Triple Youden Rectangles: A New Class of Fully Balanced Combinatorial Arrangements

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Abstract. Triple Youden rectangles are defined and examples are given. These combinatorial arrangements constitute a special class of $k \times v$ row-and-column designs, $k < v$, with superimposed treatments from three sets, namely a single set of v treatments and two sets of k treatments. The structure of each of these row-and-column designs incorporates that of a symmetrical balanced incomplete block design with v treatments in blocks of size k. Indeed, when either of the two sets of k treatments is deleted from a $k \times v$ triple Youden rectangle, a $k \times v$ double Youden rectangle is obtained; when both are deleted, a $k \times v$ Youden square remains. The paper obtains an infinite class of triple Youden rectangles of size $k \times (k+1)$. Then it presents a 4×13 triple Youden rectangle which provides a balanced layout for two packs of playing-cards, and a 7×15 triple Youden rectangle which incorporates a particularly remarkable 7×15 Youden square. Triple Youden rectangles are fully balanced in a statistical as well as a combinatorial sense, and those discovered so far are statistically very efficient.

1. Introduction, with definitions

This paper defines a new type of fully balanced combinatorial arrangement and gives examples. For our description of the arrangements and their construction, we use standard definitions of a set of mutually orthogonal Latin squares (Hall, 1967, Chapter 12), the incidence matrix of a balanced incomplete block design (Hall, 1967, Chapter 10), a symmetric balanced incomplete block design (SBIBD, or symmetric 2-design) (Hall, 1967, Chapter 10), a Youden square (Preece, 1990), and a double Youden rectangle (Bailey, 1989, p. 40; Preece, 1991). In particular, the incidence matrix $\underset{\sim}{n}$ of an SBIBD with parameters (v, k, λ) is a $v \times v$ matrix which has each of its elements equal to 0 or 1 and which satisfies

$$\underset{\sim}{n}\underset{\sim}{n}' = \underset{\sim}{n}'\underset{\sim}{n} = (k - \lambda)\underset{\sim}{I} + \lambda \underset{\sim}{J}$$

where $\underset{\sim}{I}$ is the identity matrix and $\underset{\sim}{J}$ is the matrix all of whose elements are 1, and where k and λ are integers with $k < v$ and $\lambda = k(k-1)/(v-1)$. Also, in its customary non-square representation, a $k \times v$ Youden square ($k < v$) is a $k \times v$ row-and-column design where each of v treatments occurs exactly once in each row and where the treatment subsets in the columns constitute the blocks of an SBIBD with parameters (v, k, λ).

We define a 'triple Youden array' to be a $v \times v$ array with these properties:

(i) in each row and each column of the array, k of the entries are ordered triples of elements from a set T of k elements, and the other $v-k$ entries are blank;

(ii) if each blank entry is replaced by 0 and each ordered triple by 1, the array becomes the incidence matrix of an SBIBD;

(iii) in each row and each column of the array, each element from T occurs exactly once in the ith position of a triple ($i = 1,2,3$);

(iv) if x and y are any elements (not necessarily distinct) from T, the triples containing x in the ith position ($i = 1,2,3$) include exactly s or $s+1$ triples having y in the jth position ($j \neq i$), where s is the integral part of v/k;

(v) if $\underset{\sim}{N}_{ij}$ is the $k \times k$ matrix whose (p,q)th element is the number of times (namely s or $s+1$) that the pth element of T occurs in position i in conjunction with the qth element of T in position j ($i,j = 1,2,3; i \neq j$), then $\underset{\sim}{N}_{ij} - s\underset{\sim}{J}$ is either (a) the incidence matrix of an SBIBD with parameters (k, m, μ) where $m = v - sk$ and $\mu = m(m-1)/(k-1)$, or (b) a matrix where each row and each column contains one element equal to 1 and $(k-1)$ elements equal to 0;

(vi) the matrix $\underset{\sim}{N}_{ij}\underset{\sim}{N}_{jh}\underset{\sim}{N}_{hi} + \underset{\sim}{N}_{ih}\underset{\sim}{N}_{hj}\underset{\sim}{N}_{ji}$ is of the form $f\underset{\sim}{I} + g\underset{\sim}{J}$ for some integers f, g ($i,j,h = 1,2,3; i \neq j \neq h \neq i$) with, perforce, $f = 2v^3 - kg$.

Sometimes T is most conveniently taken to be $\{1,2,\ldots,k\}$, but sometimes $\{0,1,\ldots,k-1\}$ is preferable.

A simple example of a triple Youden array, with rows labelled $A, B, \ldots, F$, is

A	---	543	425	352	234	111
B	345*	---	154	531	413	222*
C	524*	451	---	215	142	333*
D	253*	135	512	---	321	444*
E	432*	314	241	123	---	555*
F	111	222	333	444	555	---

(1)

Here $v = 6, k = 5, s = 1, \underset{\sim}{N}_{ij} = \underset{\sim}{I} + \underset{\sim}{J}$ ($i,j = 1,2,3; i \neq j$), $f = 2, g = 86$. Another triple Youden array can be obtained from (1) by swapping the starred elements in each row that contains such elements.

Property (vi) of a triple Youden array is statistically motivated (see Section 5 below). A simple example of an array that satisfies properties (i) to (v) but not (vi) is

---	254	142	323	535	411
245	---	351	434	113	522
422	511	---	155	344	233
553	332	415	---	221	144
131	443	524	212	---	355
314	125	233	541	452	---

(2)

This array, based on display (15) of Preece (1966), has $v = 6, k = 5, s = 1$, $\underset{\sim}{N}_{32} = \underset{\sim}{I} + \underset{\sim}{J}$,

$$\underset{\sim}{N}_{31} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \underset{\sim}{N}_{21} = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

All triple Youden arrays so far discovered satisfy property (v)(b), which may be regarded merely as (v)(a) with $m = 1$ and therefore $\mu = 0$. For these triple Youden arrays, property (vi) implies that the elements in each of the three positions in the ordered triples can be labelled so that $\underset{\sim}{N}_{ij} = \underset{\sim}{I} + s\underset{\sim}{J}$ $(i, j = 1, 2, 3; i \neq j)$. Then $f = 2$ and

$$g = 2s(3 + 3sk + s^2k^2) = 2(v^3 - 1)/k.$$

We now define a 'triple Youden rectangle' to be a $k \times v$ row-and-column design obtained from a triple Youden array by interchanging the role of the rows with that of the symbols from the ith position in the triples, where i takes any one of the values $1, 2, 3$. To illustrate this definition, we obtain the 5×6 triple Youden rectangle (3) from the triple Youden array (1) by making the interchange with $i = 1$:

1	*F*11	*D*35	*B*54	*E*23	*C*42	*A*11	
2	*D*53	*F*22	*E*41	*C*15	*A*34	*B*22	
3	*B*45	*E*14	*F*33	*A*52	*D*21	*C*33	(3)
4	*E*32	*C*51	*A*25	*F*44	*B*13	*D*44	
5	*C*24	*A*43	*D*12	*B*31	*F*55	*E*55	

In any column of (3), the 5 entries come from the 5 non-blank entries in the corresponding column of (1). In column 1, for example, the entry *C*24 in row 5 of (3) comes from the entry 524 in row *C* of (1). In terminology used originally for Latin squares, the row-and-column design (3) could be described as an 'adjugate' or 'conjugate' of (1).

A triple Youden rectangle is thus a particular type of $k \times v$ row-and-column design for three superimposed sets of treatments, these sets being a single set of v treatments (here, letters) and two sets of k treatments (here, digits). If one of the two sets of k treatments is omitted, we obtain a double Youden rectangle; if both are omitted, a Youden square is obtained.

In standard statistical terminology, a triple Youden rectangle can also be regarded as a 5-factor design with two factors (columns and letters) each having v levels, and three factors (rows, first digits, second digits) each having k levels. The incidence matrix for any two of the five factors, these two being taken in either order, is either that of a SBIBD, as in condition (ii), or a matrix $\underset{\sim}{N}_{ij}$ as in condition (v), which is a condition of balance, or a $k \times v$ or $v \times k$ matrix $\underset{\sim}{J}$ (all elements equal to 1), which represents the mutual orthogonality of the two factors. As orthogonality can be viewed as a special case of balance, a triple Youden rectangle can therefore be described as 'fully balanced' in a combinatorial sense.

For convenience, a triple Youden rectangle is often best represented by a corresponding triple Youden array.

2. General constructions for $k \times (k+1)$ triple Youden rectangles

The construction of triple Youden array (1) is achieved by a method available for all values of $k = v - 1$ such that there are at least 3 mutually orthogonal $k \times k$ Latin squares that have a common transversal. For convenience, we take the symbols in each of the 3 squares to be $\{1, 2, \ldots, k\}$, we order the rows and columns of the squares so that the common transversal is the main diagonal, and we assign the symbols of each square so that they appear in the order $1, 2, \ldots, k$ down the main diagonal. The elements of the main diagonal of each square are now removed and reintroduced in the corresponding positions in a $(k+1)$th row and a $(k+1)$th column, as in (1); the position where the $(k+1)$th row and $(k+1)$th column intersect is left blank. This gives a triple Youden array with $\underset{\sim}{N}_{ij} = \underset{\sim}{I} + \underset{\sim}{J}$ $(i, j = 1, 2, 3; i \neq j)$, which can now be converted into a $k \times (k+1)$ triple Youden rectangle as already described.

The requirement for 3 mutually orthogonal $k \times k$ Latin squares with a common transversal is, of course, met if there exist 4 mutually orthogonal $k \times k$ Latin squares.

The construction just given is a refinement of one described by Preece (1966, p.19), which was in turn based on one devised by Clarke (1963) for double Youden rectangles. The 1966 paper did not, however, note that its method of construction leads to (vi) being satisfied. In outcome, the difference between the refined method and the less easily described 1966 method is merely the difference between array (1) and the array obtained from (1) by swapping starred elements as above.

3. A 4×13 triple Youden rectangle

Ever since Clarke (1963) gave the first of them, the discovery of double Youden rectangles has proceeded slowly (Preece, 1991, 1992). The search is indeed still in its early days. Consequently, much of the groundwork needed for the discovery of triple Youden rectangles remains to be done. However, triple Youden rectangles of sizes 4×13 and 7×15 have been found. Those of size 4×13 are obtained by observing the combinatorial structure of the 4×13 double Youden rectangles of Preece (1982) and by extending it to three 4-level factors. The triple Youden array obtained in this way is in Fig. 1, which has $v = 13$, $k = 4$, $s = 3$, $\underset{\sim}{N}_{ij} = \underset{\sim}{I} + 3\underset{\sim}{J}$ $(i, j = 1, 2, 3; i \neq j)$, $f = 2$, $g = 1098$.

A	000	111	222	333	-	-	-	-	-	-	-	-	-
2	111	-	-	-	203	-	-	032	-	-	320	-	-
3	222	-	-	-	-	301	-	-	013	-	-	130	-
4	333	-	-	-	-	-	102	-	-	021	-	-	210
5	-	302	-	-	020	-	-	-	-	-	-	211	133
6	-	-	103	-	-	030	-	-	-	-	211	-	322
7	-	-	-	201	-	-	010	-	-	-	133	322	-
8	-	230	-	-	-	-	-	-	121	313	002	-	-
9	-	-	310	-	-	-	-	121	-	232	-	003	-
t	-	-	-	120	-	-	-	313	232	-	-	-	001
J	-	023	-	-	-	112	331	200	-	-	-	-	-
Q	-	-	031	-	112	-	223	-	300	-	-	-	-
K	-	-	-	012	331	223	-	-	-	100	-	-	-

Figure 1: Triple Youden array with $v = 13, k = 4$.

The triple Youden array in Fig. 1 has the following properties:

(i) Within each $1 \times 3, 3 \times 1$ and 3×3 sub-array, each non-blank entry can be used to generate 2 further entries by repeated use of the cyclic permutation (123) for each position in the triples. The entries are generated horizontally, vertically and diagonally within, respectively, the $1 \times 3, 3 \times 1$ and 3×3 sub-arrays.

(ii) If each non-blank entry x, y, z is replaced by y, z, x (i.e. if the three 4-level factors are permuted cyclically), the array can then readily be retrieved by permuting rows and columns.

(iii) If the array is transposed about its main diagonal, it can then readily be retrieved by replacing each non-blank entry x, y, z by z, y, x (i.e. by swapping the first and third of the 4-level factors).

The triple Youden array in Fig. 1 is thus rich in automorphisms. However, it is not isomorphic to the triple Youden array obtained from it merely by swapping two of the three 4-level factors.

A 4×13 triple Youden rectangle obtained from Fig. 1 is shown in Fig. 2.

0	A00	J23	Q31	K12	520	630	710	232	313	421	802	903	t01
1	211	A11	603	t20	Q12	J12	402	921	821	K00	733	330	533
2	322	830	A22	701	203	K23	Q23	J00	t32	932	611	511	410
3	433	502	910	A33	K31	301	J31	t13	Q00	813	220	722	622

Figure 2: 4×13 triple Youden rectangle obtained from the array in Fig. 1

The set of 13 treatments is taken to be $\{A, 2, 3, ..., 9, t, J, Q, K\}$ to signal that Fig. 2 can be used to set up a balanced 4×13 layout for two packs of playing-cards, with each card from the second pack layed on top of a first-pack card having the same value (i.e. Ace on Ace, Queen on Queen, etc.). To obtain such a layout, the digits $\{0, 1, 2, 3\}$ in the second position of each triple from Fig. 2 must be assigned in some order to the suit-names {Spade, Diamond, Club, Heart}, as must the digits in the third position. Suppose we adopt {0=Spade, 1=Diamond, 2=Club, 3=Heart} for each of the two relevant positions. Then, for example, the entry $J23$ in row 0 of Fig. 2 means that the Jack of Clubs from the first pack lies under the Jack of Hearts from the second. In the layout as a whole, each pack is arranged in a 4×13 double Youden rectangle, and the 13 first-pack cards of any suit lie under 4 second-pack cards of that suit and under 3 second-pack cards of each of the other suits.

4. A 7×15 triple Youden rectangle

Figure 3 gives a triple Youden array with $v = 15, k = 7, s = 2, \underset{\sim}{N}_{ij} = \underset{\sim}{I} + 2\underset{\sim}{J}$ $(i,j = 1,2,3; i \neq j)$, $f = 2, g = 964$. Inspection of Fig. 3 immediately reveals the cyclic structure of the array, with cycles of length 7.

*	-	111	222	333	444	555	666	777	-	-	-	-	-	-	-
A	111	-	-	-	657	-	734	426	-	-	-	573	-	342	265
B	222	537	-	-	-	761	-	145	376	-	-	-	614	-	453
C	333	256	641	-	-	-	172	-	564	417	-	-	-	725	-
D	444	-	367	752	-	-	-	213	-	675	521	-	-	-	136
E	555	324	-	471	163	-	-	-	247	-	716	632	-	-	-
F	666	-	435	-	512	274	-	-	-	351	-	127	743	-	-
G	777	-	-	546	-	623	315	-	-	-	462	-	231	154	-
a	-	-	-	-	725	-	453	632	111	546	274	-	367	-	-
b	-	743	-	-	-	136	-	564	-	222	657	315	-	471	-
c	-	675	154	-	-	-	247	-	-	-	333	761	426	-	512
d	-	-	716	265	-	-	-	351	623	-	-	444	172	537	-
e	-	462	-	127	376	-	-	-	-	734	-	-	555	213	641
f	-	-	573	-	231	417	-	-	752	-	145	-	-	666	324
g	-	-	-	614	-	342	521	-	435	163	-	256	-	-	777

Figure 3: Triple Youden array with $v = 15, k = 7$.

In these cycles, the digits in each position in the triples are permuted in accordance with the cyclic permutation (1234567). Consistently with the cyclic structure, the rows are labelled $\{*, A, B, \ldots, G, a, b, \ldots, g\}$. The structure inherent here is similar to that of the analogous arrays given by Preece (1971, p. 428) for some 7×15 double Youden rectangles, but those particular double Youden rectangles cannot be obtained from Fig. 3.

Figure 4 gives a 7×15 triple Youden rectangle obtained from Fig. 3 by interchanging the role of the rows in Fig. 3 with that of the digits from the first position in each of the triples from Fig. 3. Structurally, the rectangle in Fig. 4 falls into a single column followed by two 7×7 arrays of triples. Within the initial column, each triple is obtained from the previous one by simultaneous use of the cyclic permutations (ABCDEFG) and (1234567). Within each of the two 7 × 7 arrays, the triples are obtained similarly by moving down diagonals and broken diagonals, with simultaneous use of the permutations (ABCDEFG), (abcdefg) and (1234567).

1	A11	*11	c54	e27	E63	b36	C72	B45	a11	g63	f45	F27	d72	G54	D36
2	B22	C56	*22	d65	f31	F74	c47	D13	E47	b22	a74	g56	G31	e13	A65
3	C33	E24	D67	*33	e76	g42	G15	d51	B76	F51	c33	b15	a67	A42	f24
4	D44	e62	F35	E71	*44	f17	a53	A26	g35	C17	G62	d44	c26	b71	B53
5	E55	B37	f73	G46	F12	*55	g21	b64	C64	a46	D21	A73	e55	d37	c12
6	F66	c75	C41	g14	A57	G23	*66	a32	d23	D75	b57	E32	B14	f66	e41
7	G77	b43	d16	D52	a25	B61	A34	*77	f52	e34	E16	c61	F43	C25	g77

Figure 4: A 7 × 15 triple Youden rectangle obtained from the array in Fig. 3.

In Fig. 4, the 7 × 15 Youden square $Y1$ that incorporates the treatments $\{*, A, B, \ldots, G, a, b, \ldots, g\}$ is the very remarkable 7 × 15 Youden square reported by Preece (1990), which is equivalent to any one of the 8 heptads described by Edge (1954, p. 332) and by Hirschfeld (1985, Section 17.5). One of the many special characteristics of $Y1$ is that 35 Latin squares of size 3 × 3 are embedded in it. However, if a triple Youden rectangle is obtained from Fig. 3 by interchanging the role of rows with that of the digits from either the second or third position, the incorporated Youden square $Y2$ or $Y3$ contains no 3 × 3 Latin squares. In a sense described by Preece (1990), $Y1$ is 'self-dual' whereas $Y2$ is the 'dual' of $Y3$. Thus the triple Youden array in Fig. 3 has a different pattern of automorphisms from that for the triple Youden array in Fig. 1.

5. The statistical efficiency of triple Youden rectangles

Property (vi) of a triple Youden array implies that a triple Youden rectangle, when regarded as a row-and-column design for three non-interacting sets of treatments, is 'fully balanced' in a statistical as well as combinatorial sense. [The mathematics lying behind this is similar to that produced for other multi-factor designs by Preece (1966, Section 2.3), and does not merit inclusion here.] An important aspect of this full statistical balance is that a triple Youden rectangle has a single 'efficiency factor' E_k for any one of the three k-level factors, when the other two k-level factors are taken into account. [Again the supporting mathematics is a straightforward extension of that given by Preece (1966).] A general formula for E_k is

$$E_k = \frac{v^3 - 3v(m - \mu) + f}{v^3 - v(m - \mu)}$$

There is also a single efficiency factor E_v for either of the v-level factors, when the other is taken into account; this is the standard efficiency factor for a $k \times v$

Youden square, namely

$$E_v = \frac{1 - 1/k}{1 - 1/v}$$

All triple Youden rectangles so far discovered have $m = 1$ and therefore $\mu = 0$ and $f = 2$; thus they have

$$E_k = (v - 1)(v + 2)/v(v + 1).$$

All these known triple Youden rectangles have $v > 4$, so their values of E_k are close to 1, the upper limit for an efficiency factor; these triple Youden rectangles are thus statistically very efficient. The efficiency factor E_k for the 4×13 triple Youden rectangle in Fig. 2 is $90/91 = 0.989$, whereas that for the 7×15 triple Youden rectangle in Fig. 4 is $119/120 = 0.992$.

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