The Hall-Condition Number of a Graph

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Abstract. The Hall-condition number $s(G)$ of a graph G is defined and some of its fundamental properties are derived. This parameter, introduced in [6], bears a certain relation to the chromatic number $\chi(G)$ and the choice number $c(G)$ (see [3] and [7]).

One result here, that $\chi(G) - s(G)$ may be arbitrarily large, solves a problem posed in [6].

1. Definitions and background

The parameter of the title, to be defined below, is related to a class of problems variously designated by the terms *choosability* (see [3]), *list-colorings* (see [1], [2], [5] and [6]), and *coloring with prescribed colors* (see [7]). In these problems the vertices of a simple graph G are supplied with finite sets or lists of colors—one list per vertex—and it is desired to choose for each vertex a color from its list in such a way that adjacent vertices are colored differently. We will refer to such colorings, chosen from lists, as vertex list colorings.

The two best-known parameters used in discussing these problems are the *choice number* $c(G)$, introduced in [3] and [7], and the familiar (vertex) chromatic number $\chi(G)$. The choice number of G is the smallest positive integer among those m with the property that the desired coloring can always be found if the lists are at least of length m. Since $\chi(G)$ is defined similarly, but with the restriction that the lists must all be the same, it is clear that $\chi(G) \leq c(G)$. It is shown in [3] and in [7] that $c(G)$ can be very much larger than $\chi(G)$.

There is a conjecture of recent vintage (see [1] and [5]) that $\chi(G) = c(G)$ when G is a line graph (of a simple graph). This is known as the *edge list-coloring conjecture*. The restriction to the case when G is $L(K_{n,n})$, the line graph of $K_{n,n}$, is called Dinitz's Conjecture (or Problem), originally posed by Jeff Dinitz in the form of a problem about Latin-like squares. (Make the edges of $K_{n,n}$ correspond to the cells of an $n \times n$ array, each equipped with a list of length n. Can these cells be filled, from their respective lists, so that no symbol occurs twice in the same row or column?)

The inspiration for [5] and [6] was the realization, due to Vizing and Hilton, independently, and probably others, that Hall's Theorem [4], when interpreted properly, gives a necessary and sufficient condition for the existence of a vertex list-coloring of G in the special case when $G = K_n$. The condition is that for each set $U \subseteq V(K_n)$ of vertices, the cardinality of the union of the lists associated with the vertices in U is at least $|U|$. To understand the definition of "Hall's condition", below, it may help to observe that the condition given by Hall's theorem in the case $G = K_n$ is a condition on the assignment of lists to the vertices of G, more than on G itself. Note also that the independence number $i(H)$ of any non-empty induced subgraph H of K_n is one. (The independence number of a graph will be, as usual, the greatest size of an independent, i.e., mutually non-adjacent, set of vertices in the graph.) Finally, note that for $U \subseteq V(K_n)$, the cardinality of the union of the lists associated with members of U is obtained by tallying one for each color appearing in that union, and each of those ones is the independence number of the subgraph of K_n induced by those vertices in U that harbor that color in their lists.

Perhaps the time has come to be more formal. Let $G = (V, E)$ be a simple graph, $\mathcal{F}$ a collection of finite sets, and $C : V \to \mathcal{F}$ a function, to be called a *(vertex color) list assignment.* The restriction of C to any subset of V will continue to be denoted by C. Let S be the union of all the sets in the collection $\mathcal{F}$. For a subgraph H of G and $\sigma \in S$, we define $t(H, C, \sigma) = i$ [the subgraph induced, in H, by the vertices $\{v \in V(H); \sigma \in C(v)\}$]. That is to say, in order to compute $t(H, C, \sigma)$, you single out all the vertices of H that have the color σ on their list, consider the subgraph of H induced by those vertices, and find the independence number of that subgraph. The letter t is chosen to denote the result of this ghastly procedure in order to suggest the word "transversal", for reasons it would be tedious to explain.

As in [6], we will say that the pair (G, C) *satisfies Hall's condition* iff

$$(*) \qquad \sum_{\sigma \in S} t(H, C, \sigma) \geq |V(H)|$$

for each subgraph H of G.

Since deleting edges does not decrease independence numbers, (G, C) will satisfy Hall's condition if the inequality $(*)$ holds for all induced subgraphs H of G. From this observation it is not too terrible a task to verify that in the case $G = K_n$, (G, C) satisfies Hall's condition iff the hypothesis of Hall's theorem referred to earlier is satisfied, and thus iff there is what we will call a *C-coloring* of G, a choice of representatives from the sets $C(v)$, $v \in V$, so that the representatives of sets on adjacent vertices are distinct. [That is a C-coloring of G is a vertex list coloring of G, when the lists are dictated by C.]

For arbitrary G, it is quite easy to see that Hall's condition is necessary for the existence of a C-coloring of G, but, as shown in [6], it is rarely sufficient. (Indeed, with G fixed, the satisfaction of Hall's condition is a sufficient condition on C for the existence of C-coloring of G iff every block of G is a clique. This is Theorem 3 of [6].)

As in [6], the *Hall number* of G, $h(G)$, is the smallest positive integer m such that the satisfaction of Hall's condition and the additional requirement that

$|C(v)| \geq m$ for each $v \in V$ are sufficient for the existence of a C-coloring of G. It is obvious that $h(G) \leq c(G)$. It is shown in [6] (Corollary 2) that if either $\chi(G) \leq h(G)$ or $\chi(G) < c(G)$ then $h(G) = c(G)$. For these reasons, the authors of [5] and [6] entertain the hope that if, as seems likely, the edge list-coloring conjecture is false, we might be led to a counter-example by wrestling with the parameter $h(G)$ with G restricted to line graphs. In the case of Dinitz's problem, the question would be settled in the affirmative if $h(L(K_{n,n})) = n$, and in the negative if $h(L(K_{n,n})) > n$.

At last we come to the parameter that this paper is about. The *Hall-condition number* of G, denoted $s(G)$ as in [6], is the smallest positive integer among those m with the property that if C is a list assignment to the vertices of G, and $|C(v)| \geq m$ for all $v \in V$, then (G, C) satisfies Hall's condition. While we are at it, let $s_0(G)$ be defined similarly, except that the list assignments C are confined to constant maps (i.e., the same list is to be supplied to the various vertices of G). Clearly $s_0(G) \leq s(G)$. It is shown in [6] (Theorem 1) that $s(G) \leq \chi(G)$. [This will be proven below, as well.] Following one's nose through the definitions, and keeping in mind the fact that satisfaction of Hall's condition is necessary for the existence of a C-coloring of G, it is easy to see that if $s(G) \geq h(G)$, then $s(G) = c(G)$ (and, therefore, $c(G) = \chi(G)$). The converse, that $s(G) = c(G)$ implies $s(G) \geq h(G)$, is a trivial consequence of the previous remark that $h(G) \leq c(G)$.

It is also elementary to see that a sort of dual statement holds: if $h(G) \geq s(G)$, then $h(G) = c(G)$, and conversely.

It is these relations with the other parameters, $\chi(G)$, $c(G)$, and $h(G)$, that make $s(G)$ worth looking at, and one other thing: by Theorem 1, below, $s(G)$ is straightforwardly (although not efficiently) computable by looking at the graph G alone, with no lists stacked on the vertices, and the same cannot, so far as we know at present, be said of $h(G)$ and $c(G)$.

The result of Theorem 1 here is "sort of" proven, in haste, at the tail end of [6], with a reference to a proof elsewhere in that paper. Also, graphs G for which $s(G) = \chi(G) - 1$ are described in that paper, and it is wondered how much less than $\chi(G)$ the value of $s(G)$ can be. My aim here is to give an orderly proof of Theorem 1, to deduce some other potentially useful properties of $s(G)$, to show that $\chi(G) - s(G)$ can be as large as you want, and to posit a conjecture about $\chi(G)/s(G)$ that now seems reasonable.

2. Results

Throughout, G will be a simple graph. The clique number of G will be denoted $\omega(G)$.

Theorem 1.

$$s_0(G) = s(G) = \max\left\{\left\lceil \frac{|V(H)|}{i(H)} \right\rceil ; H \textit{ is a subgraph of } G\right\}$$
$$= \max\left\{\left\lceil \frac{|V(H)|}{i(H)} \right\rceil ; H \textit{ is an induced subgraph of } G\right\}.$$

Corollary 1. $\omega(G) \le s(G) \le \chi(G)$.

Corollary 2. *If H is a subgraph of G, then $s(H) \le s(G)$.*

If G_1 and G_2 are simple graphs on disjoint sets of vertices, let G_1G_2 be the graph obtained by joining all G_1 vertices to all G_2 vertices; i.e.,

$$V(G_1G_2) = V(G_1) \cup V(G_2) \text{ and}$$
$$E(G_1G_2) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1), v \in V(G_2)\}.$$

It is easy to see that $\chi(G_1G_2) = \chi(G_1) + \chi(G_2)$ and that $\omega(G_1G_2) = \omega(G_1) + \omega(G_2)$.

Theorem 2. *If G_1 and G_2 are graphs then $s(G_1G_2) \le s(G_1) + s(G_2)$.*

Corollary 3. *If U is an independent set of vertices in G, then*

$$s(G) - 1 \le s(G - U) \le s(G).$$

[$G - U$ denotes the graph obtained from G by deleting the vertices in U, and all edges incident to them.]

Theorem 3. *Suppose that r and t are positive integers, $4 \le m_1 \le \cdots \le m_r$ are integers, and that $G = K_t \prod_{j=1}^{r} C_{m_j}$, a product of a complete graph and the cycles C_{m_j}. Then*

$$t + 2r \le s(G) \le \max\left\{t + 2r, 2r + \left\lceil \frac{t+r}{2} \right\rceil\right\}.$$

Furthermore, if $m_1 = \cdots = m_r = m$, then

$$s(G) = 2r + t \quad \textit{if } m \textit{ is even}$$
$$\textit{and } s(G) = \max\left\{2r + t, 2r + \left\lceil \frac{2}{m-1}(r+t) \right\rceil\right\}$$

if m is odd.

Corollary 4. *$\{\chi(G) - s(G); G$ is a simple graph$\}$ is unbounded.*

Corollary 5. *If the m_j are all even, then $s(G) = t + 2r$.*

Corollary 4 solves a problem posed in [6], but in a way that raises a new question.

Problem. *How large can* $\chi(G)/s(G)$ *be?*

Conjecture. $\chi(G)/s(G) < \frac{3}{2}$ *for all* G.

Taking r and m (odd) large in Theorem 3 shows that $\frac{3}{2}$ is the smallest, best hope for this conjecture, compatible with what little is known about $s(G)$ so far.

3. Proofs

Proof of Theorem 1: Let

$$M(G) = \max\left\{\left\lceil\frac{|V(H)|}{i(H)}\right\rceil; H \text{ is a subgraph of } G\right\}.$$

Then $M(G) \geq \max\{\lceil\frac{|V(H)|}{i(H)}\rceil; H$ is an induced subgraph of $G\}$. The inequality the other way arises from the observation that deleting edges does not decrease the independence number.

Suppose that C is a constant color list assignment to G, such that the constant list C contains $s_0(G)$ colors. Suppose that H is a subgraph of G. By the definition of $s_0(G)$, (*) holds; clearly $t(H,C,\sigma) = i(H)$ if $\sigma \in C$, and $t(H,C,\sigma) = 0$ otherwise. Thus, from (*),

$$|V(H)| \leq \sum_{\sigma\in S} t(H,C,\sigma) = i(H)s_0(G)$$

It follows that $M(G) \leq s_0(G)$.

It remains to be seen that $s(G) \leq M(G)$. Let C be a color list assignment to G with $|C(v)| \geq M(G)$ for all $v \in V$. Suppose that H is a subgraph of G. We aim to show that (*) holds. Let U be a set of $i(H)$ independent vertices in H. Let U also denote the graph consisting of these vertices and no edges, an induced subgraph of H. We have

$$\begin{aligned}\sum_{\sigma\in S} t(H,C,\sigma) &\geq \sum_{\sigma\in S} t(U,C,\sigma)\\ &= \sum_{u\in U} |C(u)|\\ &\geq M(G)|U| = M(G)i(H)\\ &\geq |V(H)|\end{aligned}$$

∎

Proof of Corollary 1: The inequality $\omega(G) \leq s(G)$ follows from the theorem and the observation that $i(H) = 1$ for any complete subgraph H of G.

For any subgraph H of G,

$$|V(H)| \leq \chi(H) i(H) \leq \chi(G) i(H); \text{ thus } \chi(G) \geq s(G).$$

∎

Corollary 2 is immediate from Theorem 1. It is also easy, but tedious, to extract from the definitions. Corollary 1 was also proven in [6].

Proof of Theorem 2: Each induced subgraph of $G_1 G_2$ is of the form $H_1 H_2$ with H_j an induced subgraph of G_j $j = 1,2$. [It is possible that $V(H_j)$ is empty, for one $j \in \{1,2\}$.] Clearly $i(H_1 H_2) = \max\{i(H_1), i(H_2)\}$, so

$$\frac{|V(H_1 H_2)|}{i(H_1 H_2)} = \frac{|V(H_1)|}{i(H_1 H_2)} + \frac{|V(H_2)|}{i(H_1 H_2)} \leq \frac{|V(H_1)|}{i(H_1)} + \frac{|V(H_2)}{i(H_2)}.$$

The desired conclusion now follows from Theorem 1. ∎

Proof of Corollary 3: G is a subgraph of $U \cdot (G - U)$, so, by Corollary 2 and Theorem 2,

$$s(G) \leq s(U \cdot (G - U)) \leq s(U) + s(G - U) = 1 + s(G - U).$$

The inequality $s(G - U) \leq s(G)$ follows from Corollary 2. ∎

Proof of Theorem 3: The inequality $t + 2r \leq s(G)$ follows from Corollary 1 and the observation that

$$\omega(G) = \omega(K_t) + \sum_{j=1}^{r} \omega(C_{m_j}) = t + 2r.$$

In what follows, it will be useful to keep in mind that if H is an induced subgraph of a cycle C_m and if $|V(H)| < m$, then H is an induced subgraph of a path P_{m-1} and $|V(H)| \leq 2i(H)$. If $H = C_m$ then $i(H) = m/2$ if m is even, and $(m-1)/2$ if m is odd, so $|V(H)| = 2i(H)$ or $|V(H)| = 2i(H) + 1$. Thus $|V(H)| \leq 2i(H) + 1$, in any case.

Suppose H is an induced subgraph of G. If $i(H) = 1$ then H is complete, whence

$$|V(H)|/i(H) = |V(H)| \leq \omega(G) = t + 2r.$$

So, suppose that $i(H) \geq 2$. Let $H_0 = \langle V(H) \cap V(K_t) \rangle$, the subgraph of H induced by $V(H) \cap V(K_t)$, and $H_j = \langle V(H) \cap V(C_{m_j}) \rangle$, $j = 1, \ldots, r$; then $H = \prod_{j=0}^{r} H_j$ and $i(H) = \max_{0 \leq j \leq r} i(H_j) = \max_{1 \leq j \leq r} i(H_j)$. [$H_0$ is complete, so $i(H_0) \leq 1 \leq i(H)$.] We have

$$\begin{aligned} |V(H)|/i(H) &= (\sum_{j=0}^{r} |V(H_j)|)/i(H) \\ &\leq t/2 + i(H)^{-1}\sum_{j=1}^{r}|V(H_j)| \\ &\leq t/2 + i(H)^{-1}\sum_{j=1}^{r}(2i(H_j)+1) \\ &\leq t/2 + 2r + \sum_{j=1}^{r} i(H)^{-1} \\ &\leq 2r + (t+r)/2 \end{aligned}$$

This completes the proof of the inequality

$$s(G) \leq \max\left\{t+2r, 2r + \left\lceil \frac{t+r}{2} \right\rceil\right\},$$

in view of Theorem 1.

Now suppose that $m_1 = \cdots = m_r = m$. Suppose first that m is even. Then $\chi(G) = t + 2r$ so the result $s(G) = t + 2r$ follows from Corollary 1 and the first part of this theorem.

Now suppose that m is odd, and H is as above. If $i(H) < (m-1)/2$, then $|V(H_j)| \leq 2i(H), j = 1, \ldots, r$, so, again,

$$|V(H)|/i(H) \leq 2r + t/i(H) \leq t + 2r.$$

So suppose that $i(H) = (m-1)/2$; then

$$|V(H)|/i(H) \leq \frac{2t}{m-1} + \frac{2m}{m-1}r = 2r + \frac{2}{m-1}(r+t),$$

with equality if $H = G$. The conclusion follows. ∎

Proof of Corollary 4: Take the m_j all odd in Theorem 3. Then $\chi(G) = \chi(K_t) + \sum_{j-1}^{r} \chi(C_{m_j}) = t + 3r$. If $t \geq r$, then $s(G) = t + 2r$, so

$$\chi(G) - s(G) = r.$$

∎

Proof of Corollary 5:

$$t + 2r \leq s(G) \leq \chi(G) = \chi(K_t) + \sum_{j=1}^{r} \chi(C_{m_j}) = t + 2r.$$

∎

Note that the result of Theorem 3 shows that the inequality of Theorem 2 may be strict.

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