 7_1,\ 0_3 6_3 9_1,\ 7_1 9_1 5_2;$$
$$0_3 7_3 8_1,\ 4_1 3_2 7_3,\ 4_1 8_1 10_2;\quad 0_1 1_1 5_2,\ 10_1 13_1 5_2,\ 1_1 10_1 4_2.$$

Similarly, the orbit determined by the base block $0_1 1_1 5_2$ is easily seen to decompose into five 3-snakes (into five huts, respectively). From the remaining 21 orbits we can choose 7 sets of "base" 3-snakes as follows: $0_1 5_1 0_3$, $0_1 3_1 10_2$, $3_1 12_1 6_2$;

$$0_2 2_2 8_2,\ 0_2 1_2 0_3,\ 5_1 2_2 9_3;\quad 0_3 1_3 5_2,\ 0_3 3_3 6_2,\ 3_3 5_3 9_1;$$
$$0_3 2_2 7_2,\ 0_3 4_3 14_1,\ 7_1 14_1 0_2;\quad 0_2 3_2 6_3,\ 1_2 5_2 6_3,\ 0_1 0_2 2_3;$$
$$0_3 5_3 7_1,\ 0_3 6_3 9_1,\ 9_1 11_1 0_2;\quad 0_3 7_3 8_1,\ 4_1 8_1 10_2,\ 11_1 10_2 14_3.$$

From these, we can get 7 sets of "base" huts by simply adding 1 modulo 15 to the *third* line of each of the first four configurations given above, and adding 2 modulo 15 to the third line of each of the last three. To complete the proof we observe that any STS(45) is decomposable into 3-windmills and into 3-PPCs (see [HR]). ∎

A *group divisible design* (GDD) is a triple $(V, \mathbf{G}, \mathbf{B})$ where V is a set of elements, $\mathbf{G}$ is a collection of subsets of V called *groups* that partition V, and $\mathbf{B}$ is a collection of subsets of V called *blocks* with the property that any two elements that belong to distinct groups are contained in exactly one block, and any two elements belonging to the same group are contained in no block (cf. [BJL], [BSH], [CHR]). If all blocks have the same size k, we refer to the GDD as a k-GDD. As is customary, we use exponential notation to indicate the *type* of the GDD, i.e. the number of groups of a given size. For example, $g^{\alpha} h^{\beta} i^{\gamma} \ldots$ indicates that there are α groups of size g, β groups of size h, γ groups of size i, etc.

Lemma 2.5. *There exists a 3-GDD with $9s$ elements of type $(3s)^3$ such that its block set $\mathbf{B}$ is decomposable into 3-snakes, and $\mathbf{B}$ is also decomposable into triangles.*

Proof: Any 3-GDD of type $(3s)^3$ corresponds to a latin square of order $3s$. In particular, a 3-GDD of type 3^3 corresponds to a latin square of order 3. It is an easy exercise to see that the latter can be decomposed into 3-snakes, and also into triangles. To complete the proof it suffices to consider the latin square of order $3s$ which is the direct product of a latin square of order 3 and a latin square of order s. ∎

The following lemma which is a variant of Wilson's fundamental construction (cf. [BJL]) will be useful in this and in the next section.

Lemma 2.6. *If there exists a group divisible design $GD(V, \mathbf{G}, \mathbf{B})$ such that $w|G| + \delta \in S(\mathbf{C})$ for all $G \in \mathbf{G}$ where w is a positive integer and $\delta \in \{0, 1\}$, and if for each $B \in \mathbf{B}$ there exists a $GD(V', \mathbf{G}', \mathbf{B}')$ with $|\mathbf{G}'| = |B|$, $|G'| = w$ for all $G' \in \mathbf{G}'$, $|B'| = 3$ for all $B' \in \mathbf{B}'$ and such that $\mathbf{B}'$ is decomposable into copies of C for all $C \in \mathbf{C}$ then $w|V| + \delta \in S(\mathbf{C})$.*

Proof: Give every element of $GD(V, \mathbf{G}, \mathbf{B})$ weight w. Let ∞ be a new point. For every $G \in \mathbf{G}$, put on the set wG if $\delta = 0$ [on the set $wG \cup \{\infty\}$ if $\delta = 1$] a copy of an STS($w|G| + \delta$) decomposable into copies of C for all $C \in \mathbf{C}$. For each $B \in \mathbf{B}$, replace B with a copy of a $GD(V', \mathbf{G}', \mathbf{B}')$ decomposable into copies of C for all $C \in \mathbf{C}$. The result is an STS($w|V| + \delta$) decomposable into copies of C for all $C \in \mathbf{C}$. ∎

We refer to the $GD(V, \mathbf{G}, \mathbf{B})$ in Lemma 2.6 as the "master GDD", and to the $GD(V', \mathbf{G}', \mathbf{B}')$ as the "ingredient GDDs".

We are now ready to state the result for simultaneous decompositions into three-line configurations.

Theorem 2.7. $S(\mathbf{C}_3) = \{v: v \equiv 1 \text{ or } 9 \pmod{18}, v \geq 19\}$.

Proof: Consider first the case when $v \equiv 9 \pmod{18}$. Write $v = 18t + 9 = 9(2t + 1)$. By Lemmas 2.2 and 2.4, we may assume $v \geq 63$. When $t \equiv 0$ or $1 \pmod 3$, consider an STS($2t + 1$) as a 3-GDD of type 1^{2t+1}. Since the unique STS(9) is decomposable into 3-snakes and also into triangles, we may apply Lemma 2.6 with weight $w = 9$, $\delta = 0$ and with the GD from Lemma 2.5 as the ingredient GDD. When $t \equiv 2 \pmod 3$, $2t + 1 \equiv 5 \pmod 6$. In this case consider a 3-GDD of type $5^1 1^{2t-4}$ which is well known to exist (see [BJL]). Since $45 \in S(\mathbf{C}_3)$ by Lemma 2.4, we may again apply Lemma 2.6 with $w = 9$, $\delta = 0$ and with the GDD from Lemma 2.5 as the ingredient GDD.

In the case when $v \equiv 1 \pmod{18}$, $v = 18t+1$, we may assume by Lemmas 2.1 and 2.3 that $v \geq 55$. When $t \equiv 0$ or $1 \pmod 3$, consider a maximum packing of triples on $2t$ elements as a 3-GDD of type 2^t, and apply Lemma 2.6 with $w = 9$, $\delta = 1$ taking as the ingredient GDD a 3-GDD of type 9^3 decomposable into 3-snakes and also into triangles; the existence of the latter GDD is immediate from Lemma 2.5. When $t \equiv 2 \pmod 3$, we take instead a 3-GDD of type $4^1 2^{2t-4}$ as the master GDD, taking into account that $37 \in S(\mathbf{C}_3)$ by Lemma 2.3. This completes the proof. ∎

Let us remark that the above result (Theorem 2.7) should be perhaps viewed as an interim result. It has been conjectured (see [HR]) that *every* STS(v), $v \in$

$A(\mathrm{C}_3)$, can be decomposed into 3-snakes, and also into triangles. An eventual proof of these two conjectures would clearly render Theorem 2.7 obsolete (at least for $v \geq 55$).

3. Simultaneous decompositions into 4-line configurations

There are 16 nonisomorphic four-line configurations with three points per line denoted as in Fig. 2. Clearly, we have $A(\mathrm{C}_4) = \{v : v \equiv 1 \text{ or } 9 \pmod{24}\}$, where, of course, $\mathrm{C}_4 = \{C_i : i = 1, 2, \ldots, 16\}$. In [GRR], the spectrum $S(C_i)$ was completely determined for 10 configurations, and just one value ($v = 81$) has been left in doubt for the remaining 6 configurations.

In what follows we pursue the determination of $S(\mathrm{C}_4)$. Unlike the situation for three-line configurations discussed in Section 2, the results in this section are certainly no interim results since both C_{14} and $C_{16} = P$ are known not to be v_0-universal for any v_0: there are infinite classes of STSs which even completely avoid one of the configurations (see [GRR]).

As in the previous section, we start with several examples that are needed later.

Lemma 3.1. $25 \in S(\mathrm{C}_4)$.

Proof: Consider the following cyclic STS(25) (see [GRR], Lemma 3.2): $V = Z_{25}$ and base triples are $0\ 1\ 6, 0\ 9\ 11, 0\ 3\ 10, 0\ 4\ 12$. Decomposition of this STS(25) into each of the configurations of C_4 are easy, and are also given in [GRR]. ∎

Lemma 3.2. $33 \in S(\mathrm{C}_4)$.

Proof: Consider the following STS(33) (cf. [GRR], Lemma 3.3 (d)): $V = Z_{11} \times \{1, 2, 3\}$, and base triples $0_1 0_2 0_3$, $0_1 6_2 10_3$, $0_1 8_2 7_3$, $0_1 3_2 8_3$, $0_1 1_1 3_1$, $0_1 7_1 9_2$, $0_1 5_1 3_3$, $0_1 10_2 6_3$, $0_2 1_2 4_2$, $0_2 5_2 2_3$, $0_2 2_2 6_1$, $0_2 3_3 10_1$, $0_3 5_3 7_3$, $0_1 1_3 2_3$, $0_3 3_3 5_2$, $0_1 4_2 5_3$. Decompositions of this STS(33) into each of the configurations of C_4 are given in the Appendix. ∎

Lemma 3.3. $49 \in S(\mathrm{C}_4)$.

Proof: Consider the following cyclic STS(49) (cf. [GRR], Lemma 3.4 (c)): $V = Z_{49}$, and base triples are $0\ 1\ 12, 0\ 2\ 10, 0\ 3\ 20, 0\ 4\ 18, 0\ 6\ 21, 0\ 9\ 22, 0\ 16\ 23, 0\ 19\ 24$. Decompositions of this STS(49) into each of the configurations of C_4 are given in the Appendix. ∎

Lemma 3.4. $57 \in S(\mathrm{C}_4)$.

Proof: Consider the following STS(57) (cf. [GRR], Lemma 3.5 (d)): $V = Z_{19} \times \{1, 2, 3\}$, and base triples are $0_1 1_1 7_2, 0_1 2_1 16_2, 0_1 3_1 5_2, 0_1 4_1 15_2, 0_1 5_1 8_2, 0_1 6_1 18_2, 0_1 7_1 17_2, 0_1 8_1 9_2, 0_1 9_1 13_2, 0_2 1_2 4_3, 0_2 2_2 13_3, 0_2 3_2 2_3, 0_2 4_2 12_3, 0_2 5_2 14_3, 0_2 8_2 5_3, 0_2 9_2 15_3, 0_2 12_2 10_3, 0_2 13_2 1_3, 0_3 1_3 9_1, 0_3 2_3 12_1, 0_3 3_3 14_1, 0_3 4_3 3_1, 0_3 5_3 7_1, 0_3 6_3 4_1, 0_3 7_3 1_1, 0_3 8_3 5_1, 0_3 9_3 15_1, 0_1 0_2 0_3$. Decompositions of this STS(57) into each of the configurations of C_4 are given in the Appendix. ∎

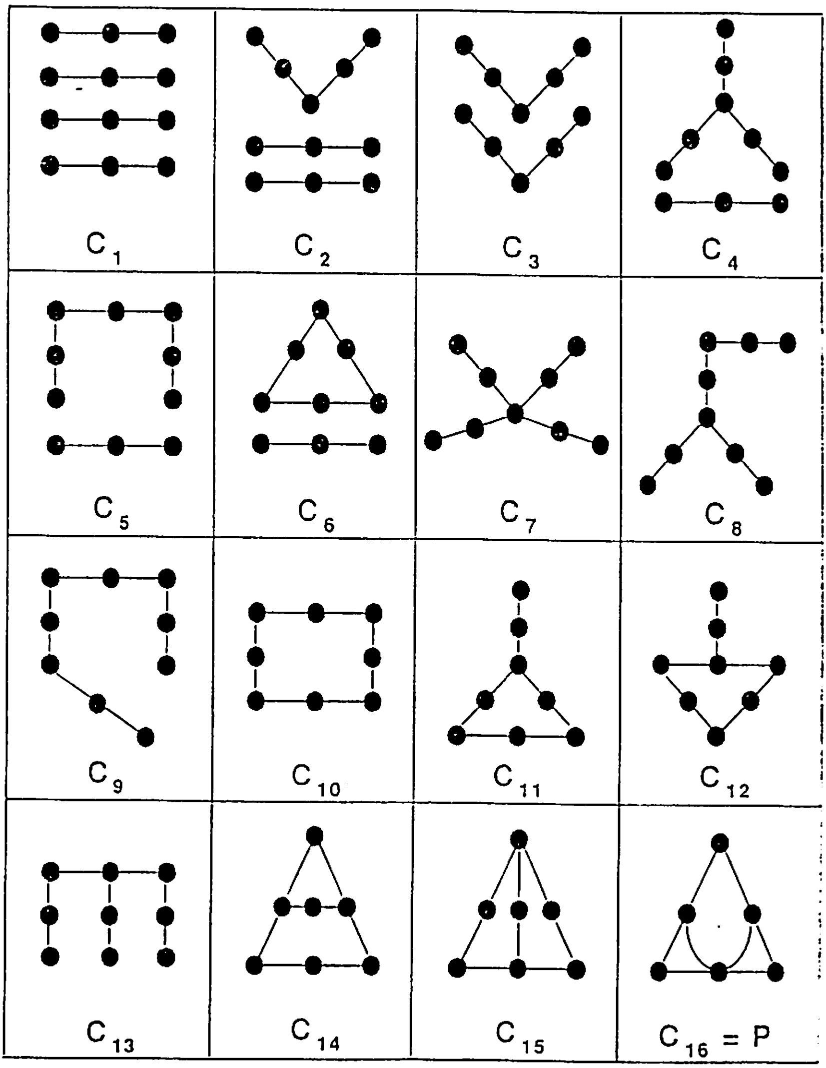

Figure 2

Lemma 3.5. $73 \in S(C_4)$.

Proof: Consider the following cyclic STS(73): $V = Z_{73}$, and the base triples are $0\,1\,4$, $0\,5\,32$, $0\,6\,22$, $0\,7\,28$, $0\,9\,33$, $0\,10\,30$, $0\,13\,42$, $0\,14\,25$, $0\,15\,17$, $0\,19\,37$, $0\,23\,35$, $0\,26\,34$. Again, decompositions of this STS(73) into each of the configurations of C_4 are given in the Appendix. ∎

Lemma 3.6. *There exists a 3-GDD with 16 elements of type 4^4 whose block-set is decomposable into copies of C_i for all $C_i \in C_4$.*

Proof: All 32 blocks of the design No.23 in the listing of [FMR] together with its decompositions into each of the configurations of C_4 are given in the Appendix. ∎

Lemma 3.7. $\{v: v \equiv 1 \pmod{24}, v \geq 25, v \neq 97\} \subset S(C_4)$.

Proof: We have $\{25, 49, 73\} \subset S(C_4)$ by Lemmas 3.1, 3.3 and 3.5. Thus we may assume $v \geq 121$. There exists a 4-GDD of type 6^t for all $t \geq 5$ (see [BSH]). Apply now Lemma 2.6 with weight $w = 4$ and $\delta = 1$ using this 4-GDD as the master GDD, and the 3-GDD of type 4^4 of Lemma 3.6 as the ingredient GDD, taking into account that $25 \in S(C_4)$ by Lemma 3.1. ∎

Lemma 3.8. $\{v: v \equiv 33 \pmod{96}\} \subset S(C_4)$.

Proof: By Lemma 3.2, $33 \in S(C_4)$. There exists a 4-GDD of type 8^{3t+1} for all $t \geq 1$ (see [BSH]). Apply Lemma 2.6 with $w = 4$ and $\delta = 1$ using this 4-GDD as the master GDD, and the GDD of Lemma 3.6 as the ingredient GDD. ∎

One of the difficulties encountered when trying to determine $S(C_4)$ is due to the lack of 4-GDDs that could serve as master GDDs when applying Lemma 2.6. The need to use 4-GDDs rather than 3-GDDs is, in turn, due to the fact that neither of the two 3-GDDs of type 4^3 (i.e. latin squares of order 4) decomposes into copies of C_4 or into copies of C_{15}. As every STS(v) with $v \geq 169$ can be decomposed into copies of C_4 (see [GRR]), the only real problem is caused by the configuration C_{15}. It is therefore much easier to determine the spectrum for $C_4' = C_4 \backslash \{C_{15}\}$.

Lemma 3.9. *There exists a 3-GDD of type 4^3 decomposable into copies of C for all $C \in C_4 \backslash \{C_1, C_4, C_{15}\}$.*

Proof: The 3-GDD in question corresponds to the latin square which is the Cayley table of the cyclic group of order 4. To verify the assertion of the lemma is an easy exercise. ∎

Lemma 3.10. $\{v: v \equiv 9 \pmod{24}, v \geq 33, v \neq 81\} \subset S(C_4 \backslash \{C_4, C_{15}\})$.

Proof: There exists a 3-GDD of type $6^t 8^1$ for all $t \geq 3$ (see [CHR]). Use Lemma 2.6 with weight $w = 4$ and $\delta = 1$ taking the above 3-GDD as the master GDD, the 3-GDD of Lemma 3.9 as the ingredient GDD and using the fact that $25, 33 \in S(C_4)$ by Lemmas 3.1 and 3.2, respectively, as well as taking into account the fact that *every* STS(v) with $v \geq 25$ can be decomposed into copies of C_1. This proves our assertion for $v \geq 105$. Finally, $57 \in S(\mathrm{C}_4)$ by Lemma 3.3. ∎

In spite of the difficulties caused by the fact that the 3-GDD of Lemma 3.9 (the "ingredient" GDD) does not decompose into copies of C_1 or into copies of C_4, we can now state an "almost complete" result on the spectrum for C_4'.

Theorem 3.11. $A(\mathrm{C}_4)\setminus\{81, 97, 105, 153\} \subset S(\mathrm{C}_4')$.

Proof: *Every* STS(v) with $v \in A(\mathrm{C}_4)$, $v \geq 169$, can be decomposed into copies of C_4 [GRR]. The rest follows from Lemma 3.7, 3.8 and 3.10. ∎

Concerning the spectrum for the set of *all* four-line configurations C_4, we can state a complete result for the residue class $v \equiv 1 \pmod{24}$, with the single exception of $v = 97$, but at present only a partial result for the class $v \equiv 9 \pmod{24}$. The result is summarized in the following theorem.

Theorem 3.12. $\{v : v \equiv 1 \pmod{24}$ *or* $v \equiv 33 \pmod{96}$ *or* $v \equiv 57 \pmod{168}$, $v \neq 97\} \subset S(\mathrm{C}_4)$.

Proof: Apply Lemma 2.6 with $w = 4$ and $\delta = 1$ using a 4-GDD of type 14^{3t+1} (see [BSH]) as the master GDD and the 3-GDD of Lemma 3.6 as the ingredient GDD, taking into account that $57 \in S(\mathrm{C}_4)$ by Lemma 3.4. This shows $\{v : v \equiv 57 \pmod{168}\} \subset S(\mathrm{C}_4)$. The rest follows from Lemma 3.7 and 3.8. ∎

4. The general problem

Assume now that C and C' are two configurations of lines, not necessarily of the same size, with three points per line.

Lemma 4.1. *The following statements are equivalent:*

(i) $S(\{C, C'\}) \neq \emptyset$.

(ii) *There exists v such that there is a partial Steiner triple system of order v whose block-set is simultaneously decomposable into copies of C and copies of C'.*

(iii) *There exists an integer $v_0 = v_0(C, C')$ such that* $\{v : v \in A(\{C, C'\}), v \geq v_0\} \subset S(\{C, C'\})$.

Proof: Implications (i) → (ii) and (iii) → (i) are trivial (since $A(\{C, C'\}) \neq \emptyset$). Wilson's theorem on the asymptotic existence of graph decompositions [W] yields (ii) → (iii). ∎

Thus to show that the spectrum for $\{C, C'\}$ contains all sufficiently large orders in $A(\{C, C'\})$, it suffices to produce a *single* partial STS simultaneously decomposable into copies of C and C'. The next lemma shows that this is always possible for a large class of configurations. Before we can formulate this lemma, we need a definition.

A *strong colouring* of a partial STS(V, $\mathbf{B}$) is a colouring of its elements such that for any $B \in \mathbf{B}$, the three elements of B are coloured with three distinct colours. If no more than k colours are used in a strong colouring, the partial STS is *strongly k-colourable* (cf. [RC]). In other words, $C = (V, \mathbf{B})$ is strongly k-colourable if there exists a 3-GDD $(V, \mathbf{G}, \mathbf{B}')$ with $|\mathbf{G}| = k$ and $\mathbf{B}' \supseteq \mathbf{B}$. We denote the collection of all strongly k-colourable configurations by $\mathbf{C}^k$.

If $(V, \mathbf{B}), (V', \mathbf{B}')$ are two partial STSs, their (*weak*) product $(VV', \mathbf{BB}')$ is given by $VV' = V \times V', \mathbf{BB}' = \{\{(a,x),(b,y),(c,z)\} : \{a,b,c\} \in \mathbf{B}, \{x,y,z\} \in \mathbf{B}'\}$.

Lemma 4.2. *If C, C' are strongly 3-colourable configurations then there exists a partial STS which is simultaneously decomposable into copies of C and C'.*

Proof: Obviously, it suffices to prove that if $(V, \mathbf{B})$ is a strongly 3-colourable partial STS and $(T, \mathbf{T})$ contains a single triple, say, $T = \{1,2,3\}, \mathbf{T} = \{\{1,2,3\}\}$ then $(VT, \mathbf{BT})$ can be decomposed into copies of $(V, \mathbf{B})$. Let the set of colours be $\{1,2,3\}$. If ϕ is a strong 3-colouring of $(V, \mathbf{B})$ and $R = \{(x, \phi(x)) : x \in V\}$, $\mathbf{S} = \{\{(x,\phi(x)),(y,\phi(y)),(z,\phi(z))\} : \{x,y,z\} \in \mathbf{B}\}$ (where $\phi(u)$ is the colour of $u \in V$), then clearly $(R, \mathbf{S})$ is isomorphic to $(V, \mathbf{B})$. Further if $\pi \in S_3$, the symmetric group acting on the set of colours $\{1,2,3\}$, let $\pi(R) = \{x, \pi(\phi(x)) : x \in V\}$, and $\pi(\mathbf{S}) = \{\{(x,\pi(\phi(x))),(y,\pi(\phi(y))),(z,\pi(\phi(z)))\} : \{x,y,z\} \in \mathbf{B}\}$. Also for all $\pi \in S_3$ it is clear that both $(\pi(R), \pi(\mathbf{S})$ is isomorphic to $(V, \mathbf{B})$ and $\pi(R) \subseteq VT$, $\pi(\mathbf{S}) \subseteq \mathbf{BT}$. Moreover $VT = \bigcup_{\pi \in S_3} \pi(R)$ and $\mathbf{BT} = \bigcup_{\pi \in S_3} \pi(\mathbf{S})$.

Finally if $\pi, \rho \in S_3$, $\pi \neq \rho$ and $B \in \pi(\mathbf{S})$, $B' \in \rho(\mathbf{S})$ then $|B \cap B'| \leq 1$. It follows that $\{(\pi(R), \pi(\mathbf{S})) : \pi \in S_3\}$ is a decomposition of $(VT, \mathbf{BT})$ into six copies of $(V, \mathbf{B})$.

The statement of the lemma now follows by observing that if $(V, \mathbf{B}), (V', \mathbf{B}')$ are both strongly 3-colourable then $(VV', \mathbf{BB}')$ decomposes into $|\mathbf{C}|$ copies of $(VT, \mathbf{BT})$, and also into $|\mathbf{B}'|$ copies of $(V'T, \mathbf{B}'\mathbf{T})$. ∎

Lemma 4.3. *If $C = (V, \mathbf{B})$ is strongly 3-colourable and $C' = (V, \mathbf{B}')$ is any configuration then $CC' = (VV', \mathbf{BB}')$ is strongly 3-colourable.*

Proof: If ϕ is a strong 3-colouring of C, colour VV' by ϕ' where ϕ' is given by $\phi'(a, a') = \phi(a)$. Consider a triple $\{a'', b'', c''\} \in \mathbf{BB}'$, where $a'' = (a, a'), b'' = (b, b'), c'' = (c, c')$. Then $|\{\phi'(a''), \phi'(b''), \phi'(c'')\}| = |\{\phi(a), \phi(b), \phi(c)\}| = 3$. ∎

Lemma 4.2 applies to a very wide class of configurations. For example, all four-line configurations with the exception of C_{15}, all path-like, cycle-like, tree-like and bipartite-like configurations are strongly 3-colourable. For all pairs of such configurations, Lemma 4.2 in conjunction with Lemma 4.1 says, in effect, that the spectrum of such a pair $\mathbf{P}$ is "asymptotically" the admissible set $A(\mathbf{P})$. Lemma 4.3 allows us to go from a pair to any finite set of strongly 3-colourable configurations. We can state this formally as follows.

Theorem 4.4. *If* C *is any finite subset of* $\mathbf{C}^3$, *there exists a* v_0 *such that* $S(\mathbf{C}) \cap \{v: v \geq v_0\} = A(\mathbf{C}) \cap \{v: v \geq v_0\}$.

Nevertheless, it seems still of interest to determine exactly the spectrum for, say, pairs of cycle-like configurations.

The following remains an open problem.

Given two *arbitrary* configurations C, C' with three points per line, does there exist a partial Steiner triple system that is simultaneously decomposable into copies of both C and C'? We conjecture that the answer is in the affirmative.

Acknowledgements

Thanks to Nabil Shalaby for providing the STS(73) in Lemma 3.5. This paper was written while the first author was visiting Department of Mathematics and Statistics, McMaster University; he would like to thank the Department for its hospitality. Second and third author's research was supported by NSERC of Canada Grant No.A7681 (EM) and A7268 (AR).

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