ON ORTHOGONAL DOUBLE COVERS OF K_n

BERNHARD GANTER
Technische Universität Dresden
Institut für Algebra
Mommsenstr. 13, 01062 Dresden, Germany

HANS-DIETRICH O.F. GRONAU
Universität Rostock
Fachbereich Mathematik
Universitätsplatz 1, 18051 Rostock, Germany

RONALD C. MULLIN
University of Waterloo
Department of Combinatorics & Optimization
Waterloo, Ontario, N2L 3G1, Canada

April 25, 1994

Abstract

An orthogonal double cover of the complete graph K_n is a collection of n spanning subgraphs $G_1, G_2, ..., G_n$ of K_n such that
- every edge of K_n belongs to exactly 2 of the G_i's and
- every pair of G_is intersect in exactly one edge.

It is proved that an orthogonal double cover exists for all $n \geq 4$, where the G_i's consist of short cycles; this result also proves a conjecture of Chung and West.

1 Introduction

Let $n \geq 2$ be an integer and let K_n be the complete graph on the n-element vertex set V. We consider collections $\mathcal{G} = \{G_1, G_2, ..., G_n\}$ [1] of spanning subgraphs of K_n, i.e. graphs on V. We call $\mathcal{G}$ an *orthogonal double cover of* K_n if
(i) every edge of K_n belongs to exactly 2 of the G_i's and
(ii) every pair of G_i's intersect in exactly one edge.

Of course, we also may restrict the set of graphs from which we choose the G_i's, e.g., graphs which consist of disjoint cliques only, graphs with maximum degree 2, or graphs which consist of cycles of small length, etc.

Note that every graph must have exactly $n - 1$ edges.

The general problem is to determine all values of n for which such a orthogonal double cover exists.

For illustration we start with the solutions for $2 \leq n \leq 7$ given in Figures 1,2,3,4, and 5.

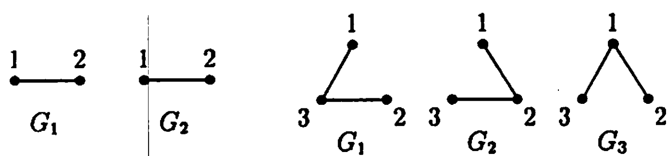


Figure 1: Solutions for $n = 2$ and $n = 3$

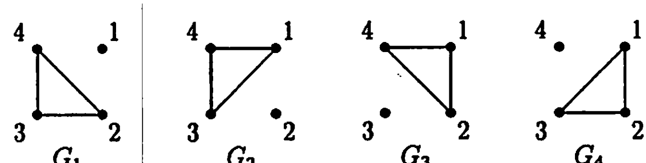
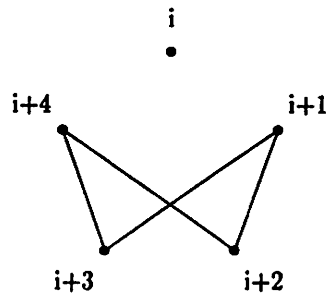


Figure 2: Solution for $n = 4$

In [4] and [3] DEMETROVICS, FÜREDI, AND KATONA studied extremal problems in relational data bases and arrived at our problem (in a different terminology) in the case of graphs which consist of distinct cliques. They conjectured that an orthogonal double cover of K_n exists in this case for all

[1] Note that the number of graphs coincides with the size of V.



G_i; $i = 1, 2, 3, 4, 5$

Figure 3: Solution for $n = 5$

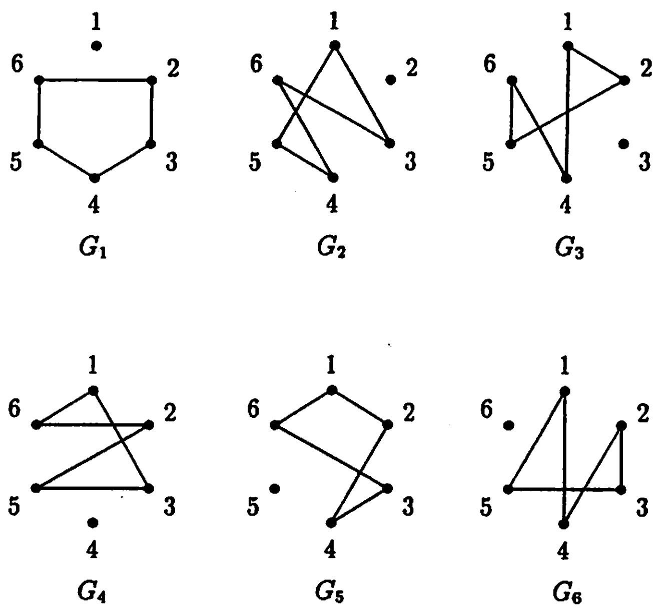


Figure 4: Solution for $n = 6$

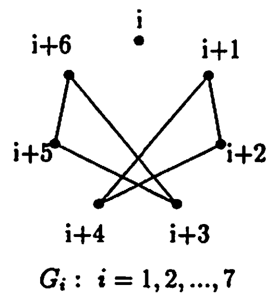


Figure 5: Solution for $n = 7$

$n \geq 7$. Actually they studied the case $n \equiv 1 \bmod 3$, where each G_i consists of one isolated point and $\frac{n-1}{3}$ disjoint 3-cliques, i.e., K_3s, and eventually they asked the same question in the directed case. Their conjecture that such an orthogonal double cover in the directed case exists for all $n \geq 4$, $n \equiv 1 \bmod 3$ was proved by them for all $n \equiv 1$ *or* $4 \bmod 12$. RAUSCHE [11] proved that there is no solution for the latter problem for $n = 10$ in the undirected case and this implies trivially that there is also no solution for $n = 10$ in the directed case. GANTER AND GRONAU [5] proved the latter conjecture in the directed case for all $n \equiv 1 \bmod 3$, $n \neq 10$, and the general conjecture for all sufficiently large n. They also proved that the general conjecture fails in the case $n = 8$. A complete confirmation of the first conjecture (with $n \neq 8$) was given in BENNETT AND WU [1] and GRONAU AND MULLIN [8].

In late 1991 a paper of CHUNG AND WEST [2] came to our attention in which the same question was asked for graphs G_i having maximum degree 2, i.e., consisting of cycles or cycles and just one path.

They proved the existence of such orthogonal double covers for 6 residue classes mod 12, namely, for $n \equiv 1, 2, 5, 7, 10, 11 \bmod 12$. Their constructions give solutions with cycles of certain length (depending on number-theoretic properties of n) if $n \equiv 1, 5 \bmod 6$ and containing a path in the remaining cases.

The aim of this paper is to study a strengthening of the problem, namely, the existence of orthogonal double covers of K_n where the graphs consist of disjoint cycles and one isolated point. Moreover, we answer this question where the cycles are all short, i.e., the cycles have length at most 5.

We call a set $\{G_1, G_2, ..., G_n\}$ of graphs *idempotent* if i is an isolated point in G_i for $i = 1, 2, ..., n$. Idempotent solutions play a crucial role in our constructions.

One also may strengthen the problem in the following way.

Let $\vec{K}_n$ be the complete digraph on the vertex set V. We consider collections $\vec{\mathcal{G}} = \{ \vec{G}_1, \vec{G}_2, ..., \vec{G}_n \}$ of spanning subgraphs of $\vec{K}_n$, i.e., digraphs on V. We call $\vec{\mathcal{G}}$ an *orthogonal double cover of* $\vec{K}_n$ *in the directed case* if the associated collection $\mathcal{G} = \{ G_1, G_2, ..., G_n \}$ which is obtained from $\vec{\mathcal{G}}$ by replacing the directed edges by undirected edges is an orthogonal double cover of K_n and has the additional property that the two common edges of the G_i and G_j have different orientations in $\vec{G}_i$ and $\vec{G}_j$ for any pair {i,j} $\subset \{1, 2, ..., n\}$.

This problem was studied in GANTER AND GRONAU [5] in case of directed 3-cycles and more generally in GRONAU AND MULLIN [9].

2 The main results

The aim of this paper is to prove the following results.

Theorem 1 *There exists an idempotent orthogonal double cover* $\{G_1, G_2, ..., G_n\}$ *of* K_n *in which each of the graphs* G_i $(i = 1, 2, ..., n)$ *consists of an isolated point* i *and the union of edge-disjoint cycles of length 3, 4, or 5 only, for all* $n \geq 4$, $n \neq 8$. *For* $n = 2, 3, 8$ *there is no solution.*

The proof of this theorem will be given in section 4.

An immediate consequence of the above theorem is the following result:

Theorem 2 *There exists an idempotent orthogonal double cover* $\{G_1, G_2, ..., G_n\}$ *of* K_n *in which each of the graphs* G_i $(i = 1, 2, ..., n)$ *consists of an isolated point* i *and the union of edge-disjoint cycles (without size restrictions), for all* $n \geq 4$.

In order to prove the last theorem we need only to refer to Theorem 1 and to add a solution for $n = 8$.

n	solution
8	$G_1 = \{(1),(2,8,7,6,5,4,3)\}$[1]
	$G_2 = \{(2),(1,4,6,7,3,8,5)\}$
	$G_3 = \{(3),(1,5,7,8,4,2,6)\}$
	$G_4 = \{(4),(1,6,8,2,5,3,7)\}$
	$G_5 = \{(5),(1,7,2,3,6,4,8)\}$
	$G_6 = \{(6),(1,8,3,4,7,5,2)\}$
	$G_7 = \{(7),(1,2,4,5,8,6,3)\}$
	$G_8 = \{(8),(1,3,5,6,2,7,4)\}$

Table 1: Solution for $n = 8$

Remark 1 *A complete computer search showed that there is no solution in case $n = 8$, where the graphs have all cycle structure 4,3. Thus, the only idempotent solution for $n = 8$ needs 7-cycles.*

The cases $n \equiv 2 \mod 4$ are the most difficult ones. There are no cyclic solutions at all, as Lemma 1 shows.

Before presenting Lemma 1 we will mention necessary and sufficient conditions, Claims 1 and 2, for a graph being a starter in a cyclic solution.

For every edge $\{a, b\} \subset \{0, 1, 2, ..., n-1\}$, the *Lee-distance* is defined as

$$l(a, b) = min\{|a-b|, n-|b-a|\}.$$

For every $k \in \{1, 2, ..., \lfloor\frac{n}{2}\rfloor\}$ let the *rotation-distance* denote the unique number $r(k) \in \{1, 2, ..., \lfloor\frac{n}{2}\rfloor\}$ such that G_i and $G_{i+r(k)}$ (index mod n) coincide in an edge of *Lee-distance* k.

Remark 2 *In Figure 5 we have the following rotation-distances in that example for the case $n = 7$*

Lee-distance k	*1*	*2*	*3*
rotation-distance $r(k)$	*3*	*1*	*2*

[1]This notation means that G_1 has exactly the edges (2,8), (8,7), (7,6), (6,5), (5,4), (4,3), and (3,2)

Claim 1 *For every $k \in \{1, 2, ..., \lfloor \frac{n-1}{2} \rfloor\}$ the graph G must contain exactly 2 edges of Lee-distance k and exactly one edge of Lee-distance $\frac{n}{2}$ if n is even.*
Claim 2 *The rotation-distances $r(k)$ form a permutation of $\{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$.*

Claims 1 and 2 are neccessary and sufficient for G being a starter for a solution.

Lemma 1 *If a spanning subgraph G in K_n is a starter for a cyclic solution, then $n \not\equiv 2 \bmod 4$.*

Proof. Assume the contrary. Let $n \equiv 2 \bmod 4$ and let G be a solution for K_n. For each edge ($n-1$ of them in total), we assign the Lee-distance to it, and sum them up. We get $2(1 + 2 + ... + \frac{n}{2} - 1) + \frac{n}{2}$ which is odd; see Claim 1. If $a, b \in \{0, 1, 2, ..., n-1\}$ the Lee-distance of the edge (a, b) is

$$l(a, b) \equiv \pm(a - b) \bmod n.$$

For a cycle $C = x_1, x_2, ..., x_k$ of length k in G, the sum of the Lee-distances of edges in C is

$$\sum_{i=1}^{k} \pm(x_i - x_{i-1})$$

which is even, since that is true for all the cycles in G. So the sum of the Lee-distances of all the edges in G is even, which contradicts the fact that this sum must be odd.

Theorem 2 together with Figure 1 answers the question of CHUNG AND WEST [2] completely.

Theorem 3 *An orthogonal double cover $\{G_1, G_2, ..., G_n\}$ of K_n, where the graph G_i $(i = 1, 2, ..., n)$ of the collection has maximum degree 2 exists if $n \geq 2$.*

3 PBD-closure

The main idea for our construction is to show first that the set of orders for which idempotent solutions exist is PBD-closed. Then the set of orders for which appropriate *pairwise balanced designs* (PBD) exist gives the desired solutions.

Lemma 2 *The set of integers n for which an idempotent orthogonal double cover of K_n exists is PBD-closed.*

Proof. Let B be the set of blocks of a PBD on V, and suppose that for every $b \in B$ we have an idempotent solution

$$\mathcal{G}^b = \{G^b_{x_1}, G^b_{x_2}, ..., G^b_{x_{|b|}}\}, b = \{x_1, x_2, ..., x_{|b|}\}.$$

Then, for every $x \in V$, define

$$G_x := \bigcup_{x \in b \in B} G^b_x.$$

We show that $\{G_x : x \in V\}$ is again an idempotent double cover for K_n.

It is obvious that $\{G_x : x \in V\}$ is idempotent, i.e., it contains the singleton $\{x\}$.

To see that $\{G_x : x \in V\}$ is a double cover, we need to prove two things:

1. $|G_x \cap G_y| = 1$ *for $x, y \in V$ and $x \neq y$.*
   Consider G_x for an arbitrary $x \in V$. The only component of G_x containing x is $\{x\}$. $\{b - \{x\} : x \in b \in B\}$ is a partition of $V - \{x\}$; so, if a member of G_x contains any element y different from x, then it must by contained in the block b through x and y and thus it comes from G^b_x. Since this is partition, no two such classes can intersect.

2. For any $u, v \in V$ and $u \neq v$, there is a unique pair G_x and G_y which both contain the edge (u, v).
   Now let $u, v \in V, v \neq u$, and let b be the block containing u and v. Then there is precisely one block b which contains both u and v, and this block contains a unique pair of elements x and y such that G^b_x and G^b_y intersect in (u, v). As a consequence, there is a unique pair G_x and G_y which intersect in (u, v).

4 Proof of Theorem 1

Let K be a set of positive integers. For brevity let $B(K)$ denote the set of orders for which a pairwise balanced design with block sizes from K exists. We will use the following two theorems.

Theorem 4 ([10],[7])

$$B(\{4,5,6,7\}) = \{4,5,6,7,13,16,17,20,21,22\} \cup \{n : n \geq 24\}.$$

LENZ [10] proved the above result with one undecided case: $n = 23$. This was resolved by GRONAU, METSCH, AND MÜLLIN [7].

In order to prove Theorem 1, we first give solutions for $n = 4,5,6,7$:

n	solution	fig.
4	$G_1 = \{(1),(2,3,4)\}$ $G_2 = \{(2),(1,3,4)\}$ $G_3 = \{(3),(1,2,4)\}$ $G_4 = \{(4),(1,2,3)\}$	2
5	$G_i = \{(i),(1+i,2+i,4+i,3+i)\}$ mod 5, $i = 1,2,3,4,5$	3
6	$G_1 = \{(1),(2,3,4,5,6)\}$ $G_2 = \{(2),(1,3,6,4,5)\}$ $G_3 = \{(3),(1,2,5,6,4)\}$ $G_4 = \{(4),(1,3,5,2,6)\}$ $G_5 = \{(5),(1,2,4,3,6)\}$ $G_6 = \{(6),(1,4,2,3,5)\}$	4
7	$G_i = \{(i),(1+i,2+i,4+i),(3+i,5+i,6+i)\}$ mod 7, $i = 1,2,...,7$	5

Table 2: Solutions for $n = 4,5,6,7$

Applying Theorem 4 and Lemma 1 we only have to present solutions for $n = 9,10,11,12,14,15,18,19,23$ in order to complete the proof of Theorem 1, which will be given in Table 3.

Remark 3 *A complete computer search showed that there is no cyclic solution in case $n = 9$ and no solution at all in case $n = 11$ with cycle structure 4,3,3.*

The most surprising case was $n = 10$. RAUSCHE [11] showed by a direct but laborious case analysis that there is no solution if each of these graphs is idempotent and consists of three disjoint 3-cycles. Obviously, the triangles in a solution would give rise to a simple 2-(10,3,2) design. A solution in

n	solution
9	$G_1 = \{(1),(2,4,6,8),(3,5,7,9)\}$ $G_2 = \{(2),(1,7,6,9),(3,4,8,5)\}$ $G_3 = \{(3),(1,8,7,2),(4,5,9,6)\}$ $G_4 = \{(4),(1,3,6,8),(2,7,5,6)\}$ $G_5 = \{(5),(1,2,9,4),(3,8,6,7)\}$ $G_6 = \{(6),(1,3,2,5),(4,9,7,8)\}$ $G_7 = \{(7),(1,4,3,6),(2,8,9,5)\}$ $G_8 = \{(8),(1,5,4,7),(2,6,3,9)\}$ $G_9 = \{(9),(1,6,5,8),(3,7,4,2)\}$
10	$G_1 = \{(1),(2,3,4),(5,6,7),(8,9,10)\}$ $G_2 = \{(2),(1,3,4,5,8),(6,9,7,10)\}$ $G_3 = \{(3),(1,2,5,9),(4,6,8,10,7)\}$ $G_4 = \{(4),(1,2,8,7),(3,5,10,9,6)\}$ $G_5 = \{(5),(1,3,8,6,7),(2,9,4,10)\}$ $G_6 = \{(6),(1,4,2,10),(3,7,8,5,9)\}$ $G_7 = \{(7),(1,6,3,10),(2,5,4,8,9)\}$ $G_8 = \{(8),(1,5,10,4,6),(2,3,9,7)\}$ $G_9 = \{(9),(1,4,7,5),(2,6,10,3,8)\}$ $G_{10} = \{(10),(1,8,4,9),(2,6,5,3,7)\}$
11	$G_i = \{(i),(1+i,5+i,3+i,4+i,9+i),$ $(2+i,7+i,8+i,6+i,10+i)\}$ mod 11, $i = 1,2,...,11$
12	$G_i = \{(i),(1+i,7+i,10+i,3+i),(4+i,8+i,9+i,11+i),$ $(2+i,5+i,6+i)\}$ mod 12, $i = 1,2,...,12$
14	$G_i = \{((i,0)),((1+i,0),(3+i,0),(2+i,0),(i,1)),((4+i,0),$ $(1+i,1),(5+i,1)),((5+i,0),(4+i,1),(6+i,1)),((6+i,0),$ $(2+i,1),(3+i,1))\}$ $G_{i+7} = \{((i,1)),((i,0),(1+i,0),(6+i,1),(4+i,0)),((2+i,0),$ $(6+i,0),(2+i,1)),((3+i,0),(5+i,0),(5+i,1)),((1+i,1),$ $(3+i,1),(4+i,1))\}$ mod$(7,-)$, $i = 1,2,...,7$
15	$G_i = \{(i),(1+i,7+i,4+i,6+i),(3+i,11+i,12+i,13+i),$ $(2+i,5+i,9+i),(8+i,10+i,14+i)\}$ mod 15, $i = 1,2,...,15$

Table 3: Solutions for $n = 9,10,11,12,14,15$

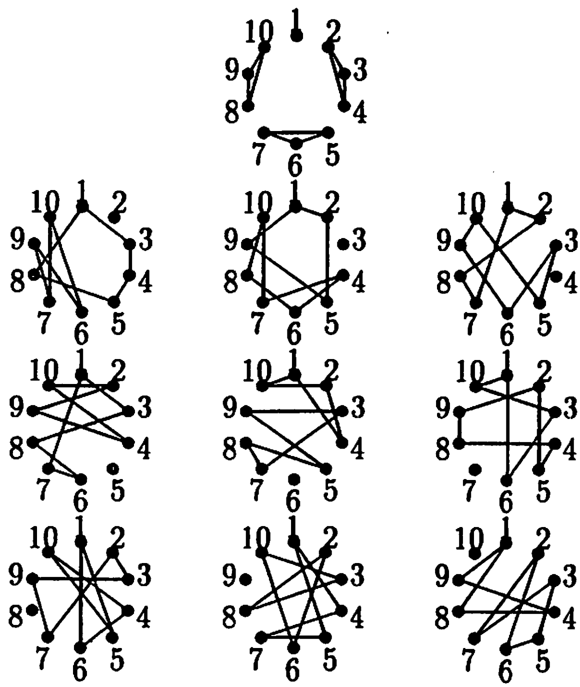


Figure 6: Solution for $n = 10$

n	solution
18	$G_i = \{((i,0)),((1+i,0),(2+i,0),(3+i,0),(5+i,1),(i,1)),$ $((4+i,0),(7+i,0),(1+i,1)),((6+i,0),(2+i,1),(4+i,1)),$ $((8+i,0),(3+i,1),(7+i,1)),((5+i,1),(6+i,1),(8+i,1))\}$ $G_{i+9} = \{((i,1)),((i,0),(3+i,0),(7+i,0),(5+i,1),(5+i,0)),$ $((1+i,0),(2+i,1),(6+i,1)),((2+i,0),(3+i,1),(4+i,1)),$ $((4+i,0),(1+i,1),(7+i,1)),((6+i,0),(8+i,0),(8+i,1))\}$ $\text{mod}(9,-),\ i=1,2,...,9$
19	$G_i = \{(i),(1+i,7+i,11+i),(2+i,14+i,3+i),$ $(4+i,9+i,6+i),(5+i,16+i,17+i),$ $(8+i,18+i,12+i),(10+i,13+i,15+i)\}$ $\text{mod}19,\ i=1,2,...,19$
23	$G_i = \{(i),(1+i,19+i,8+i),(2+i,17+i,11+i),$ $(3+i,4+i,6+i),(5+i,15+i,13+i,20+i),$ $(7+i,18+i,21+i),(9+i,14+i,10+i),$ $(12+i,16+i,22+i)\}\ \text{mod}\ 23,\ i=1,2,...,23$

Table 4: Solutions for $n = 18, 19, 23$

the corresponding problem for digraphs (see the introduction) would give rise to Mendelsohn triple systems. All Mendelsohn triple systems are known [6]. BENNETT AND WU [1] checked all these designs for partitions with ten classes consisting each of 3 disjoint 3-cycles which could form a solution for the directed problem, and, of course, their approach also gave another proof of its nonexistence.

Then we looked for solutions where every idempotent graph consists of one 4-cycle and one disjoint 5-cycle. A complete computer search using a backtracking algorithm showed that there is also no solution. Finally, we started looking for a solution where the graphs are mixed, i.e., some consist of three 3-cycles and others consist of one 4-cycle and one 5-cycle. This search was successful; see Figure 6. This was the first solution where the graphs are not all of the same cycle structure.

Acknowledgement

The authors would like to thank Prof. P.J. Schellenberg for his assistance with some computational aspects of this paper.

References

[1] F.E. Bennett and L. Wu. On minimum matrix representation of closure operations. *Discrete Appl. Math.*, **26**(1990), 25-40.

[2] M.S. Chung and D.B. West. The p-intersection number of a complete bipartite graph and orthogonal double coverings of a clique. *Combinatorica*, submitted, 1991.

[3] J. Demetrovics, Z. Füredi, and G.O.H. Katona. Minimum matrix representation of closure operations. *Discrete Appl. Math.*, **11**(1985), 115-128.

[4] J. Demetrovics and G.O.H. Katona. Extremal combinatorial problems in relational data base. In *Fundamentals of Computation Theory*, volume **117** of *Lecture Notes in Computer Science*, pages 110–119, Szeged, 1981. Springer, Berlin.

[5] B. Ganter and H.-D.O.F. Gronau. On two conjectures of Demetrovics, Füredi, and Katona on partitions. *Discrete Math.*, **88**(1991), 149-155.

[6] B. Ganter, R. Mathon, and A. Rosa. A complete census of (10,3,2)-block designs and of Mendelsohn triple systems of order ten I, Mendelsohn triple systems without repeated blocks. *Congressus Numerantium*, **20**(1977), 383-398.

[7] H.-D.O.F. Gronau, K. Metsch, and R.C. Mullin. Some results on pairwise balanced designs with block sizes 4, 5, and 7, 1993. in preparation.

[8] H.-D.O.F. Gronau and R.C. Mullin. On a conjecture of Demetrovics, Füredi and Katona. unpublished, June 1990.

[9] H.-D.O.F. Gronau, R.C. Mullin, and P.J. Schellenberg. On orthogonal covers of $\vec{K}_n$, 1993. in preparation.

[10] H. Lenz. Some remarks on pairwise balanced designs. *Mitt. Math. Sem. Giessen*, **165**(1984), 49-62.

[11] A. Rausche. On the existence of special block designs. *Rostock. Math. Kolloq.*, **35**(1988), 13-20.