Intersection-representation by connected subgraphs of some n-cyclomatic graph

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Abstract. A hypergraph H is called *connected over* a graph G with the same vertex set as H if every hyperedge of H induces a connected subgraph in G. A graph F is *representable* in the graph G if there is some hypergraph H which is connected over G and has F as its intersection graph. Generalizing the well-known problem of representability in forests, the following problems are investigated: Which hypergraphs are connected over some n-cyclomatic graph, and which graphs are representable in some n-cyclomatic graph, for any fixed integer n? Several notions developed in the theory of subtree hypergraphs and chordal graphs (i.e. in the case $n = 0$) yield necessary or sufficient conditions, and in certain special cases even characterizations.

1 Introduction

A *hypergraph* $H = (V, Y)$ consists of a *vertex set* V and a family $Y = (y_i/i \in I)$ of nonempty subsets of V — the so - called *hyperedges* of H — such that every vertex $b \in V$ lies in some hyperedge. The hypergraph is finite if both V and Y are finite.

A *graph* $G = (V, E)$ consists, as usual, of a vertex set V and an edge set E, which is a subset of the set of all two-element subsets of V. For $V' \subseteq V$, the *induced subgraph* $G[V']$ of G is the graph with vertex set V' and those edges of G whose both endpoints lie in V'.

The *cyclomatic number* $\beta_1(G)$ of the graph G denotes the first modulo 2 Betti cardinal number of the simplicial complex formed by the vertices and edges of G. If G is finite, it equals $|E| - |V| + q(G)$, where $q(G)$ is the number of connected components of G. A graph is called *n-cyclomatic* if its cyclomatic number is at most the natural number n.

All graphs and hypergraphs are finite unless explicitely stated otherwise.

A hypergraph $H = (V, (y_i/i \in I))$ is *connected over* the graph $G = (V, E)$ if every hyperedge y_i induces a connected subgraph $G[y_i]$ in G. The

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intersection graph or *representative graph* $\Omega(H)$ of H has I as vertex set, and two distinct elements $i \neq j$ of I are adjacent in $\Omega(H)$ if $y_i \cap y_j \neq \emptyset$. If H is connected over G, then $\Omega(H)$ is called *representable* in the graph G, and H is a *representation* of $\Omega(H)$ in G.

The following problems are investigated in this paper: Given any integer n, which hypergraphs are connected over some n-cyclomatic graph, and which graphs are representable in some n-cyclomatic graph? Of course, the representability in an n-cyclomatic graph implies representability in higher-cyclomatic graphs, thus it suffices to determine the smallest such number.

The case $n = 0$, representability in forests, has been studied extensively. In the early seventies, Buneman [6], Gavril [11], and Walter [24] independently showed that a (finite!) graph is representable in some forest if and only if it is *chordal* or *triangulated*, that is, if the only induced subcycles are triangles. These chordal graphs had been previously characterized by Dirac [7], Lekkerkerker and Boland [17], and Rose [21], and they obey many nice properties. To mention only two, they are perfect (see [3,14]), and most of the problems which are in general NP-complete can be solved in polynomial time for chordal graphs. Furthermore, the famous interval graphs are chordal.

The other stream of research in the representability in forests arose in statistics and in the computer sciences (see [16,2] for details). Hypergraphs which are connected over some forest are called *subtree hypergraphs* and their duals are called *acyclic hypergraphs*. Many characterizations of acyclic hypergraphs (and thus of subtree hypergraphs also) are known, and again, these hypergraphs obey many desirable properties. But indeed, there is a very strong connecteion between acyclic hypergraphs and chordal graphs, see [9,10]. Almost every result about chordal graphs has its counterpart in the theory of acyclic hypergraphs, and conversely.

In the case $n = 1$, a proper subclass of the class in question, the class of *circular-arc graphs* (i.e. graphs representable in some cycle) has been investigated, see [22,12].

This paper is arranged as follows: In section 2 we give several sufficient conditions for representability in n-cyclomatic graphs. Most of them are corollaries of a generalization of a result in [1]. In section 3 three necessary conditions for a hypergraph to be connected over some n-cyclomatic graph are given. Such necessary hypergraph conditions do not automatically imply necessary conditions for representability of graphs, but in these three special cases, they do. In section 4 we show that one of these necessary conditions is also sufficient for special classes of hypergraphs or graphs.

Let us close this section by giving some standard notations and definitions: The numbers of vertices and edges of a graph are denoted by α_0 and α_1 respectively. The *dual* H^* of a hypergraph $H = (V, (y_i/i \in I))$ is the hypergraph $(I, (v^*/v \in V))$, where v^* is the set of those indices i for which y_i contains v in H. For $V' \subseteq V$, the *subhypergraph* $H[V']$ is $(V', (y_i \cap V'/i \in J))$, where J denotes the set of all indices i with $y_i \cap V' \neq \emptyset$. Every subfamily $Y' = (y_i/i \in I')$ of the hyperedge family generates a *partial hypergraph* $H\langle Y'\rangle$ of H, which is defined as $H^*[I']$. A *transversal* of the hypergraph H is a subset V' of V that meets each hyperedge at least once. The *transversal number* $\tau(H)$ is the cardinality of a smallest transversal. *Cliques* in a graph are (vertex sets of) maximal complete subgraphs. The *clique hypergraph* $\kappa(F)$ of the graph $F = (V, E)$ has V as vertex set, and all cliques of F as hyperedges. Of course $F = \Omega((\kappa(F))^*)$.

2 Sufficient conditions

First we show that for every representation of a graph F in some minimal-cyclomatic graph G, this G must be triangle-free. For, if G contains a triangle $\{a, b, c\}$, then we can construct a lower-cyclomatic graph G' in which F is representable also: We add a new vertex t and the edges ta, tb, tc, and delete the edges ab, ac, bc. We make this more precise:

Proposition 2.1 *Let the graph F be representable in the graph G, and let G contain k pairwise edge-disjoint triangles. Then F is representable in some $(\beta_1(G) - k)$-cyclomatic graph.*

Proof: Let $H = (V, (y_i/i \in I))$ be some representation of F in $G = (V, E)$, and let $T_1, T_2, \ldots, T_k$ be these mentioned triangles. We choose k new vertices $t_1, t_2, \ldots, t_k$ and define some modified hypergraph $H' := (V \cup \{t_1, t_2, \ldots, t_k\}, (y_i'/i \in I))$, where every y_i' is the union of y_i and the set of those vertices t_p for which at least two vertices of T_p lie in y_i. Any two new hyperedges y_i' and y_j' (i $\neq j$) intersect if the old ones y_i, y_j had nonempty intersection. But conversely, if $y_i' \cap y_j' \neq \emptyset$, then y_i and y_j have nonempty intersection, or y_i' and y_j' contain some common member of the form t_p. In this second case, again, y_i and y_j intersect in some vertex of T_p. So we have shown that F is the representative graph of H' also.

Next we construct a new graph G' from G by deleting all $3k$ edges of the triangles $T_1, T_2, \ldots, T_k$, adding the k new vertices $t_1, t_2, \ldots, t_k$, and joining every t_p by an edge with each vertex of the corresponding triangle T_p. The graph G' has as many edges as G, and the same number of connected components, but it has k more vertices. Thus G' is $(\beta_1(G) - k)$-cyclomatic. It is easily seen that H' is connected over G'. □.

Acharya and Las Vergnas introduced in [1] the hypergraph cyclomatic number. This has nothing to do with the first Betti numbers, but for finite graphs it coincides with the graph cyclomatic number β_1. The *weighted underlying graph* $G_w(H)$ of the hypergraph $H = (V, (y_1, y_2, \dots, y_m))$ has vertex set V and two distinct vertices b and c are joined whenever they lie in some common hyperedge of H. The *weight* $w(bc)$ of this edge is the number of such common hyperedges. Let T be a maximal spanning tree of $G_w(H)$ — if this weighted graph is not connected, it can be made so by adding some edges of weight 0. The *cyclomatic number* $\mu(H^*)$ of the dual of H is defined by

$$\mu(H^*) := \sum_{i=1}^{m}(|y_i| - 1) - w(T).$$

Acharya and Las Vergnas [1] and Lewin [19] showed that a hypergraph is a subtree hypergraph if and only if $\mu(H^*) = 0$. One direction can be generalized:

Theorem 2.2 *Every hypergraph H is connected over some $\mu(H^*)$-cyclomatic graph.*

Proof: Let $H = (V, (y_1, y_2, \dots, y_m))$, and let T be some maximum spanning tree of the weighted underlying graph $G_w(H)$ of H. Let, for every $1 \le i \le m$, q_i denote the number of connected components of the forest $T[y_i]$. We construct a finite sequence $T = G_0 \subseteq G_1 \subseteq \dots \subseteq G_m$ of graphs, all with V as vertex set, and all with the property that $G_j[y_i]$ is connected for all $1 \le i \le j \le m$. This can be done by adding at most $q_i - 1$ new edges in step i to transform G_{i-1} into G_i. Thus G_m is $\sum_{i=1}^{m}(q_i - 1)$-cyclomatic, and H is connected over G_m. Of course

$$\sum_{i=1}^{m}(q_i - 1) = \sum_{i=1}^{m}(|y_i| - \alpha_1(T[y_i]) - 1)$$

and

$$w(T) = \sum_{i=1}^{m} \alpha_1(T[y_i]).$$

From these two equations there follows that G_m is $\mu(H^*)$-cyclomatic. □.

The result is sharp. For every positive integer n there is a hypergraph H_n — namely the dual of an arbitrary n-cyclomatic graph F_n — which is not connected over any $(n-1)$-cyclomatic graph. But $\mu(H_n^*) = \beta_1(F_n) = n$, see [1].

On the other hand, $\mu(H^*)$ is not useful for necessary conditions. Look at the hypergraph $H_{m,n}$ whose vertex set contains the n vertices of the

n-cycle C_n and whose hyperedges are formed by (the vertex sets of) all m-vertex subpaths of C_n, where $m < n$. By construction, each $H_{m,n}$ is connected over the 1-cyclomatic C_n. But $\mu(H^*_{m,n} = \mu(H_{m,n}) = n(m-1) - (m-1)(n-1) = m-1$.

In [20] two polynomial heuristic algorithms are presented, that construct for any input hypergraph H some low-cyclomatic graph G over which H is connected. For one of these algorithms the resulting graph is even $\mu(H^*)$-cyclomatic.

Theorem 2.2 can be improved by defining $\mu'(H^*)$ as

$$\mu'(H^*) := \sum_{i=1}^{m}(|y_i| - 1) - max\{w(J) - \beta_1(J)\}$$

where J runs over all connected subgraphs of $G_w(J)$ without cycles of length $l \leq r(H) := max\{|y_i|/1 \leq i \leq m\}$. As in the proof of Theorem 2.2 one can show:

Proposition 2.3 *Every hypergraph H is connected over some $\mu'(H^*)$-cyclomatic graph.* □

Now we apply Theorem 2.2 to obtain sufficient conditions for a graph to be representable in some n-cyclomatic graph.

Proposition 2.4 *Let Δ be some set of triangles of the graph $F = (Y, E_F)$, and let E_0 denote the set of those edges of F that lie in no member of Δ. Then F is representable in some $(2|\Delta|+|E_0|-|Y|+q(F))$-cyclomatic graph.*

Proof: Consider the dual H of the hypergraph $H^* = (Y, E_0 \cup \Delta)$. F is the representative graph of $H = (E_0 \cup \Delta, \{y^\circ / y \in Y\})$, where y° denotes the set of those triangles of Δ or edges of E_0 that contain the vertex y of F. Each maximal spanning tree T of $G_w(H)$ has $|\Delta|+|E_0|$ vertices and weight at least $|\Delta| + |E_0| - q(F)$, where $q(F)$ denotes the number of connected components of F, or equivalently, of H. Then

$$\mu(H^*) \leq \sum_{y \in Y}(|y^\circ| - 1) - (|\Delta| + |E_0| - q(F)).$$

Since

$$\sum_{y \in Y}(|y^\circ|) = \sum_{t \in \Delta \cup E_0} |t| = 3|\Delta| + 2|E_0|,$$

H is connected over some $(2|\Delta| + |E_0| - |Y| + q(F))$ -cyclomatic graph G; it is a representation of F in G. □.

This implies a result, that looks at the first sight rather similiar than Proposition 2.1:

Corollary 2.5 *Every graph F with k pairwise edge-disjoint triangles is representable in some $(\beta_1(F) - k)$-cyclomatic graph.*

Proof: Let Δ be the set of this pairwise edge-disjoint triangles. Then $|E_0| = \alpha_1(F) - 3k$. Now

$$2|\Delta| + |E_0| - |Y| + q(F) = 2k + \alpha_1(F) - 3k - |Y| + q(F) = \beta_1(F) - k,$$

and we apply Proposition 2.4. □.

3 Necessary conditions

In [20] various necessary conditions for hypergraphs connected over some n-cyclomatic graph were given. In the present paper we mention only those that can be used to obtain necessary conditions for representability of graphs.

The first approach generalizes definitions and results of Rose [21] for graphs, and Graham [13] and Beeri et al. [2] for hypergraphs. Assume that the hypergraph $H = (V, (y_1, y_2, \ldots, y_m))$ is given. In the following H_i denotes the partial hypergraph of H generated by those hyperedges h_j with $j \geq i$ and nonempty intersection with y_i. The ordering $y_1, y_2, \ldots, y_m$ of the hyperedges is called an *n-hypergraph elimination ordering (n-heo)* provided all $H_i, 1 \leq i \leq m$ have transversals of cardinalities at most n.

The dual version of some results of Graham [13] and Beeri et al. [2] is: A hypergraph is a subtree hypergraph if and only if it has some 1-heo. One direction can be generalized:

Theorem 3.1 *Every hypergraph which is connected over some $2n$-cyclomatic graph for an integer n, has some $(n+1)$-heo.*

Proof: The proof is by induction on n. The case $n = 0$ was mentioned above. Let now for a fixed integer $n > 0$ the statement be true for all smaller numbers, and let $H = (V, Y)$ be connected over the $2n$-cyclomatic graph G. W.l.o.g. we may assume G connected.

Case 1: The cycles of G are pairwise vertex-disjoint. Let J be some spanning tree of the *block graph* (that is the intersection graph of the set of all blocks) of G, and let the blocks of G (the vertices of J) be numbered as $B_0, B_1, \ldots, B_t$ such that $J[\{B_0, B_1, \ldots, B_t\}]$ is connected for every $0 \leq i \leq t$. For a hyperedge y of H , the integer $b(y)$ denotes the smallest index i for which y contains vertices of B_i, and $A(y)$ is defined as the set of these vertices — the nonempty intersection $V(B_{b(y)}) \cap y$. It is possible to order the hyperedges of H as $y_1, y_2, \ldots, y_m$ so that $b(y_i)$ is nonincreasing for increasing i, and, furthermore, $A(y_j)$ is not a proper subset of $A(y_k)$

for $k < j$ and $b(y_k) = b(y_j)$. Since every block of G must be a cycle or an edge, every $A(y_i)$ induces some (possibly trivial) path in G. We take the end vertices of this path as the set T_i of cardinality 1 or 2. It is easy to see that $y_1, y_2, \ldots, y_m$, together with these sets T_i, form a 2-heo of H.

Case 2: We can find a vertex x such that $G' := G - x$ is $2(n-1)$-cyclomatic. Let Y' be the family of those hyperedges of H that do not contain x. Then $H' := H(Y')$ is connected over G'. By the induction hypothesis, H' has some n-heo $y_1, y_2, \ldots, y_t$ with transversals $T_1, T_2, \ldots, T_t$. We order the hyperedges containing x arbitrarily as $y_{t+1}, \ldots, y_m$. Then $T_1 \cup \{x\}, T_2 \cup \{x\}, \ldots, T_t \cup \{x\}, \{x\}, \ldots, \{x\}$ are transversals of $H_1, H_2, \ldots, H_m$, whence $y_1, y_2, \ldots, y_m$ forms an $(n+1)$-heo of H. □.

An *n-graph elimination ordering (n-geo)* of a graph F is an ordering $y_1, y_2, \ldots, y_t$ of its vertices which is an n-heo of the hypergraph $(\kappa(F))^*$. In other words, the complements of all subgraphs F_i induced by y_i and its higher indexed neighbors are at most n-chromatic. Rose has shown in [13] that a graph is chordal if and only if it has an 1-geo. From Theorem 3.1 there follows:

Corollary 3.2 *If a graph is representable in some $2n$-cyclomatic graph, then it has an $(n+1)$-geo, for every positive integer n.*

Proof: Let $H = (V, Y)$ be a representation of the graph $F = (Y, E_F)$ in the $2n$-cyclomatic graph $G = (V, E_G)$. By 3.1, H has an $(n+1)$-heo $y_1, y_2, \ldots, y_m$. The graphs F_i mentioned in the definition of the n-geo are the representative graphs of the hypergraphs H_i. Thus, for every transversal $T_i = \{x_i^1, x_i^2, \ldots, x_i^k\}$ of H_i the set $\{x_i^{1*}, x_i^{2*}, \ldots, x_i^{k*}\}$ is a vertex cover of F_i by complete subgraphs (where x_i^{s*} denotes the set of those hyperedges containing x_i^s). □.

The following two examples show that 3.1 and 3.2 are sharp for $n \leq 3$. Consider first the graph $G = (V, E)$ which is obtained from the complete graph K_4 by substituting every edge by a 4-vertex path. Let Y denote the set of all connected induced subgraphs of G which contain exactly one "old" vertex, and where this vertex has also degree 3 in this subgraph. Then the hypergraph $H = (V, Y)$, which is connected over the 3-cyclomatic graph G by construction, has no 2-heo. Moreover the graph $\Omega(H)$ has no 2-geo.

For the second example, let $G = (V, E)$ be the graph obtained from the cube graph $P_2 \times P_2 \times P_2$ by replacing every edge by a 4-vertex path. Let Y be the set of all connected, induced subgraphs of G which have exactly 2 "old" vertices, and where both have also degree 3 in this subgraph. The hypergraph $H = (V, Y)$ is connected over the 5-cyclomatic graph G, but it can be shown that it has no 3-heo. The representative graph $\Omega(H)$ has no 3-geo either.

On the other hand, 3.1 and 3.2 can not be converted. The hypergraph H_n of Figure 1 has a 2-heo, but it is not connected over any $(2n+1)$-cyclomatic graph. see Theorem 3.6. Its representative graph $\Omega(H) = P_3 \times P_{2+n}$ has a 2-geo, but it is not representable in any $(2n+1)$-cyclomatic graph, see Corollary 3.7.

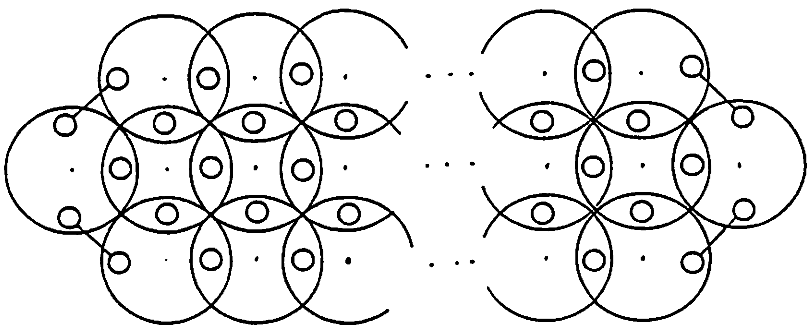

Figure 1
A hypergraph H_n with $7+5n$ vertices and $6+3n$ hyperedges

Our second approach is inspired by a Theorem of Dirac in [7]. A subset Y' of the hyperedge set of a hypergraph H forms a *relative minimal cutset*, if there are hyperedges r and s which are separated by Y' (that is, if they lie in different connected components of the partial hypergraph $H\langle Y \setminus Y' \rangle$), and if Y' is minimal with this property.

Theorem 3.3 *Let for a nonnegative integer n the (possibly infinite) hypergraph $H = (V, Y)$ be connected over some (possibly infinite) n-cyclomatic graph. Then $\tau(H\langle Y' \rangle) \leq n+1$ for every relative minimal cutset Y' of H.*

Proof: Let H be connected over the graph $G = (V, E)$. A hyperedge y of H *covers* an edge of G if it contains both vertices of that edge. We may assume that every edge of G is covered by some hyperedge of H — otherwise edges could be deleted without loosing connection of H. Let Y' be some relative minimal cutset of H, minimally separating the hyperedges r and s. Let E' denote the set of those edges of G that are not covered by members of $Y \setminus Y'$. Of course $H\langle Y \setminus Y' \rangle$ is connected over $G\langle E \setminus E' \rangle$, and r and s are subsets of distinct connected components of this graph. Let E° be some minimal subset of E' separating these vertex sets r and s. Surely all these edges must lie in some common block B of G. As a finite-cyclomatic block, B must be finite. Deleting all edges of E° in B results in a graph M with the same vertices as B and exactly two connected components. Now

$$n \geq \beta_1(B) = \alpha_1(B) - \alpha_0(B) + 1 =$$

$$= \alpha_1(M) + |E^\circ| - \alpha_0(M) + 1 = |E^\circ| - 1 + \beta_1(M),$$

thus $|E^\circ| \leq n + 1$.

Each hyperedges of Y' covers at least one edge of E°, otherwise a proper subfamily of Y' would separate r from s. Thus every transversal of $G\langle E^\circ\rangle$ is also a transversal of $H\langle Y'\rangle$, but, by the computation above, $\tau(G\langle E^\circ\rangle) \leq n + 1$. □.

We have indeed shown a bit more: The statement of the theorem is also true if H is connected over some graph whose blocks are all n-cyclomatic.

A subset V' of the vertex set V of a graph F is a *relative minimal cutset* if there are two vertices u, v that are separated by V' but by no proper subset of V'. $Y' \subseteq Y$ is a relative minimal cutset of a hypergraph $H = (V, Y)$ if and only if the corresponding vertex set of $\Omega(H)$ forms a relative minimal cutset there.

Corollary 3.4 *Let the (possibly infinite) graph $F = (Y, E_F)$ be representable in some (possibly infinite) n-cyclomatic graph, for $n \in \mathcal{N}$. Then every relative minimal cutset of F can be covered by at most $n+1$ complete subgraphs of F.*

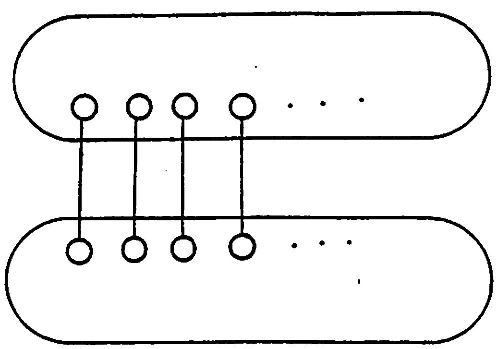

Figure 2
A hypergraph H_n° with $2n + 2$ vertices and $n + 3$ hyperedges

The results of 3.3 and 3.4 are sharp: The set of all two-element hyperedges of the hypergraph H_n° in Figure 2 forms a relative minimal cutset without m-element transversals for $m \leq n$. But H_n° is connected over an n-cyclomatic block, namely over the 'ladder graph' $P_2 \times P_{n-1}$.

For $n \geq 1$, the converses of 3.3 and 3.4 are false: Look at the hypergraph H_n in Figure 1. Every relative minimal cutset of H_n or of $\Omega(H_n)$ has at most $n+2$ elements, but H_n is not connected over any $(2n+1)$-cyclomatic graph, and $\Omega(H_n)$ is not representable in any $(2n+1)$-cyclomatic graph. Only for $n = 0$, Dirac [7] showed that a graph is representable in some tree if and only if every relative minimal cutset is complete. But the converse of 3.3 fails even here: The hypergraphs $H_{n-1,n}, n \geq 3$, defined in Section 2, are no subtree hypergraphs, but they have no cutsets.

Thirdly we present some topological approach. To every (possibly infinite) hypergraph H we can associate a simplicial complex $S(H)$: The vertices of $S(H)$ are those of H, and a nonempty finite set of vertices forms a simplex in $S(H)$ if and only if it is contained in some hyperedge of H. Two simplicial complexes are called ***homologically equivalent*** if the corresponding homology groups are isomorphic. The following proposition follows from a Theorem of Dowker in [8]:

Proposition 3.5 *For every possibly infinite hypergraph H, $S(H)$ and $S(H^*)$ are homologically equivalent.*

Proof: Let $H = (V, (y_i/i \in I))$. We define a relation $R \subseteq V \times I$ by $(v, i) \in R$ iff $x \in y_i$. According to [8], the two complexes, which are built by all finite nonempty subsets of V and I repectively with some common relative in R, are homologically equivalent. But these two complexes are indeed $S(H)$ and $S(H^*)$. □.

Let now in the sequel $\beta_1(S)$ denote the first modulo 2 Betti cardinal number of the simplicial complex S.

Theorem 3.6 *Let P be a partial hypergraph of the (possibly infinite) hypergraph H, and let n be some cardinal number smaller than $\beta_1(S(P))$. Then H is not connected over any n-cyclomatic graph.*

Proof: Let $H = (V, (y_i/i \in I))$ be connected over the (possibly infinite) graph G, and let V_P denote the vertex set of P. P is connected over the induced subgraph $G[V_P]$ of G; obviously $\beta_1(G[V_P]) \leq \beta_1(G)$. We are going to show that every 1-cycle $\sum \langle a_k, b_k \rangle$ of $S(P)$ is in $S(P)$ homological to some 1-cycle z of $G[V_P]$; here and in the following all sums are meant modulo 2. For every index k of the sum, we can find some hyperedge y_k of P that contains both a_k and b_k. Since $G[V_P][y_k]$ is connected, it contains an $a_k - b_k$ path $a_k = d_k^0, d_k^1, \ldots, d_k^{t(k)} = b_k$. So

$$d := \sum_k \sum_{j=1}^{t(k)-1} \langle d_k^0, d_k^j, d_k^{j+1} \rangle$$

is a 2-chain of $S(P)$, and

$$z := \sum_k \sum_{j=1}^{t(k)-1} \langle d_k^j, d_k^{j+1} \rangle$$

is an 1-chain of $G[V_P]$, and also of $S(P)$. Since the difference of z and $\sum \langle a_k, b_k \rangle$ is the boundary of d, z is such an 1-cycle we were looking for. Therefore $H_i(S(P))$ is a subgroup of $Z_i(G[V_P]) = H_i(G[V_P])$. □.

In section 4 we shall see that this result is sharp. But the converse of the statement fails in general. Look for example at the hypergraph $H_{3,4}$, defined in section 2, which is not a subtree hypergraph. But for every partial hypergraph P of $H_{3,4}$ we get $\beta_1(S(P)) = 0$.

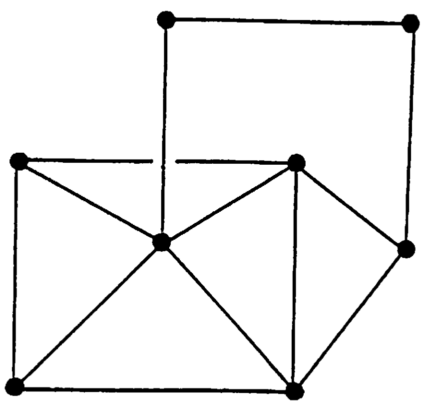

Figure 3
A graph F not representable in any 1-cyclomatic graph

Corollary 3.7 *A possibly infinite graph F is representable in no n-cyclomatic graph (where n denotes any cardinal number) if F has an induced subgraph J for which $n < \beta_1(S(\kappa(J)))$.*

Proof: Assume F were representable in some n-cyclomatic graph $G = (V, E_G)$, with n as above. Then there is some representation $H = (V, Y)$ of $J = (Y, E_J)$ in G. Now $S(\kappa(J))$ and $S(H^*)$ have the same 1-cycle groups. But since every 2-chain of $S(H^*)$ is also a 2-chain of $S(\kappa(J))$, $H_i(S(\kappa(J)))$ is a subgroup of $H_i(S(H^*))$. Then $n < \beta_1(S(\kappa(J))) < \beta_1(S(H^*)) = \beta_1(S(H))$, a contradiction to Theorem 3.6. □.

This result is again sharp (see 4.4), but for $n \geq 1$, again the converse fails. The graph F of Figure 3 obeys $\beta_1(S(\kappa(J))) \leq 1$ for every induced subgraph J of F. It is easy to show that for any two induced cycles of length ≥ 4 of a graph which is representable in an 1-cyclomatic graph, each vertex of the one must be adjacent to some vertex of the second cycle. But this does not hold for the graph in Figure 3, thus it is not representable in any 1-cyclomatic graph. Only for finite graphs and $n = 0$ the converse also holds (see the characterization of Buneman, Gavril, and Walter [6,11,24]).

4 Characterizations

For a *regionally subtree hypergraph* $H = (V, (y_1, y_2, \ldots, y_m))$, every subhypergraph $H[y_i], 1 \leq i \leq m$, is a subtree hypergraph. In this section we shall determine the smallest integer n for which a regionally subtree hypergraph is connected over some n-cyclomatic graph. First we need two results. As an immediate consequence of a theorem of Leray [18, p. 138], we get the following improvement of Theorem 3.6:

Lemma 4.1 *Let the hypergraph $H = (V, (y_1, y_2, \ldots, y_m))$ be connected over the graph G. If there are subtrees $T_1, T_2, \ldots, T_m$ of G with vertex sets $y_1, y_2, \ldots, y_m$ respectively, such that all intersections of these trees are (possibly empty) trees, then $S(H^*)$ and $\bigcup_{i=1}^{m} T_i$ are homologically equivalent*

Theorem 4.2 *([5]) A subtree hypergraph H is connected over exactly all maximum spanning trees of the weighted underlying graph $G_w(H)$.*

Theorem 4.3 *Every regionally subtree hypergraph H is connected over some $\beta_1(S(H))$-cyclomatic graph.*

Proof: Assume $H = (V, (y_1, y_2, \ldots, y_m))$. Let us call a graph $G_i = (V, E_i)$ *i-admissible*, if it obeys:

(i) G_i is the union of the graphs $G_i[y_j], j < i$;

(ii) For every nonempty subset J of $\{1, 2, \ldots, i-1\}$ the (possibly empty) set $V_J := \bigcap_{j \in J} y_j$ induces a tree in G_i over which $H[V_J]$ is connected.

The edgeless graph $G_1 := (V, \emptyset)$ is 1-admissible. If we have eventually found some graph G_{m+1} that is $(m+1)$-admissible, then H is connected over G_{m+1} and, by (ii) and 4.1, $S(H^*)$ and G_{m+1} (as a 1-dimensional simplicial complex) are homologically equivalent. Then Proposition 3.5 completes the proof.

All what remains to show is how to construct an $(i+1)$-admissible graph provided 1-admissible, ..., i-admissible graphs $G_1, \ldots, G_i$ respectively are given.

We call the hyperedges $y_1, y_2, \ldots, y_{i-1}$ the *old* hyperedges, and the others the *new* ones.

Recall that for every vertex $v \in V$ the set of those hyperedges that contain v is denoted by v^*.

Claim 1: For every path $v_0, v_1, \ldots, v_p$ of $G_i[y_i]$ holds

$$v_0^* \cap v_1^* \supseteq v_0^* \cap v_2^* \supseteq \ldots \supseteq v_0^* \cap v_p^*.$$

In other words, every hyperedge containing two vertices of $G_i[y_i]$ contains all vertices of all $G_i[y_i]$-paths between them.

Assume that this is not the case, let $W = u_0, u_1, \ldots, u_t$ with $t \geq 2$ be a minimal path where the corresponding inclusion sequence fails. Thus, by the minimality of W, there is some hyperedge y_k that covers u_0 and u_t, but no other vertex of W. Applying (i), we can find a covering $y_{\ell(1)}, y_{\ell(2)}, \ldots, y_{\ell(s)}$ of the edges of W by old hyperedges (i.e. all $\ell(j) < i$), and we choose one with s minimum.

Assume first $s = 1$. Then W is part of the tree $G_i[y_{\ell(1)}]$ over which $H[y_{\ell(1)}]$ is connected by (ii). But the hyperedge $y_k \cap y_{\ell(1)}$ of $H[y_{\ell(1)}]$ contains u_0 and u_t but no further vertex of W, a contradiction (since $t \geq 2$).

Hence $s \geq 2$. By the minimality of the path W chosen, $V(W) \cap y_{\ell(j)}$ is a consecutive part $u_{f(j)}, u_{f(j)+1}, \ldots, u_{g(j)}$ of W, for every $1 \leq j \leq s$. Let the indices be ordered such that $f(1) \leq f(2) \leq \ldots \leq f(s) \leq g(s) = t$. By the minimality of s, we get $g(j) < f(j+2)$ for each $1 \leq j \leq s-2$. Thus $y_k \cap y_i, y_{\ell(1)} \cap y_i, y_{\ell(2)} \cap y_i, \ldots, y_{\ell(s)} \cap y_i, y_k \cap y_i$ is a cycle of the graph $F_i := \Omega(H[y_i])$. But this graph must be chordal, since $H[y_i]$ is a subtree hypergraph. We distinguish two cases:

Case 1: $y_k \cap y_i, y_{\ell(1)} \cap y_i, y_{\ell(2)} \cap y_i$ form a triangle in F_i. Every subtree hypergraph is a Helly hypergraph (see [9] or [10]), thus there must be some vertex z_0 in the intersection of the three hyperedges $y_k \cap y_i, y_{\ell(1)} \cap y_i$, and $y_{\ell(2)} \cap y_i$ of $H[y_i]$. By (ii), the tree $G_i[y_{\ell(1)} \cap y_{\ell(2)}]$ contains both vertices z_0 and $u_{f(2)}$, so it contains also some path $z_0, z_1, \ldots, z_p = u_{f(2)}$. Then $z_0, z_1, \ldots, z_p = u_{f(2)}, u_{f(2)-1}, \ldots, u_0$ is a path in $G_i[y_{\ell(1)}]$. Note that all these vertices are distinct, since $u_{f(2)-1}, \ldots, u_0 \notin y_{\ell(2)}$. Now the vertices z_0 and u_0 lie in y_k, but $u_{f(2)}$ does not, that is, $H[y_{\ell(1)}]$ can not be connected over the tree $G_i[y_{\ell(1)}]$, a contradiction.

Case 2: There is some $j \in \{1, 2, \ldots, s-2\}$ such that $y_{\ell(j)} \cap y_i, y_{\ell(j+1)} \cap y_i, y_{\ell(j+2)} \cap y_i$ form a triangle in F_i. This case can be led to a contradiction quite similiar as the first case.

Claim 2: $G_i[y_i]$ is a forest.

For otherwise, assume this graph contains some cycle $u_0, u_1, \ldots, u_t, u_0$. By (i), there must be some $p < i$ with $u_0, u_t \in y_p$. Applying Claim 1, y_p would contain all vertices of the cycle, a contradiction to (ii) with $J = \{p\}$.

Claim 3: $G_i[y_i]$ can be extended to some maximum spanning tree of $G_w(H[y_i]) = G_w(H)[y_i]$.

Let T be such a maximal spanning tree of this weighted graph having the maximum number of common edges with $G_i[y_i]$. Assume $G_i[y_i]$ is not a subgraph of T, then there is some edge uv in $G_i[y_i]$ but not in T. Let $u = u_0, u_1, \ldots, u_r = v$ be the $u-v$ path in T. Since $G_i[y_i]$ is a forest, some edge $u_s u_{s+1}$ is no edge of $G_i[y_i]$. By the properties of T mentioned above, there follows $w(u_s u_{s+1}) > w(uv)$. By (i), some old hyperedge h_k contains u and v. $H[y_i]$ is connected over the tree T, according to Theorem 4.2, so y_k contains the whole path $u = u_0, u_1, \ldots, u_r = v$. But $G_i[y_k]$ is not a maximum spanning tree of $G_w(H[y_k]) = G_w(H)[y_k]$, by applying Kruskal's argument [15] to $w(u_s u_{s+1}) > w(uv)$. So Theorem 4.2 implies that $H[y_k]$ is not connected over $G_i[y_k]$, a contradiction to (ii) in the i-admissiblity of G_i for $J = \{k\}$.

Claim 4: The graph G_{i+1} obtained from G_i by adding the edges necessary in Claim 3 is (i+1)-admissible.

(i) is quite obvious by our construction: all new edges have been drawn inside y_i.

Let now J' be some nonempty subset of $\{1, 2, \ldots, i-1\}$. Since G_i is i-admissible, $H[V_{J'}]$ is connected over the tree $G_i[V_{J'}]$, and this implies that $G_i[y_i \cap V_{J'}]$ is connected. So no edge of $G_{i+1} - G_i$ joins two vertices in $V_{J'}$. Thus $G_i[V_{J'}] = G_{i+1}[V_{J'}]$ and $G_i[y_i \cap V_{J'}] = G_{i+1}[y_i \cap V_{J'}]$ and (ii) is true for $i+1$ for all sets J of the form J' or $J' \cup \{i\}$. The case $J = \{i\}$ is obvious by (3). □.

A graph is called *locally chordal*, if the neighborhood of each vertex induces a chordal graph; or equivalently, if it has no wheel W_n for $n \geq 4$ as induced subgraph. A graph F is locally chordal if and only if $(\kappa(F))^*$ is a regionally subtree hypergraph. Consequently from Theorem 4.3 there follows:

Corollary 4.4 *Every locally chordal graph F is representable in some $\beta_1(S(\kappa(F)))$-cyclomatic graph.*

Theorem 4.3 is also useful for obtaining sufficient conditions for arbitrary graphs. Every graph F is the representative graph of several regionally subtree hypergraphs. An example is the hypergraph dual F^* of F. For every such hypergraph H, the graph F is representable in some $\beta_1(S(H))$-cyclomatic graph. We give the following example without proof:

Corollary 4.5 *Let the K_4-free graph F have n triangles such that every (not necessarily induced) wheel $W_m, m \geq 4$ contains at least one of these triangles. Then F is representable in some $\beta_1(S(\kappa(F))) + n$-cyclomatic graph.*

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