The Induced Path Number of Bipartite Graphs

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Abstract. The induced path number of a graph G is the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a path. The induced path number is investigated for bipartite graphs. Formulas are presented for the induced path number of complete bipartite graphs and complete binary trees. The induced path number of all trees is determined. The induced path numbers of meshes, hypercubes, and butterflies are also considered.

1. Introduction

The *arboricity* of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces a forest. In what must be considered one of the major results in graph theory, Nash-Williams [10] proved that the arboricity of a nonempty graph G is given by $\max\lceil |E(H)|/(|V(H)|-1)\rceil$, where the maximum is taken over all nontrivial induced subgraphs H of G. Harary [6] specialized this concept when he defined the linear arboricity of a nonempty graph G as the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces a linear forest (a forest in which every component is a path). He specialized this concept even further in [6] when he defined the path number of a nonempty graph G as the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces a path. A number of results on path numbers of graphs were obtained by Stanton, Cowan, and James [13].

In this paper we are interested in partitioning the vertex set of a graph (rather than the edge set). It is this situation that gives rise to colorings. The *chromatic number* of a graph G is the minimum number of colors that can be assigned to the vertices of G so that adjacent vertices are assigned distinct colors. Equivalently, the chromatic number of G is the minimum number of independent subsets into

[1] Research supported in part by Office of Naval Research Contract N00014-91-J-1060.
[2] Research supported in part by the ACM/SIGDA Scholarship 1990, and by National Science Foundation Grant USE-9052346.

which $V(G)$ can be partitioned. A color class of G consists of those vertices of G that are assigned the same color. Thus, each color class is independent and so induces an empty subgraph.

The idea of generalizing colorings of graphs was described in detail by Harary [4] and by Mynhardt and Broere [9]. Many generalizations and variations of the chromatic number have been given; for example, see [2], [8], and [12], to name just a few. In particular, let P be a graphical property possessed by the trivial graph K_1. Then the *P-chromatic number* of a graph G is the minimum number of subsets in a partition of $V(G)$ so that each subset induces a subgraph having property P. Thus, for the ordinary chromatic number, P is the property of being independent. In this article, we consider the property P of being connected but in the simplest possible manner, namely that of being a path. More formally, the *induced path number* $\rho(G)$ of a graph G is the minimum number of subsets in a partition of $V(G)$ so that each subset induces a path. For the graph G of Figure 1, $\rho(G) = 2$. A partition of $V(G)$ into two subsets, each of which induces a path, is also shown in Figure 1.

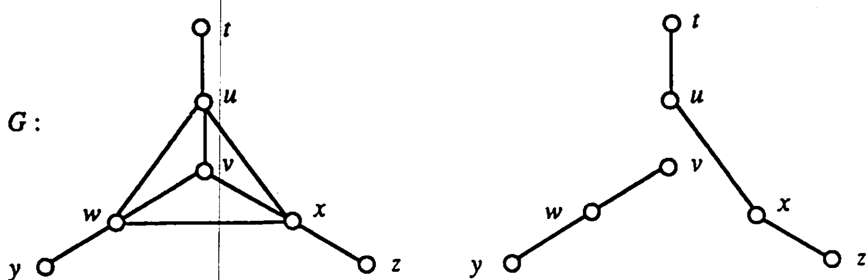


Figure 1: a graph G with $\rho(G) = 2$

The induced path number of a graph G is thus the P-chromatic number of G, where P is the property of being a path. For this reason, we say that a graph G is *n-path colorable* if $\rho(G) \leq n$. In [3] Chartrand, Kronk, and Wall defined the *vertex-arboricity* $a(G)$ of a graph G as the minimum number of subsets in a partition of $V(G)$ so that each subset induces a forest while in [7] Harary introduced the *linear vertex arboricity* $lva(G)$ of a graph G, defined as the minimum number of subsets in a partition of $V(G)$ such that each subset induces a linear forest. Consequently, $a(G) \leq lva(G) \leq \rho(G)$ for every graph G. The minimum number of components in a spanning linear forest in a graph G was introduced by Boesch, Chen, and McHugh [1] and called the *island number* $is(G)$ of G by Slater [11]. Thus $is(G) \leq \rho(G)$ for every graph G.

The induced path number does not appear to be related to the chromatic number of a graph. The induced path number can be substantially larger than the chromatic number, as in the case of the star $K_{1,n}$, $n \geq 2$, for which $\rho(K_{1,n}) = n - 1$ while

$\chi(K_{1,n}) = 2$. On the other hand, it can be substantially smaller, as in the case of the complete graph K_n, where $\rho(K_n) = \lceil n/2 \rceil$ and $\chi(K_n) = n$.

In this paper we investigate the induced path number of bipartite graphs. If the cardinalities of the partite sets of a bipartite graph G are m and n, we refer to G as an $m \times n$ bipartite graph. If G is nontrivial and connected, then its sets are uniquely determined. In the next section we study the induced path number of complete bipartite graphs, complete binary trees, meshes, hypercubes, and butterflies. The induced path number of trees is determined in Section 3. We follow [4] for graph theory terminology.

2. The Induced Path Number of Special Classes of Bipartite Graphs

In this section we investigate the induced path number of several special classes of bipartite graphs, namely, complete bipartite graphs, complete binary trees, meshes, hypercubes, and butterflies.

2.1. The Induced Path Number of Complete Bipartite Graphs

First we present a formula for the induced path number of the complete bipartite graph $K_{m,n}$, where $m \leq n$, whose value depends only on whether $n \leq 2m$ or $n \geq 2m$.

Theorem 1. *Let m and n be positive integers. Then*

$$\rho(K_{m,n}) = \begin{cases} \lceil \frac{m+n}{3} \rceil & \text{if } m \leq n \leq 2m \\ n - m & \text{if } n \geq 2m \end{cases}$$

Proof: First we consider the case $m \leq n \leq 2m$. The largest order of an induced path in $K_{m,n}$ is 3; therefore,

$$\rho(K_{m,n}) \geq \frac{m+n}{3}.$$

Since $\rho(K_{m,n})$ is an integer, $\rho(K_{m,n}) \geq \lceil \frac{m+n}{3} \rceil$. It remains, therefore, to show that the vertex set of $K_{m,n}$ can be partitioned into $\lceil \frac{m+n}{3} \rceil$ subsets, each of which induces a path. Let the partite sets of $K_{m,n}$ be V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. Let

$$r = \left\lfloor \frac{2m-n}{3} \right\rfloor = \frac{2m-n-k}{3},$$

where k is 0, 1, or 2. By selecting r vertices in V_2 and $2r$ vertices in V_1, we obtain r induced paths of order 3. This leaves $m - 2r = (2n - m + 2k)/3$ vertices in V_1 and $n - r = (4n - 2m + k)/3 = 2(m - 2r) - k$ vertices in V_2. Consequently, there are $m - 2r - k$ additional induced paths of order 3 and k others of order 2. In all, then, this produces

$$r + (m - 2r - k) + k = m - r = \frac{m+n+k}{3} = \left\lceil \frac{m+n}{3} \right\rceil$$

induced paths in $K_{m,n}$.

Suppose next that $n \geq 2m$. There are m induced paths of order 3 in this case along with $n - 2m$ trivial paths, for a total of $n - m$ paths. This is clearly the minimum number. ∎

2.2. *The Induced Path Number of Complete Binary Trees*

Let B_h denote the complete binary tree of height h. The vertices of B_h are labeled according to the level to which they belong. In particular, the root (at level 0) is labeled $v_{0,1}$. The two vertices at level 1 are $v_{1,1}$ and $v_{1,2}$. In general, the vertices at level ℓ $(0 \leq \ell \leq h)$ are labeled $v_{\ell,1}, v_{\ell,2}, \ldots, v_{\ell,2^\ell}$ (see Figure 2). Every vertex at level i is adjacent to a unique vertex at level $i-1$, for $i \geq 1$, referred to as its parent vertex. In particular, the parent of $v_{i,j}$ is $v_{i-1,\lfloor j/2 \rfloor}$ (see Figure 2). The edges between the vertices of level i and level $i+1$ $(0 \leq i \leq h-1)$ are referred to as *level i edges* and this set of 2^{i+1} edges is denoted by E_i. Thus $\bigcup_{i=1}^{h-1} E_i = E(B_h)$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

Denote by $\mathcal{F}$ the set of spanning forests of B_h with $\rho(B_h)$ components, each of which is a path. We define a *canonical forest* $F^* \in \mathcal{F}$ as that forest in $\mathcal{F}$ with edge set $E(B_h) - \cup E_i$, where $i \in \{h-2, h-4, \ldots, k\}$ and where $k = 0$ if h is even and $k = 1$ if h is odd. Figure 3 shows canonical forests for B_2 and B_3.

We now investigate the induced path number of a complete binary tree. A formula for this is given in the following theorem.

Theorem 2. *For the complete binary tree B_h of height h,*

$$\rho(B_h) = \begin{cases} \frac{1}{3}(2^{h+1} + 1) & \text{if } h \text{ is even} \\ \frac{1}{3}(2^{h+1} - 1) & \text{if } h \text{ is odd} \end{cases}$$

Proof: The proof of this theorem consists of two parts. First, we consider an arbitrary forest $F \in \mathcal{F}$ and convert it to a canonical forest F^*. Second, we count the number of components (maximal paths) in F^*.

Let $F \in \mathcal{F}$. We modify F to produce a canonical forest F^* without changing the number of components of F. Then the induced path number of B_h equals the number of components in F^*.

We consider two levels at a time, beginning with levels h and $h-1$. Note that every vertex at level h must be the end-vertex of a nontrivial path in F or a trivial path by itself. Considering a vertex v at level $h-1$, we have four cases.

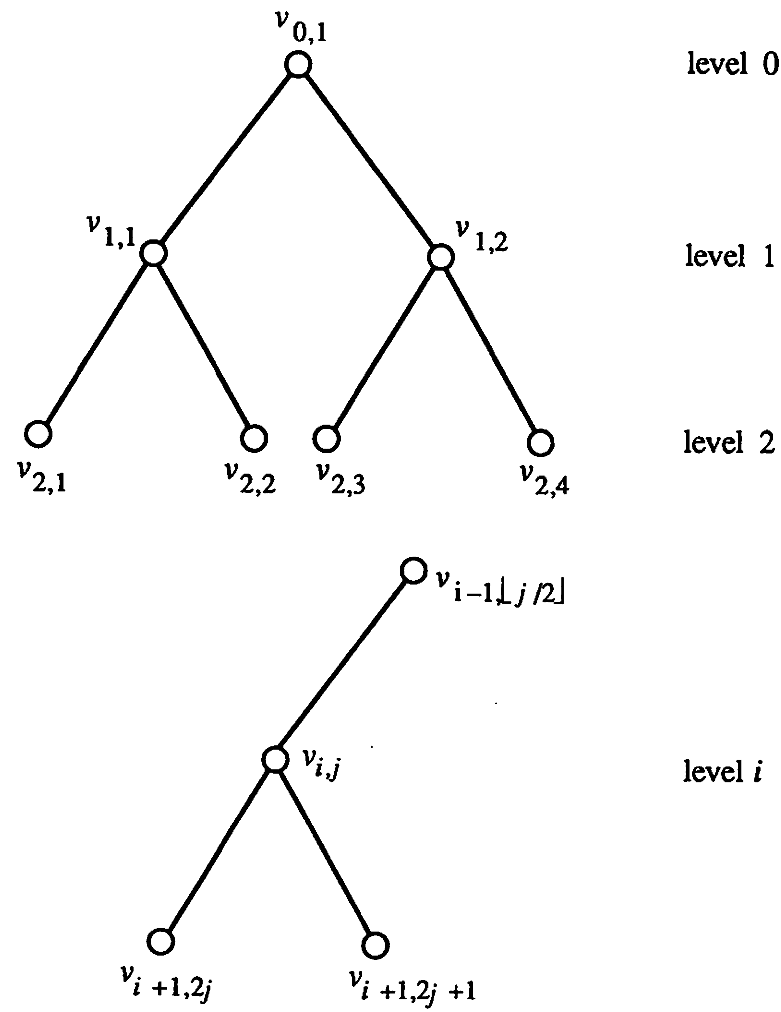


Figure 2: Portions of complete binary trees

<u>Case 1</u> *The vertex v is a path by itself.* In this case, the children of v must be trivial paths. A forest with fewer components, each of which is a path, can be obtained by simply adding edges between v and its children, reducing the number of components by 2. Therefore, $F \notin \mathcal{F}$, producing a contradiction.

<u>Case 2</u> *The vertex v is an end-vertex of a path and both children of v are trivial components.* In such a case, the path ending at v can be extended to one of its children, thereby reducing the number of components by 1. Therefore, $F \notin \mathcal{F}$, again a contradiction.

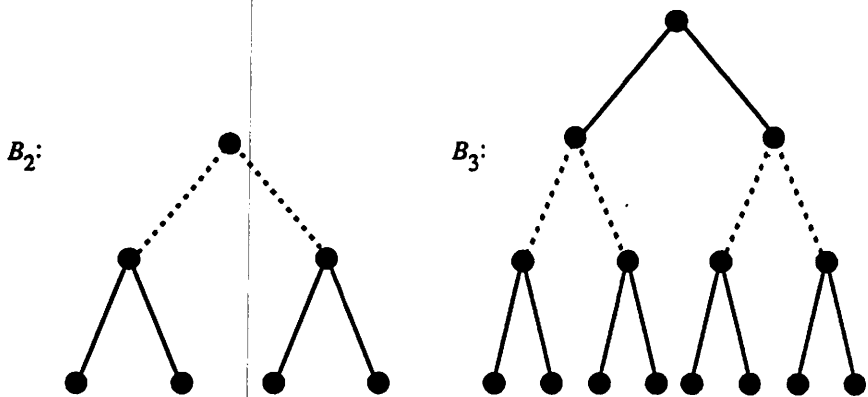


Figure 3: Canonical forests for B_2 and B_3

Case 3 *The vertex v is the central vertex of a path of length 2 whose end-vertices are its two children.* In this case, the component belongs to F^*.

Case 4 *The vertex v lies in a path, one of whose end-vertices is a child of v, while the other child is a trivial path.* In this case, we replace the path containing v by a path obtained by deleting the edge between v and its parent and introducing the edge between v and the child that was not present in F. The resulting forest contains the same number of components as F.

We repeat the above procedure for vertices at levels $h-2$ and $h-3$ and continue this for pairs of levels until there is no edge $e \in \cup E_i$ for $i \in \{h-2, h-4, \ldots, k\}$ in the resulting forest.

We have shown that $\rho(B_h)$ is the number of components in F^*. Now we count the number of components in F^*. Since there are 2^i vertices at each level i, and two vertices at level i are joined to one vertex at level $i-1$ to form a component, the total number of components formed by vertices of level i and $i-1$ are 2^{i-1} for $i > 0$. There are now two cases to consider.

Case 1 *Assume h is even.* Since the vertices at even level i are combined with the vertices at odd level $i-1$, the vertex at level 0 will form a component itself. Thus the total number of components (where $S = \{h-1, h-3, \ldots, 1\}$) is

$$\rho(B_h) = 1 + \sum_{i \in S} 2^i = \frac{1}{3}(2^{h+1} + 1).$$

Case 2 *Assume h is odd.* In this case the vertices at odd level i are combined with the vertices at even level $i-1$. Therefore, the total number of components (where $S = \{h-1, h-3, \ldots, 0\}$) is

$$\rho(B_h) = 1 + \sum_{i \in S} 2^i = \frac{1}{3}(2^{h+1} - 1).$$

∎

2.3. The Induced Path Number of Meshes

For positive integers d_1 and d_2, the 2*-dimensional mesh* M_{d_1,d_2} is defined as the product $P_{d_1} \times P_{d_2}$. In what follows, we assume that the mesh M_{d_1,d_2} is drawn and labeled as shown in Figure 4.

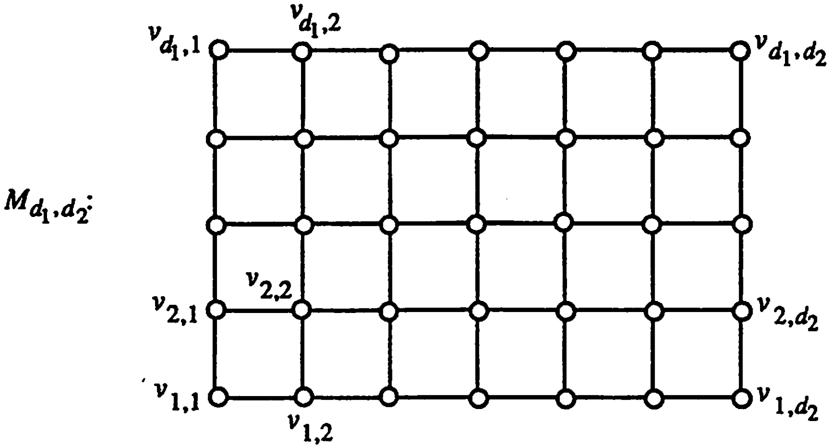


Figure 4: The mesh M_{d_1,d_2}

Certainly, $\rho(M_{d_1,d_2}) = 1$ if and only if either $d_1 = 1$ or $d_2 = 1$. What may be surprising, however, is that if neither $d_1 = 1$ nor $d_2 = 1$, then $\rho(M_{d_1,d_2}) = 2$.

Theorem 3. *The induced path number of the* 2*-dimensional mesh* M_{d_1,d_2} *is* $\rho(M_{d_1,d_2}) = 2$ *for* $d_1, d_2 \geq 2$.

Proof: The proof is constructive in nature. In particular, we describe a partition of $V(M_{d_1,d_2})$ into two sets V_1 and V_2, each of which induces a path. Assume, without loss of generality, that $d_1 \leq d_2$.

Case 1 *Assume that* $d_1 = 2$. In this case, the sets $V_1 = \{v_{1,i} \mid 1 \leq i \leq d_2\}$ and $V_2 = \{v_{2,i} \mid 1 \leq i \leq d_2\}$ give a desired partition of $V(M_{d_1,d_2})$.

Case 2 *Assume that* $d_1 = 3$. The paths shown in Figure 5 give the desired result.

Case 3 *Assume that* $d_1 > 3$ *and* d_1 *is odd.* Let P_1 and P_2 be the paths indicated in Figure 6, where the vertices $v_{1,1}$ and $v_{3,2}$ are the end-vertices of P_1, while $v_{2,1}$ and $v_{4,2}$ are the end-vertices of P_2. Further, let V_1 be the vertex set of P_1 and V_2 the vertex set of P_2.

Let G' be the subgraph induced by $V_1 \cup V_2$. Then,

$$M_{d_1',d_2'} = M_{d_1,d_2} - V(G')$$

is a mesh, where $d_1' = d_1 - 4$ and $d_2' = d_2 - 4$. We then apply the above procedure to $M_{d_1',d_2'}$ to produce two paths P_1' and P_2', where one end-vertex of P_1'

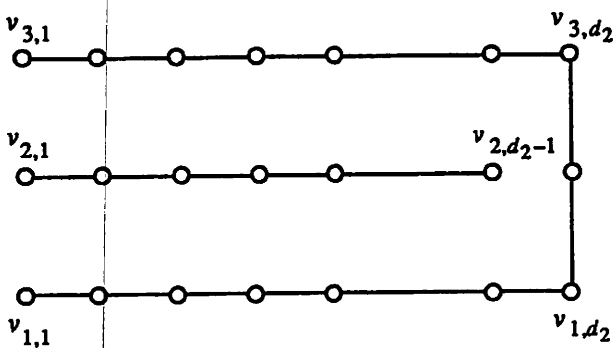


Figure 5: $\rho(M_{3,d_2}) = 2$

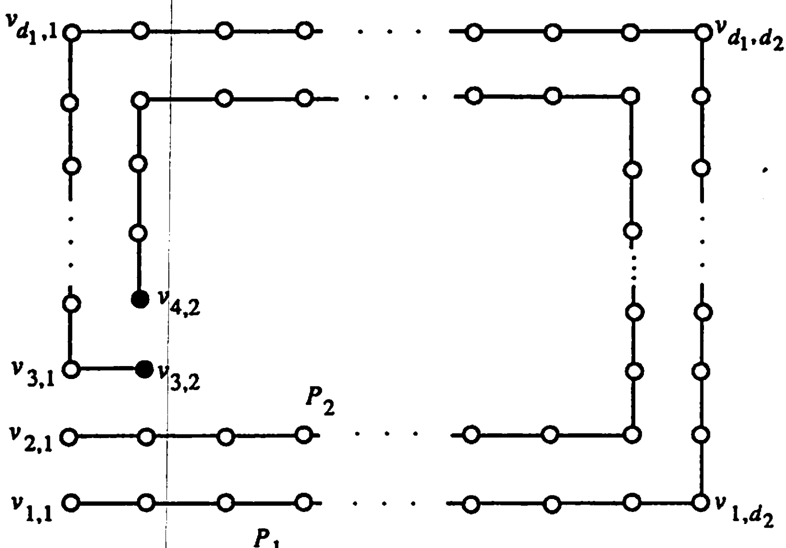


Figure 6: $\rho(M_{d_1,d_2}) = 2$ for d_1 odd and $d_1 > 3$

is $v_{3,3}$. Therefore, the end-vertex $v_{3,2}$ of P_1 is adjacent to $v_{3,3}$, and a new path P_1'' is produced from P_1 and P_1'. Similarly, a new path P_2'' can be formed from P_2 and P_2'.

We repeat this if $d_1' \geq 5$. If d_1' is either 1 or 3, then we can apply the base case as described earlier.

Case 4 *Assume that* $d_1 > 3$ *and* d_1 *is even.* Consider the vertex set $V' = V(M_{d_1,d_2}) - \{v_{1,i} \mid 1 \leq i \leq d_2\}$. The graph induced by V' is the mesh M_{d_1',d_2} where $d_1' = d_1 - 1$ and d_1' is odd.

Using Case 3, we can partition V' into subsets V_1' and V_2' that induce paths P_1'

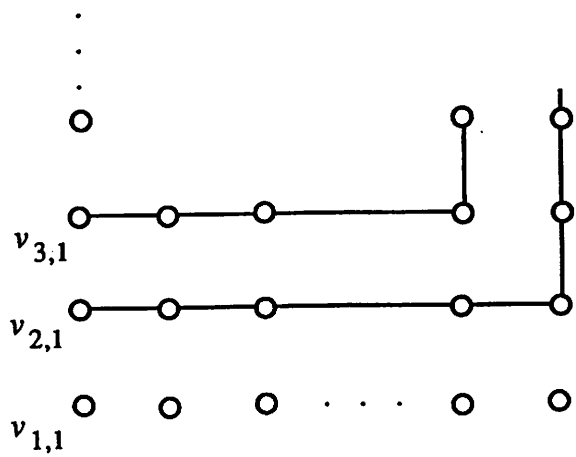
and P_2' respectively (see Figure 7a). Also note that $v_{2,1}$ and $v_{3,1}$ are end-vertices of the paths P_1' and P_2', respectively (by the construction procedure described in Case 3). We construct paths P_1 and P_2 by defining the sets V_1 and V_2 as follows:

$$V_1 = V_1' - \{v_{2,1}\}$$

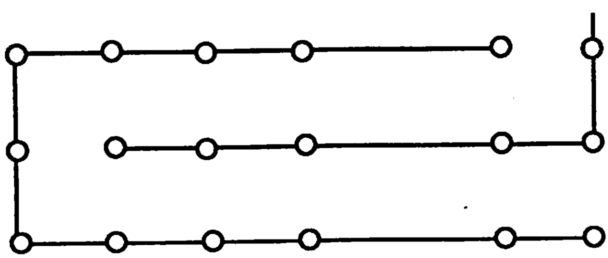
and

$$V_2 = V_2' \cup \{v_{2,1}\} \cup \{v_{1,i} \mid 1 \leq i \leq d_2\}.$$

As seen in Figure 7b, V_1 and V_2 induce paths. ∎



(a)



(b)

Figure 7: $\rho(M_{d_1,d_2}) = 2$ for $d_1 > 3$ and d_1 even

We now extend our result to 3-dimensional meshes. Given a 3-dimensional mesh M_{d_1,d_2,d_3}, we form two 2-dimensional meshes as we now describe.

For $k = 1, 2, \ldots, d_3$, let

$$V_k = \{v_{i,j,k} \mid 1 \leq i \leq d_1, 1 \leq j \leq d_2\}.$$

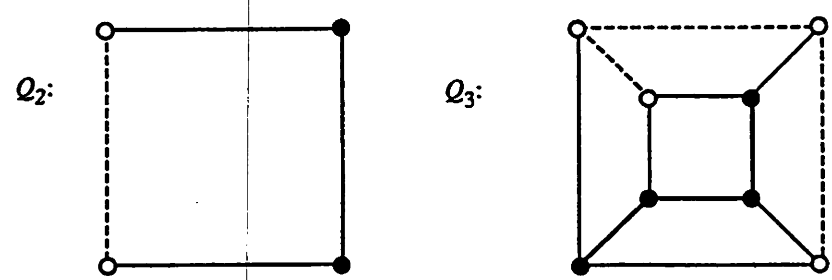


Figure 8: The hypercubes Q_2 and Q_3

Each set V_k, $1 \leq k \leq d_3$, induces a 2-dimensional mesh. Therefore, using Theorem 3, we see that each set V_k can be partitioned into subsets V_1^k and V_2^k which induce paths P_1^k and P_2^k, respectively. Observe that each vertex in P_1^k (respectively P_2^k) is adjacent to its corresponding vertex in P_1^{k+1} (respectively P_2^{k+1}) for $1 \leq k < d_3$. Therefore, the vertex sets $V_1' = \bigcup_{k=1}^{d_3} V_1^k$ and $V_2' = \bigcup_{k=1}^{d_3} V_2^k$ induce two 2-dimensional meshes. Using Theorem 3, we can color each of the two 2-dimensional meshes using two colors. Therefore, we need at most four colors to color a 3-dimensional mesh. Thus we have the following result.

Theorem 4. *The induced path number of the* 3*-dimensional mesh* M_{d_1,d_2,d_3} *is*

$$\rho(M_{d_1,d_2,d_3}) \leq 4.$$

Using the idea of decomposing a k-dimensional mesh into two $(k-1)$ dimensional meshes (as we did in Theorem 4), we have the following.

Theorem 5. *The induced path number of the k-dimensional mesh* $M_{d_1,d_2,\ldots,d_k}$ *is*

$$\rho(M_{d_1,d_2,\ldots,d_k}) \leq 2^{k-1}.$$

2.4. On the Induced Path Number of Hypercubes and Butterflies

In this section, we investigate the problem of determining the induced path number of two types of bipartite graphs that appear frequently in the study of parallel architectures, namely, hypercubes and butterflies. Recall that the d-dimensional hypercube Q_d is defined recursively by $Q_1 = K_2$ and $Q_d = Q_{d-1} \times K_2$ for $d \geq 2$. This problem appears quite complex, but we describe what is known for small values of d.

Theorem 6. *For* $d = 2, 3, 4, 5$, *the induced path number of* Q_d *is* $\rho(Q_2) = \rho(Q_3) = 2$, $\rho(Q_4) \leq 3$, *and* $\rho(Q_5) \leq 4$.

Proof: Since every hypercube Q_d where $d \geq 2$ contains cycles, $\rho(Q_d) \geq 2$. That $\rho(Q_2) = \rho(Q_3) = 2$ follows since Q_2 and Q_3 contain two induced vertex-disjoint paths that span the hypercube (see Figure 8).

That $\rho(Q_4) \leq 3$ follows from the partition of the vertex set of Q_4 shown in Figure 9. (For clarity, we have not shown all edges of Q_4.) That $\rho(Q_5) \leq 4$ follows from the partition of the vertex set of Q_5 shown in Figure 10. (Again, not all edges of Q_5 are displayed.) ∎

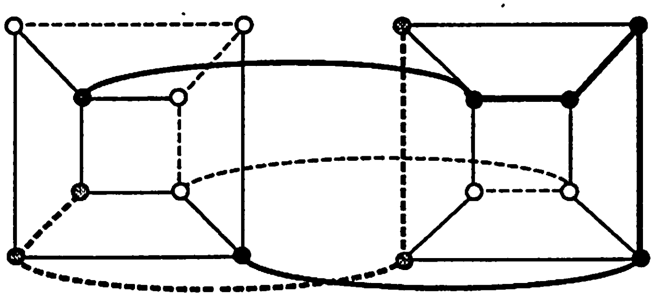


Figure 9: $\rho(Q_4) \leq 3$

By the definition of hypercubes, it follows that $\rho(Q_{d+1}) \leq 2\rho(Q_d)$. Because of this and the fact that $\rho(Q_5) \leq 4$, we can conclude that $\rho(Q_d) \leq 2^{d-3}$ for all $d \geq 5$. However, we believe that the bound 2^{d-3} is not even close to the actual value of $\rho(Q_d)$. Indeed, we conjecture the following:

Conjecture. *The induced path number of the d-dimensional hypercube Q is* $\rho(Q_d) \leq d$ *for* $d \geq 2$.

The other class of bipartite graphs that we consider in this section are the butterflies. Like the hypercubes, the butterflies are defined recursively. The first three butterflies B_1, B_2, and B_3 are shown in Figure 11.

By construction, the butterfly B_r contains 2^r induced paths of length r, so $\rho(B_r) \leq 2^r$ for all r. Since B_1 is a cycle, $\rho(B_1) = 2$. Also, for $r = 2k$, the butterfly $B_r = B_{2k}$ is a $k \cdot 2^{2k} \times (k+1) \cdot 2^{2k}$ bipartite graph. For $r = 2k-1$, the butterfly $B_r = B_{2k-1}$ is a $k \cdot 2^{2k-1} \times k \cdot 2^{2k-1}$ bipartite graph.

Every induced path in a bipartite graph contains at most one more vertex in one of its partite sets than the other. Since B_{2k} has $2k$ more vertices in one partite set than the other, it follows that $\rho(B_{2k}) \geq 2^k$. Hence, based on our earlier observation, we have the following:

Theorem 7. *The induced path number of the butterfly* B_r *is* $\rho(B_r) = 2r$ *for even positive integers* r.

Although we are unable to present a result for odd integers r, we do conjecture the following:

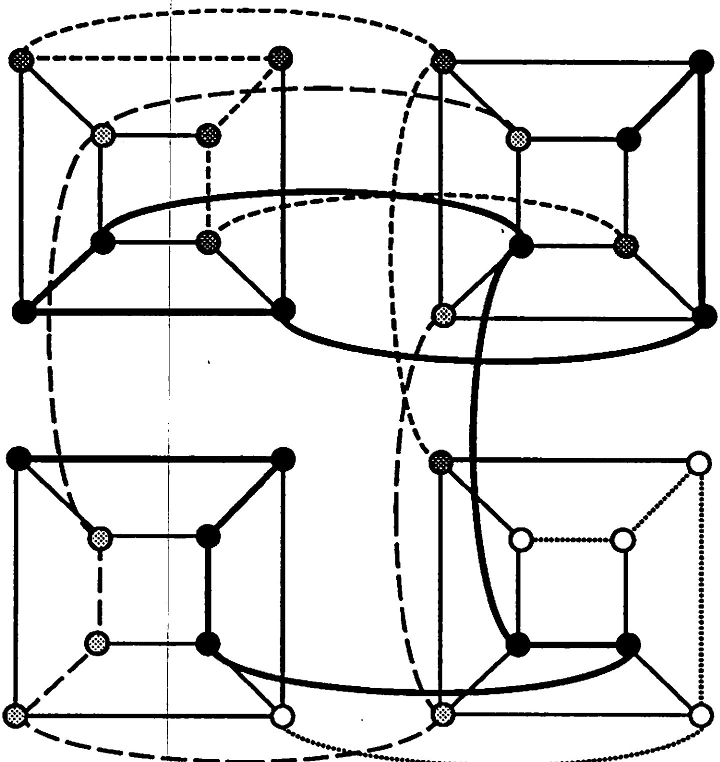


Figure 10: $\rho(Q_5) \leq 4$

Conjecture. *The induced path number of the butterfly* B_r *is* $\rho(B_r) = 2r$ *for odd positive integers* r.

3. The Induced Path Number of Trees

In this section we investigate the induced path number of trees. We begin by characterizing n-path colorable trees.

Theorem 8. *A tree* T *is* n*-path colorable if and only if there exists a set of fewer than* n *edges whose removal from* T *results in a linear forest.*

Proof: Suppose T is an n-path colorable tree. Then there is a partition $V_1, V_2, \ldots, V_k$ $(k \leq n)$ of $V(T)$ such that $\langle V_i \rangle$ is a path for all i $(1 \leq i \leq k)$. Since T is a tree, there are exactly $k - 1$ edges joining a vertex of some V_j and a vertex of some V_m, where $j \neq m$ and $1 \leq j, m \leq k$. The removal of these $k - 1 (< n)$ edges results in a linear forest.

We now consider the converse. First, observe that the removal of any edge from a forest F' results in a forest F'' containing one more component than that

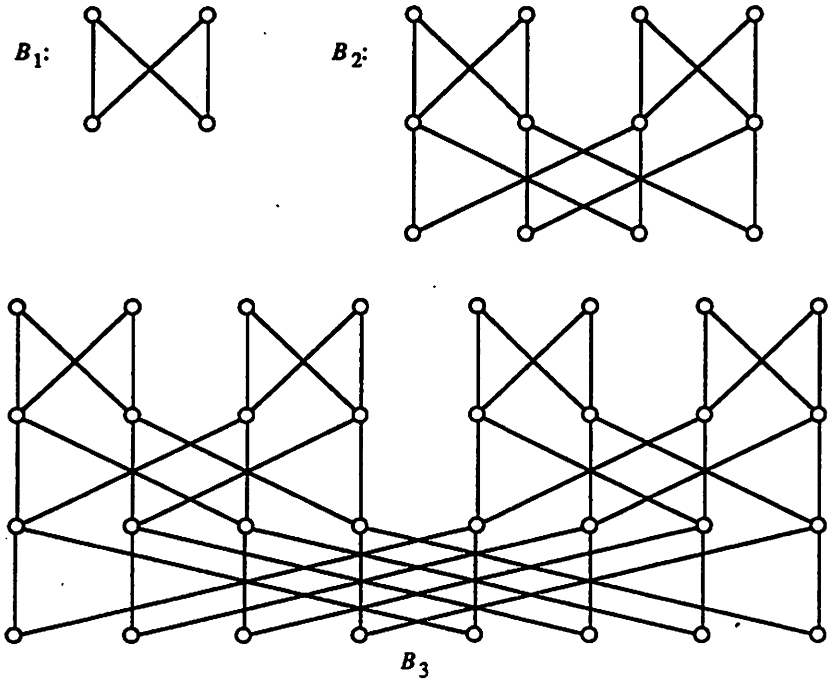


Figure 11: Butterflies

of F'. Suppose then that we remove $k(< n)$ edges from a tree T, resulting in a linear forest F. Then by the above observation, F contains $k+1(\leq n)$ components $F_1, F_2, \ldots, F_{k+1}$, each of which is a path. Consequently, the partition $V(F_1), V(F_2), \ldots, V(F_{k+1})$ of $V(T)$ shows that T is n-path colorable. ∎

We now have an immediate consequence of Theorem 8.

Corollary 9. *Let T be a tree of order p with $\rho(T) = n$. Then there exists a set E of $n-1$ edges of T but no fewer such that $T - E$ is a linear forest of size $p - \rho(T)$.*

Let T be a tree. For a vertex v of T with $\deg_T v \geq 3$, we define the *excess* degree $e(v)$ of v by

$$e(v) = \deg_T v - 2.$$

Theorem 10. *Let T be a tree, and let H be the forest induced by the vertices of T having degree 3 or more. Let H' be a spanning subforest of H of maximum size such that $\deg_{H'} v \leq \varepsilon(v)$ for every vertex v of H. Define*

$$\ell = |E(H')| + \sum_{v \in V(H)} [\varepsilon(v) - \deg_{H'} v]. \quad (1)$$

Then T is $(\ell+1)$-path colorable but not ℓ-path colorable.

Proof: First, we show that T is $(\ell+1)$-path colorable. Define H'' to be a spanning subforest of T obtained by adding, for every vertex v of H, to H' a total of $\varepsilon(v) - \deg_{H'} v$ edges of T that are incident with v but do not belong to H'. Observe that

$$|E(H'')| = |E(H')| + \sum_{v \in V(H)} [\varepsilon(v) - \deg_{H'} v] = \ell.$$

Let F be the forest obtained by deleting $E(H'')$ from T. Then for each vertex v of H, we have $\deg_F v = \deg_T v - \varepsilon(v) = 2$. Since $\deg_F v \le 2$ for all vertices v of T that do not belong to H, it follows that $\Delta(F) \le 2$. Therefore, by Theorem 8, T is $(\ell+1)$-path colorable.

Next, suppose to the contrary, that T is ℓ-path colorable. By Theorem 8 there are sets E_1 of fewer than ℓ edges of T such that $\Delta(T - E_1) \le 2$. Among all such sets E_1, let E_0 be one of minimum cardinality. Let H_0 be the spanning forest of T with edge set E_0, and let H_0' be a subforest of H_0 of maximum size such that $V(H_0')$ is the set of vertices of degree 3 or more in T and $\deg_{H_0'} v \le \varepsilon(v)$ for every vertex v of H_0'.

Then,

$$|E(H_0)| = |E(H_0')| + \sum_{v \in V(H_0')} [\varepsilon(v) - \deg_{H_0'} v]. \tag{2}$$

By the definition of H_0', we have $|E(H_0')| \le |E(H_0')|$. By (1), it follows that

$$\sum_{v \in V(H)} \varepsilon(v) = \ell + 2|E(H')|,$$

and by (2),

$$\sum_{v \in V(H_0')} \varepsilon(v) = |E(H_0)| + 2|E(H_0')|.$$

Since $V(H) = V(H_0')$, it follows that

$$\ell + 2|E(H')| = |E(H_0)| + 2|E(H_0')| < \ell + 2|E(H')|,$$

producing a contradiction. ∎

From this we have an immediate corollary.

Corollary 11. *Let T be a tree, and let H be the forest induced by the vertices of T having degree 3 or more. Let H' be a spanning subforest of H of maximum size that $\deg_{H'} v \le \varepsilon(v)$ for every vertex v of H. Then*

$$\rho(T) = 1 + |E(H')| + \sum_{v \in V(H)} [\varepsilon(v) - \deg_{H'} v].$$

With the aid of Corollary 11, it is possible to give an alternative proof of Theorem 2, but it does not have the informative nature as the proof provided for Theorem 2.

Next we determine upper and lower bounds for the induced path number of a tree. Recall that a tree T is called an $m \times n$ tree if its two partite sets have cardinalities m and n.

Theorem 12. *Let T be an $m \times n$ tree of order $p(= m + n)$ with $3 \leq m \leq n$. Then $p - \rho(T) \geq 4$.*

Proof: Because of Corollary 9, it suffices to show that T contains a linear forest of size 4. Let V_1 and V_2 be the partite sets of T. By hypothesis, $|V_i| \geq 3$ for $i = 1, 2$.

Let u be an end-vertex of T and suppose, without loss of generality, that $u \in V_1$. Then u is adjacent to exactly one vertex v. Necessarily, $v \in V_2$ and $\deg v \geq 2$. Let $w(\neq u)$ be adjacent to v. So $w \in V_1$. We now consider two cases.

Case 1 *Assume* $\deg w = 1$. Necessarily, then, $\deg v \geq 3$ and v is adjacent to a vertex y where $\deg y \geq 2$. Suppose y is adjacent to $x(\neq v)$. Since $|V_2| \geq 3$, there exists a vertex $z \in V_2$ distinct from v and x (see Figure 12). If z is adjacent to y, then $P : u, v, w$ and $Q : x, y, z$ are desired paths; otherwise, z is adjacent to a vertex t distinct from u, w, and y, and $P : u, v, y, x$ and $Q : t, z$ are desired paths.

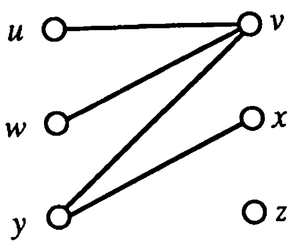


Figure 12: $\deg w = 1$

Case 2 *Assume* $\deg w \geq 2$. Let $x(\neq v)$ be adjacent to w. We consider two subcases.

Subcase 2.1 *Assume x is an end-vertex.* Then at least one of w and v has degree at least 3, say the former. Suppose w is adjacent to $y(\neq v, x)$. If y is an end-vertex, then we are in the situation described in Case 1 (with w playing the role of v in Case 1). Thus, we may assume that $\deg y \geq 2$. Since T is a tree, y is not adjacent to u. So y is adjacent to a vertex z distinct from u and w (see Figure 13). However, then $P : u, v, w, y, z$ is a path of length 4.

Subcase 2.2 *Assume x is not an end-vertex.* Since T is a tree, x is not adjacent to u. However, since $\deg x \geq 2$, there is a vertex y distinct from u and w that is

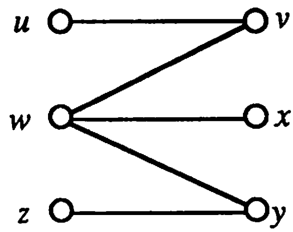


Figure 13: $\deg w \geq 2$ and x is an end-vertex

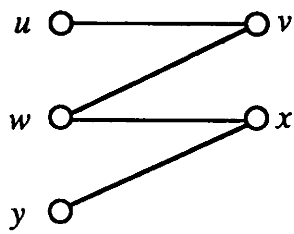


Figure 14: $\deg w \geq 2$ and $\deg x \geq 2$

adjacent to x (see Figure 14). However, then $P : u, v, w, x, y$ is a path of length 4.

This completes the proof. ∎

Theorem 13. *Let T be an $m \times n$ tree of order p with $m \leq n$. Then $p - \rho(T) \leq 2m$.*

Proof: Let V_1 be a partite set of T with $|V_1| = m$. In any linear forest F contained in T, every vertex of V_1 belongs to at most two edges of F. Therefore, $|E(F)| \leq 2m$. Then by Corollary 9, $p - \rho(T) \leq 2m$. ∎

Combining Theorems 12 and 13, we have the following result.

Corollary 14. If T is an $m \times n$ tree with $3 \leq m \leq n$, then

$$n - m \leq \rho(T) \leq m + n - 4.$$

In the special case that T is a $1 \times n$ tree (the star $K_{1,n}$), $\rho(T) = m + n - 2$; while in the case that T is a $2 \times n$ tree ($n \geq 2$), $\rho(T) \leq m + n - 3$. If T is an $m \times n$ tree for which $m = n$, then Corollary 14 gives 0 for a lower bound for $\rho(T)$ where, of course, $\rho(T) \geq 1$ for *every* tree T. Therefore, we could replace the lower bound in Corollary 14 by $\max\{1, n - m\} \leq \rho(T)$.

We now discuss the sharpness of the bounds given in Corollary 14. In order to do this, we recall some special, but familiar, classes of trees. A *caterpillar* T is a tree the removal of whose end-vertices produces a path. This resulting path is referred to as the *spine* of T. If the spine is the path $v_1, v_2, \ldots, v_t$ whose degrees are $d_1, d_2, \ldots, d_t$, respectively, then we denote T by $C(d_1, d_2, \ldots, d_t)$. If $t = 1$,

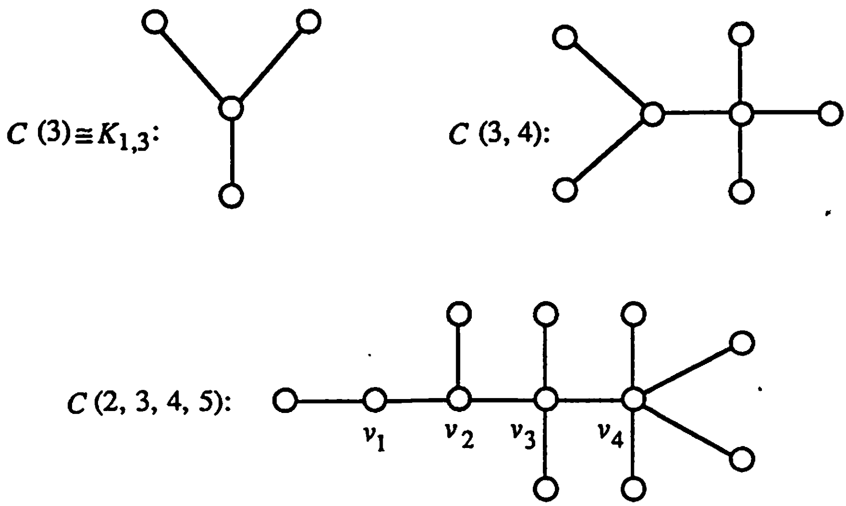


Figure 15: Three caterpillars (including a star and double star)

then T is a star, while if $t = 2$, T is a *double star*. We note that the order of $C(d_1, d_2, \ldots, d_t)$ is $\sum_{i=1}^{t} d_i - t + 2$. Examples are shown in Figure 15. The spine of the caterpillar $C(2,3,4,5)$ is v_1, v_2, v_3, v_4.

We show that for every pair m, n of integers with $3 \leq m < n$ and every value intermediate to the bounds given in Corollary 14, there exists an $m \times n$ tree T (actually a caterpillar) that attains this value.

Theorem 15. *Let k, m and n be integers such that $3 \leq m < n$ and $n - m \leq k \leq n + m - 4$. Then there exists an $m \times n$ tree T such that $\rho(T) = k$.*

Proof: Let T be the caterpillar $C(n-t, m-t, 2, \ldots, 2)$ where $0 \leq t \leq m-2$ and the spine of T has order $2t+2$. Then T is an $m \times n$ tree (of order $m+n$). By Corollary 11, when $0 \leq t \leq m-3$,

$$\rho(T) = n + m - 2t - 4. \tag{3}$$

The caterpillar $T = C(n-t-1, m-t, 2, \ldots, 2)$ where $0 \leq t \leq m-3$ and with spine of order $2t+1$ has order $m+n$ and is an $m \times n$ tree. By Corollary 11,

$$\rho(T) = n + m - 2t - 5. \tag{4}$$

For $t = m-2$, we have $T = C(n-m+1, 2, 2, \ldots, 2)$ and by Corollary 11, $\rho(T) = n - m$. Combining this result with the formulas given in (3) and (4), we have the desired result. ∎

4. Concluding Remarks

In this paper we have studied the induced path number of several classes of bipartite graphs. This, of course, suggests the investigation of the induced path number of graphs that are not necessarily bipartite, perhaps in terms of other parameters.

In [5] the induced path number was generalized, namely, the *induced tree number* $\tau(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a tree. The *induced Δ-tree number* $\tau_\Delta(G)$ requires that each induced tree has maximum degree at most Δ. Consequently, the induced 2-tree number is the induced path number.

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