Some extremal results on independent distance domination in graphs

Lowell W Beineke
Indiana University—Purdue University at Fort Wayne

Michael A Henning
University of Natal, Pietermaritzburg

ABSTRACT. Let s and r be positive integers with $s \geq r$ and let G be a graph. A set I of vertices of G is an (r,s)-set if no two vertices of I are within distance r from each other and every vertex of G not in I is within distance s from some vertex of I. The minimum cardinality of a (r,s)-set is called the (r,s)-domination number and is denoted by $i_{r,s}(G)$. It is shown that if G is a connected graph with at least $s > r \geq 1$ vertices, then there is a minimum (r,s)-set I of G such that for each $v \in I$, there exists a vertex $w \in V(G) - I$ at distance at least $s - r$ from v, but within distance s from v, and at distance greater than s from every vertex of $I - \{v\}$. Using this result, it is shown that if G is a connected graph with $p \geq s \geq 2$ vertices, then $i_{1,s}(G) \leq p/s$ and this bound is best possible. Further, it is shown that for $s \in \{1,2,3\}$, if T is a tree on $p \geq s+1$ vertices, then $i_{1,s}(T) \leq p/(s+1)$ and this bound is sharp.

1. Introduction

For graph theory terminology not presented here we follow [7]. Specifically $p(G)$ and $q(G)$ will denote, respectively, the number of vertices (also called the order) and number of edges (also called the size) of a graph G with vertex set $V(G)$ and edge set $E(G)$. If S is a set of vertices of G and v is a vertex of G, then *the distance from v to S*, denoted by $d_G(v,S)$, is the shortest distance from v to a vertex of S.

A set D of vertices of a graph G is a *dominating set (total dominating set)* of G if every vertex of $V(G) - D$ ($V(G)$, respectively) is adjacent with some vertex of D other than itself. Mo and Williams [18] extended the definition of total dominating sets to (r,s)-dominating sets of graphs. In

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[18], a set U of vertices of a graph G is called an (r, s)-dominating set of G if every vertex in $V(G) - U$ is at distance at most r from some vertex in U and every vertex in U is at distance at most s from some vertex in U other than itself. Thus a $(1,1)$-dominating set is the same as a total dominating set. In [18], various bounds on the cardinality of a smallest (r, s)-dominating set of a graph are established.

In this paper, we extend the definition of independent dominating sets in graphs. A set I of vertices of G is an *independent dominating set* of G if I is both an independent and a dominating set of G. Equivalently, I is an independent dominating set if no two vertices of I are within distance 1 from each other and every vertex not in I is within distance 1 of some vertex of I. This observation suggests a generalization of the concept of independent domination in a graph. For r and s positive integers, we define a set I of vertices of G to be an (r, s)-set if no two vertices of I are within distance r from each other and every vertex not in I is within distance s of some vertex of I. The (r, s)-domination number $i_{r,s}(G)$ of G is the minimum cardinality among all (r, s)-sets of G. Thus I is a $(1,1)$-set of G if and only if I is an independent dominating set of G. Hence $i_{1,1}(G) = i(G)$, where $i(G)$ is the independent domination number of G. The parameter $i(G)$ has received considerable attention in the literature (see, for instance, [1, 2, 5, 8, 14, 15]). For the graph G shown in Figure 1, $D = \{u, v, x, y\}$ is a $(1,2)$-set of G with $i_{1,2}(G) = |D|$ and $T = \{u, w, z, y\}$ is a $(2,2)$-set of G with $i_{2,2}(G) = |T|$.

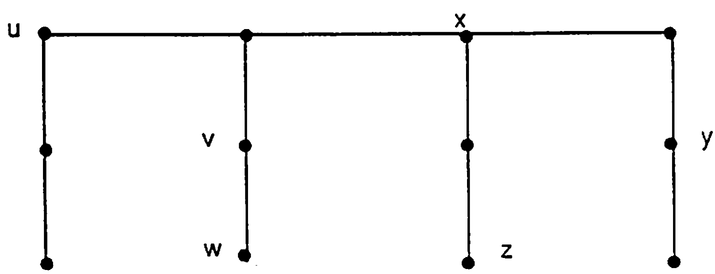

Figure 1.

This concept of distance domination in graphs finds application in many situations and structures which give rise to graphs. Consider, for instance, the following illustration related to town planning. Let G be the graph associated with the road grid of a city where the vertices of G correspond to the street intersections and where two vertices are adjacent if and only if the corresponding street intersections are a block apart. Suppose we are required to locate a minimum number of facilities (such as utilities, waste disposal dumps, hospitals, emergency medical centres, blood banks, transmission towers) such that every intersection is within s city blocks of a facility and such that no two facilities be within r blocks of each other (to avoid interference, contamination or congestion), where r and s are positive integers. Then we may site such facilities at points corresponding to vertices in a minimum (r,s)-set in G. These concepts are related to distance dominating cycles studied by Bondy and Fan [6], Fraisse [9] and Veldman [17], and also to the concepts of distance domination studied by Henning, Oellermann and Swart [12, 13] and Bacsó and Tuza [3,4].

2. Bounds on $i_{1,s}(G)$ for a graph G

In this section, we investigate good upper bounds on $i_{1,s}(G)$ for a graph G. The following theorem can be deduced from a result that was established in [11].

Theorem A. *If G is a connected graph of order $p \geq 2$, then $i(G) \leq p+2-2\sqrt{p}$, and this bound is sharp.*

That the bound given in Theorem A is sharp, may be seen by considering the graph G (indicated in Figure 2) obtained from a complete graph on $k+1$ vertices by attaching to each of its vertices k (disjoint) paths of length 1. Then $p=(k+1)^2$ and $i(G)=k^2+1$, so $i(G)=p+2-2\sqrt{p}$.

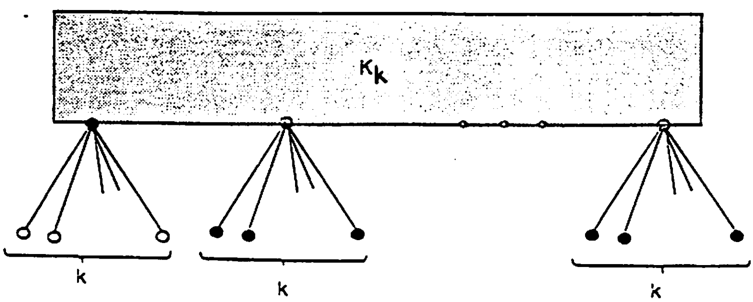

Figure 2. The graph G

Since $i_{1,1}(G) = i(G)$ for any graph G, Theorem A gives a good upper bound on $i_{1,1}(G)$ for a connected graph G. In what follows, we investigate good upper bounds on $i_{1,s}(G)$ for a connected graph G where $s \geq 2$. We begin with the following proposition which generalizes a classical result of Ore ([16], p. 206).

Proposition 1. *For $s \geq r \geq 1$, let I be an (r,s)-set of a connected graph G. Then I is a minimal (r,s)-set of G if and only if each vertex $v \in I$ has at least one of the following properties.*

$P(1,s)$: *there exists a vertex* $w \in V(G) - I$ *such that* w *is within distance* s *from* v *and at distance greater than* s *from* $I - \{v\}$;

$P(2,s)$: v *is at distance greater than* s *from* $I - \{v\}$.

Proof: Suppose, firstly, that I is a minimal (r,s)-set of G. Then for each vertex v of I, $I - \{v\}$ is not an (r,s)-set of G. Hence there is a vertex $w \in V(G) - (I \cup \{v\})$ that is at distance at least $s+1$ from every vertex of $I - \{v\}$. If $w = v$, then v has property $P(2,s)$, while if $w \notin I$, then, since every vertex not in I is within distance s of some vertex of I, w is within distance s from v and at distance greater than s from every vertex of $I - \{v\}$ and v has property $P(l,s)$. Conversely, if each vertex $v \in I$ has at least one of the properties $P(1,s)$ or $P(2,s)$, then for each such vertex v, $I - \{v\}$ is not an (r,s)-set of G.

Before proceeding further, we introduce some notation. Let S be a set of s vertices of a connected graph G. We will call a nondecreasing sequence $\ell_1, \ell_2, \ldots, \ell_s$ of integers the *distance sequence* of S in G if the vertices of S can be labelled $v_1, v_2, \ldots, v_s$ so that $\ell_i = d_G(v_i, S - \{v_i\})$ for each i. For example, for the graph G given in Figure 1, the set $\{u,v,x,y\}$ has distance sequence $2,2,2,2$ while the distance sequence of the set $\{u,y,w,z\}$ is $3,3,4,4$. Suppose $s_1 : a_1, a_2, \ldots, a_m$ and $s_2 : b_1, b_2, \ldots, b_n$ are two nondecreasing sequences of positive integers. Then we say that s_1 *precedes* s_2 in *dictionary order* if either $m < n$ and $a_i = b_i$ for $1 \le i \le m$ or if there exists an i $(1 \le i \le \min\{m,n\})$ such that $a_i < b_i$ and $a_j = b_j$ for $j < i$.

We now present the following result.

Theorem 1. *Let* $1 \le r < s$ *and let* G *be a connected graph of order at least* s. *Then* G *has a minimum* (r,s)*-set* I *such that for each* $v \in I$, *there exists a vertex* $w \notin I$ *at distance at least* $r - s$, *but at most* s, *from* v *and distance greater than* s *from* $I - \{v\}$.

Proof: Let $i_{r,s}(G) = m$. Among all the (r,s)-sets of G with cardinality m, let I be one which has the smallest distance sequence in dictionary order. Let the distance sequence of I be given by $\ell_1, \ell_2, \ldots, \ell_m$, where $I = \{v_1, v_2, \ldots, v_m\}$ and $\ell_i = d(v_i, I - \{v_i\})$ for $1 \le i \le m$.

We show firstly that each vertex of I has property $P(1,s)$. If this is not the case, then let i be the smallest integer such that the vertex v_i does not have this property. By Proposition 1, v_i then has property $P(2,s)$, and so $\ell_i \ge s+1$. Now let v_i' be adjacent with v_i. Then $d(v_i', I - \{v_i\}) \ge s > r$. We consider the set $I' = (I - \{v_i\}) \cup \{v_i'\}$. Necessarily, I' is an (r,s)-set of G with $|I'| = m$. Furthermore, the vertex v_i' is within distance s from some vertex of $I - \{v_i\}$; consequently, $\ell_i' = d(v_i', I' - \{v_i\}) = s < \ell_i$. Now let j be the largest integer for which $\ell_j < \ell_i$, and consider the value $\ell_k' = d(v_k, I' - \{v_k\})$ for each k with $1 \le k \le j$. Since $\ell_k < \ell_i$, a shortest path from the vertex v_k to a vertex of $I - \{v_i\}$ does not contain v_i. It follows, therefore, that $\ell'_k \leq \ell_k$ for all k ($1 \leq k \leq j$). This, together with the observation that $\ell'_i < \ell_t$ for all $m > t$, implies that the distance sequence of I' precedes that of I in dictionary order. This produces a contradiction. Hence every vertex of I has property $P(1,s)$.

For each vertex v_i of I, let w_i be a vertex of $V(G) - I$ at maximum distance from v_i in G such that w_i is within distance s from v_i and at distance greater than s from every other vertex of I ($1 \leq i \leq m$). We show that the distance from w_i to v_i is at least $s - r$ for all i. If this is not the case, then let i be the smallest integer for which $d(v_i, w_i) < s - r$. We observe, therefore, that every vertex at distance greater than $s - r - 1$ from v_i is within distance s from some vertex of $I - \{v_i\}$. Thus v_i is at distance at least $r + 2$ from $I - \{v_i\}$, for otherwise, if $d(v_i, I - \{v_i\}) = r + 1$, then $I - \{v_i\}$ is a (r,s)-set of G of cardinality less than m. We now consider a shortest path from the vertex v_i to a vertex of $I - \{v_i\}$ in G. Let v_i^* denote the vertex adjacent with v_i on this path. Then v_i^* is at distance at least $r+1$ from $I - \{v_i\}$. We now consider the set $I^* = (I - \{v_i\}) \cup \{v_i^*\}$. Necessarily, I^* is an (r,s)-set of G with $|I^*| = m$. Now let j be the largest integer for which $\ell_j < \ell_i$, and consider the value $\ell_k^* = d(v_k, I^* - \{v_k\})$ for each k with $1 \leq k \leq j$. Necessarily, $\ell_k^* \leq \ell_k$ for all k ($1 \leq k \leq j$). Furthermore, $d(v_i^*, I^* - \{v_i^*\}) = \ell_i - 1 < \ell_t$ for all $t > j$. It follows, therefore, that the distance sequence of I^* precedes that of I in dictionary order. This produces a contradiction. Hence $d(v_i, w_i) \geq s - r$ for all i, which completes the proof of the theorem.

As a corollary of Theorem 1, we have the following result.

Corollary 1. *For $s \geq 2$ an integer, if G is a connected graph of order $p \geq s$, then $i_{1,s}(G) \leq p/s$.*

Proof: Let $i_{1,s}(G) = k$. Among all the $(1,s)$-sets of G with cardinality equal to k, let I be one which comes first in dictionary order. Using the notation introduced in the proof of Theorem 1 (with $r = 1$), let Q_i denote a $v_i - w_i$ path of length $d(v_i, w_i)$ in G for each i with $1 \leq i \leq k$. Then the collection $\{Q_1, Q_2, \ldots, Q_k\}$ of paths is pairwise disjoint, for otherwise, for some i with $1 \leq i \leq k$, the vertex w_i is within distance s from at least two vertices of I. Thus, $\cup_{i=1}^{k} V(Q_i) \subseteq V(G)$. Hence, since each path Q_i contains at least s vertices, we have $ks \leq p$; or, equivalently $k \leq p/s$.

That the bound given in Corollary 1 is in a sense best possible may be seen by considering the connected graph G constructed as follows: for k and m very large integers, let G be obtained from a complete graph on m vertices by attaching to each of its vertices k disjoint paths of length s. (The graph G is shown in Figure 3.) Then $i_{1,s}(G) = (m-1)k + 1$ and

$p = p(G) = m(ks+1)$, so

$$\frac{i_{1,s}(G)}{p} = \frac{mk-k+1}{mks+m} = \frac{1-\frac{1}{m}+\frac{1}{mk}}{s+\frac{1}{k}} \overset{m,k}{\underset{\infty}{\longrightarrow}} \frac{1}{s}.$$

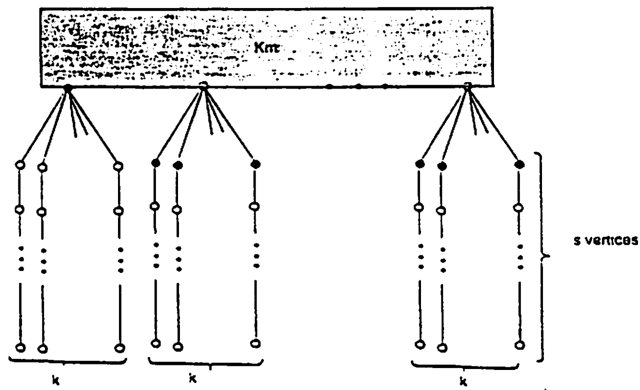

Figure 3. The graph G

3. Bounds on $i_{1,s}(T)$ for a tree T

In this section we investigate good upper bounds on $i_{1,s}(T)$ for a tree. The following result was established in [10].

Theorem B. *If T is a tree of order $p \geq 2$, then $i(T) \leq p/2$.*

That the bound given in Theorem B is sharp, may be seen by considering the tree T obtained from the union of two (disjoint) copies of $K(1,k)$ by joining their centers with an edge as shown in Figure 4. Then $i(T) = k+1$ and $p = 2(k+1)$, so $i(T) = p/2$.

Since $i_{1,1}(G) = i(G)$ for any graph G, Theorem B gives a good upper bound on $i_{1,1}(T)$ for a tree T. Hence in what follows, we investigate good upper bounds on $i_{1,s}(T)$ for a tree T and $s \geq 2$. If u and v are two adjacent vertices of T, then we will denote the two components of $T - uv$ by T_u and T_v where u is in T_u and v is in T_v.

Theorem 2. *If T is a tree of order $p \geq 3$, then $i_{1,2}(T) \leq p/3$.*

Proof: We proceed by induction on the order p of a tree T. If $p = 3$, then $T \cong P_3$ and $i_{1,2}(T) = p/3$. Let $p \geq 3$. Assume for every tree T' of order m, where $3 \leq m \leq p$, that $i_{1,2}(T') \leq m/3$, and consider a tree T of order $p+1$. Among all the $(1,2)$-sets of T of minimum cardinality, let I be one which has the smallest distance sequence in dictionary order. For $i = 0, 1, 2$, let

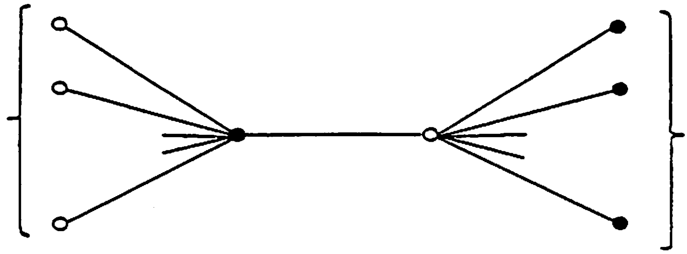

Figure 4. A tree T for which $i(T) = p(T)/2$

J_i be the set of all vertices of T at distance i from I (in particular, we have $J_0 = I$). We consider two possibilities.

Case 1.
For $i = 1$ or 2, J_i contains two adjacent vertices u and v. If T_u (or T_v) is of order 2, then the vertex of T_u, distinct from u, belongs to I and is an end-vertex of T. However replacing this vertex of I with the vertex u produces a minimum $(1,2)$-set whose distance sequence precedes that of I in dictionary order, contrary to assumption. Hence T_u and T_v are trees with $3 \le p(T_u) < p$ and $3 \le p(T_v) < p$. Let $I_u = I \cap V(T_u)$ and $I_v = I \cap V(T_v)$.

We show that I_u and I_v are minimum $(1,2)$-sets of T_u and T_v, respectively. If this is not the case, then we may assume, without loss of generality, that $i_{1,2}(T_u) < |I_u|$. Let I'_u be a minimum $(1,2)$-set of T_u. Then, since $d(u, I_v) \ge 2$, it follows that $I'_u \cup I_v$ is a $(1,2)$-set of T with $|I'_u \cup I_v| < |I_u| + |I_v| = |I| = i_{1,2}(T)$, which is impossible. We deduce, therefore, that I_u and I_v are minimum $(1,2)$-sets of T_u and T_v, respectively. Thus, by the inductive hypothesis, $|I_u| = i_{1,2}(T_u) \le p(T_u)/3$ and $|I_v| = i_{1,2}(T_v) \le p(T_v)/3$. Hence

$$i_{1,2}(T) = |I_u| + |I_v| \le p(T_u)/3 + p(T_v)/3 = p/3.$$

Case 2.
Neither J_1 nor J_2 contains two adjacent vertices. That is to say, each set J_i is independent. Let R be the set of all vertices of J_1 that are adjacent with some vertex of J_2, and let $S = J_1 - R$. Further, let S_1 be the set of all vertices of S that are at distance 2 from R, and let $S_2 = S - S_1$.

We show that every vertex of S_2 is at distance 2 from S_1. If this is not the case, then there is a vertex v of S_2 at distance 4 from S_1. Let $v = v_0$, v_1, v_2, v_3, v_4 be a shortest path from v to S_1. Then $v_1, v_3 \in J_0$, $v_2 \in S_2$ and $v_4 \in S_1$. Now let $J'_0 = (J_0 - (N(v_2) \cap J_0)) \cup \{v_2\}$. Then J'_0 is an independent set. Moreover, every vertex is within distance 2 from J'_0. To see this, observe that if there were a vertex at distance 3 from J'_0, then it must be a vertex from J_2. This means, however, that v_2 would be at distance 2 from R, which contradicts the fact that $v_2 \in S_2$. Hence J'_0 is a $(1,2)$-set of T. But the cardinality of J'_0 is less than that of J_0, which produces a contradiction. We deduce, therefore, that every vertex of S_2 is at distance 2 from S_1. This means that there is no vertex of J_0 that is adjacent only to vertices of S_2, for if there were such a vertex, then it may be removed from J_0 to produce a $(1,2)$-set of T of smaller cardinality than J_0. Hence every vertex of J_0 is adjacent with some vertex of R or S_1.

It follows from the above observations that $S_1 \cup J_2$ is a $(1,2)$-set of T, as is the set $R \cup S_2$. Hence $i_{1,2}(T) = |J_0| \leq |S_1| + |J_2|$ and $i_{1,2}(T) = |J_0| \leq |R| + |S_2|$. Thus $2|J_0| \leq |J_1| + |J_2| = p - |J_0|$; or, equivalently, $i_{1,2}(T) = |J_0| \leq p/3$. This completes the proof of the theorem.

That the bound given in Theorem 2 is sharp, may be seen by considering a tree T_s, of order p obtained from a path on k vertices by attaching a path of length s to each vertex of the path, as shown in Figure 5. Then $i_{1,s}(T_s) = k = p/(s+1)$.

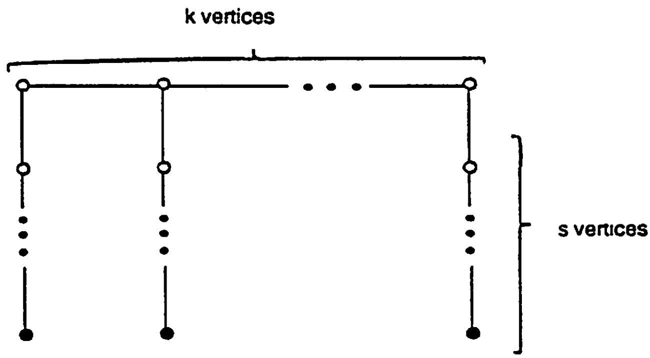

Figure 5. A tree T_s with $i(T) = p(T)/(s+1)$

Theorem 3. *If T is a tree of order $p \geq 4$, then $i_{1,3}(T) \leq p/4$.*

Proof: We proceed by induction on the order p of a tree T. If $p = 4$, then a central vertex of T is within distance 2 from every vertex of T and so $i_{1,3}(T) = p/4$. Let $p \geq 4$. Assume that for every tree T' of order m, where $4 \leq m \leq p$, that $i_{1,3}(T') \leq m/4$, and consider a tree T of order $p+1$. Among all the $(1,3)$-sets of T of minimum cardinality, let I be one which has the smallest distance sequence in dictionary order. For $i = 0, 1, 2, 3$, let J_i be the set of all vertices of T at distance i from I (in particular, we have $J_0 = I$). We consider two possibilities.

Case 1.
Some J_i contains two adjacent vertices u and v. If T_u (or T_v) is of order

less than 4, then replacing the vertex of I that is in T_u with the vertex u produces a minimum $(1,3)$-set of T whose distance sequence precedes that of I in dictionary order, contrary to assumption. Hence T_u and T_v are trees with $4 \le p(T_u) < p$ and $4 \le p(T_v) < p$. Let $I_u = I \cap V(T_u)$ and $I_v = I \cap V(T_v)$. Then, since $d(u, I_v) \ge 2$ and $d(v, I_u) \ge 2$, it is not too difficult to see that I_u and I_v must be minimum $(1,3)$-sets of T_u and T_v, respectively. Thus, by the inductive hypothesis, $|I_u| = i_{1,3}(T_u) \le p(T_u)/4$ and $|l_v| = i_{1,3}(T_v) \le p(T_v)/4$. Hence $i_{1,3}(T) = |I_u| + |I_v| \le p/4$.

Case 2.
For every two adjacent vertices of T, one of them belongs to J_i and the other to J_{i+1} for some $i \in \{0,1,2\}$. That is to say, each set J_i is independent. We now consider two further cases.

Case 2.1.
Some vertex v of J_i is adjacent with more than one vertex of J_{i-1} for some $i \in \{2,3\}$. Let u be a vertex of J_{i-1} that is adjacent with v. By the way in which the set I is chosen, it follows that each of T_u and T_v is of order at least 4. Hence T_u and T_v are trees with $4 \le p(T_u) < p$ and $4 \le p(T_v) < p$. Let $I_u = I \cap V(T_u)$ and $I_v = I \cap V(T_v)$. Now since v is adjacent with a vertex of J_{i-1}, distinct from u, and since T is a tree, it follows that neither I_u nor I_v is empty. Thus, since $d(u, I_v) \ge 3$ and $d(v, I_u) \ge 2$, it is not difficult to see that I_u and must be minimum $(1,3)$-sets of T_u and T_v, respectively. Thus, by the inductive hypothesis, $i_{1,3}(T) = |I_u| + |I_v| \le p(T_u)/4 + p(T_v)/4 = p/4$.

Case 2.2.
For $i = 2$ and 3, every vertex of J_i is adjacent with exactly one vertex of $J_{i=1}$. (Note that each vertex of J_3 is thus an end-vertex.)

Let $A_1 = J_1 \cap N(J_2)$, that is the set of all vertices in J_1 that are adjacent with some vertex of J_2, and let $B_1 = J_1 - A_1$. Then every vertex of B_1 is at distance at least 2 from A_1. We will show that this distance is always exactly 2. To this end, let S be the set of vertices of B_1 whose distance from A_1 is greater than 2, and suppose $S \ne \emptyset$. Then it is not difficult to see that every vertex of S is a distance 2 from $B_1 - S$, and therefore at distance 3 from $N(A_1) \cap J_0$. This means that $N(A_1) \cap J_0$ is a $(1,3)$-set of cardinality less than that of I, which is a contradiction. We deduce therefore that every vertex of B_1 is a distance 2 from A_1. Hence, J_2 is a $(1,3)$-set of T.

Now let $B_0 = J_0 \cap N(S)$ and let $A_0 = J_0 - B_0$. Further, let $A_2 = J_2 \cap N(J_3)$ and $B_2 = J_2 - A_2$. Also, let $A_1^{(1)} = A_1 \cap N(A_2)$ and $A_1^{(2)} = (A_1 - A_1^{(1)}) \cap N(B_0)$. Now let $C_1 = A_1^{(1)} \cup A_1^{(2)}$. We show that every vertex of $A_1 - C_1$ is at distance 2 or 4 from C_1. Supposing this not to be the case, we may assume that u_0 is a vertex of A_1 at distance 6 from C_1 and that u_0, u_1, u_2, u_3, u_4, u_5, u_6 is a shortest path from u_0 to C_1. Then u_1, u_3, $u_5 \in A_0$, u_2, $u_4 \in A_1 - C_1$ and $u_6 \in C_1$. Let $I^* = (I - (N(u_2) \cap I)) \cup \{u_2\}$. Then it is not difficult to check that I^* is a $(1,3)$-set of T of cardinality less than that of I, which is a contradiction. Therefore every vertex of $A_1 - C_1$ is at distance 2 or 4 from C_1. Let $A_1^{(3)}$ and $A_1^{(4)}$ be the sets of those vertices in $A_1 - C_1$ at distance 2 and 4, respectively, from C_1. Note that every vertex of $A_1^{(4)}$ is at distance 2 from $A_1^{(3)}$, and that the four sets $A_1^{(i)}$ form a partition of A_1.

By choice of A_1, every vertex in B_2 is an end vertex. Next, for $i = 1, 2, 3, 4$, let $B_2^{(i)} = B_2 \cap N(A_1^{(i)})$. Figure 6 shows the decomposition of T into these sets at the different levels.

It follows from the above observations that each of $A_1^{(1)} \cup A_1^{(2)} \cup A_1^{(4)}$, J_2 and $A_1^{(3)} \cup B_1 \cup J_3$ is a (1,3)-set of T. Hence, the cardinality of each of these sets is at least $|I|$. Therefore, since $I = J_0$,

$$\begin{aligned} 3|J_O| &\le |A_1^{(1)}| + |A_1^{(2)}| + |A_1^{(3)}| + |A_1^{(4)}| + |B_1| + |J_2| + |J_3| \\ &= |J_1| + |J_2| + |J_3| \\ &= p - |J_0|. \end{aligned}$$

Equivalently, $i_{1,3}(T) \le p/4$, which completes the proof of the theorem.

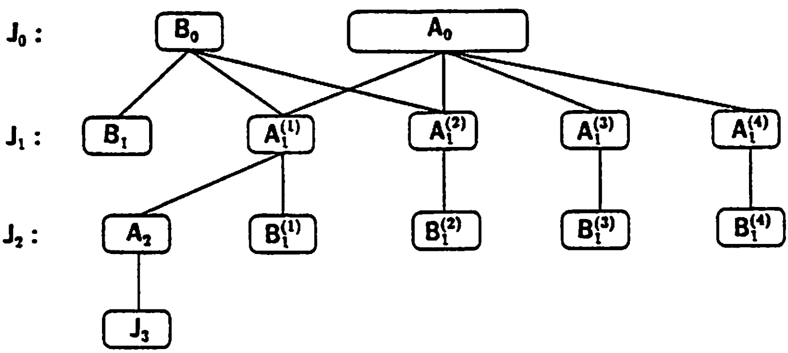

Figure 6. The structure of the tree T in Case 2.2

We close with the following:

Conjecture. *For all integers $s \ge 1$, if T is a tree of order $p \ge s$, then* $i_{1,s}(T) \le p/(s+1)$.

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