A complete solution to the packing problem with block size six and index five

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ABSTRACT. A.M. Assaf, A. Hartman and N. Shalaby determined in [1] the packing numbers $\sigma(v,6,5)$ for all integers $v \geq 6$ leaving six open cases of $v = 41$, 47, 53, 59, 62, and 71. In this paper, we deal with these open cases and thus complete the packing problem.

1. Introduction

Let v, k and λ be positive integers. A packing design with parameters v, k and λ, called a (v,k,λ)-packing, is a pair $(X, \mathcal{A})$ where X is a v-set (of points) and $\mathcal{A}$ is a collection of k-subsets of X (called blocks) such that every pair of distinct points of X occurs in at most λ blocks of $\mathcal{A}$. The packing number $\sigma(v,k,\lambda)$ is defined to be the maximum number of blocks in a (v,k,λ)-packing. The packing problem is to determine the value of $\sigma(v,k,\lambda)$. Definitions not given here can be found in [1,6,7].

Schoenheim [4] has shown that

$$\sigma(v,k,\lambda) \leq \lfloor v\lfloor \lambda(v-1)/(k-1)\rfloor/k\rfloor = \phi(v,k,\lambda) \tag{1.1}$$

where $\lfloor x\rfloor$ is the largest integer satisfying $\lfloor x\rfloor \leq x$.

Lower bounds on $\sigma(v,k,\lambda)$ are generally given by construction of (v,k,λ)-packings. A.M. Assaf, A. Hartman and N. Shalaby [1] discussed the $(v,6,5)$-packing problem. They showed the following.

Theorem 1.1. *The equality $\sigma(v,6,5) = \phi(v,6,5)$ holds for every integer $v \geq 6$ with the exception of $v = 8$ and the possible exception of $v \in \{41,47,53,59,62,71\}$. Moreover, $\sigma(8,6,5) = \phi(8,6,5) - 1$.*

In this paper, we deal with the six open cases shown in Theorem 1.1 and thus complete the packing problem.

2. Constructions

Let X be a finite set (of points). A group divisible design (GDD) of index λ is a triple $(X, \mathcal{G}, \mathcal{A})$, where

1. $\mathcal{G}$ is a collection of subsets of X (called groups) which partition X,
2. $\mathcal{A}$ is a collection of subsets of X (called blocks) such that a group and a block contain at most one common point, and
3. every pair of points from distinct groups occurs in exactly λ blocks of $\mathcal{A}$.

The group-type (or type) of a GDD is a listing of the group sizes using so-called "exponential" notation, that is, $1^i 2^j 3^k \ldots$ denotes i groups of size 1, j groups of size 2, etc. We say that a GDD is a (k, λ)-GDD if $|A| = k$ for every block $A \in \mathcal{A}$.

Three particular GDDs of which we will make use of need to be mentioned. A (k, λ)-GDD of type m^k is called a transversal design (TD), denoted by $\mathrm{TD}(k, \lambda; m)$. A (k, λ)-GDD of type 1^v is called a balanced incomplete block design (BIBD), and denoted by $\mathrm{B}(k, \lambda; v)$ and also by $(X, \mathcal{A})$. A (k, λ)-GDD of type $1^u w^1$ is referred to as an incomplete BIBD, denoted simply by $\mathrm{IB}(k, \lambda; u + w, w)$. The group of size w is thought of as a hole.

We are now in the position to give our constructions.

Lemma 2.1. *The equality $\sigma(71, 6, 5) = \phi(71, 6, 5)$ holds.*

Proof: As noted in [1], a $(v, 6, 5)$-packing with $\phi(v, 6, 5)$ blocks is essentially an $\mathrm{IB}(6, 5; v, 2)$ whenever $v \equiv 2 \pmod 3$. So we have a (6,5)-GDD of type $1^{12}2^1$ by taking $v = 14$ in Theorem 1.1. Give weight 5 to every point of such a GDD and employ Wilson's Fundamental Construction (see [5]) with the input design $\mathrm{TD}(6, 1; 5)$, which exists by [3]. This gives rise to a (6,5)-GDD of type $5^{12}10^1$. We then adjoin one infinite point to the resulting GDD and break up each group of size 5 together with the infinite point using a B(6,5;6). Filling in the group of size 10 together with the infinite point by a maximum (11,6,5)-packing from Theorem 1.1, we obtain a (71,6,5)-packing with $\phi(71, 6, 5)$ blocks. The conclusion then follows from (1.1).

Lemma 2.2. *The equality $\sigma(62, 6, 5) = \phi(62, 6, 5)$ holds.*

Proof: It is known [2] that both a B(6,1;66) and a B(6,4;61) exist. Removing one block from a B(6,1;66) yields an IB(6,1;66,6). Thus we can apply Construction 4.5 in [6] with $K = \{6\}$, $\lambda = 1$, $u = 60$, $m = e = 5$, and $q = 1$ to obtain an IB(6,5;62,2). This guarantees that a (62,6,5)-packing with $\phi(62, 6, 5)$ blocks exists. The conclusion then follows from (1.1).

Lemma 2.3. *The equality $\sigma(59, 6, 5) = \phi(59, 6, 5)$ holds.*

Proof: By (1.1), it is sufficient to construct a (59,6,5)-packing with $\phi(59,6,5)$ blocks. We proceed as follows.

First of all, we construct an IB(6,5;59,11) based on the set $Z_{48} \cup \{x_1, x_2, \ldots, x_{11}\}$ and consisting of the following 552 blocks:

x_1	j	$16+j$	$17+j$	$32+j$	$33+j$	$j = 0,1,2,\ldots,15$
x_1	j	$1+j$	$16+j$	$32+j$	$33+j$	$j = 0,1,2,\ldots,15$
x_1	j	$1+j$	$16+j$	$17+j$	$32+j$	$j = 0,1,2,\ldots,15$

0	3	5	24	27	29	mod 48 (orbit length 24)
0	2	5	13	23	x_2	mod 48
0	2	12	19	25	x_3	mod 48
0	3	20	29	35	x_4	mod 48
0	4	5	19	41	x_5	mod 48
0	4	7	13	18	x_6	mod 48
0	4	21	34	44	x_7	mod 48
0	5	11	12	20	x_8	mod 48
0	8	10	20	22	x_9	mod 48
0	11	19	20	41	x_{10}	mod 48
0	21	24	30	44	x_{11}	mod 48

Note that $\{x_1, x_2, \ldots, x_{11}\}$ is the hole.

Secondly, we fill the hole with a maximum (11,6,5)-packing with $\phi(11,6,5)$ blocks from Theorem 1.1.

Finally, it is readily checked that the above procedure provides the required packing and the result follows.

Lemma 2.4. *If* $v \in \{41, 47, 53\}$, *then* $\sigma(v,6,5) = \phi(v,6,5)$.

Proof: In view of (1.1), we can establish the result by constructing a $(v,6,5)$-packing with $\phi(v,6,5)$ blocks for each value of $v \in \{41, 47, 53\}$. These constructions are exhibited below.

For $v = 41$, the point set is $Z_{39} \cup \{x, y\}$ and the 273 blocks are

0	1	2	3	4	19	mod 39
0	1	9	17	24	29	mod 39
0	2	5	8	12	16	mod 39
0	4	9	14	19	25	mod 39
0	6	12	18	25	32	mod 39
0	2	11	24	28	x	mod 39
0	3	12	21	29	y	mod 39

For $v = 47$, the point set is $Z_{45} \cup \{x, y\}$ and the 360 blocks are

0	1	3	8	12	21	mod 45
0	1	3	16	24	27	mod 45
0	1	6	9	19	23	mod 45
0	1	6	31	33	41	mod 45
0	1	7	17	22	32	mod 45
0	2	14	28	30	39	mod 45
0	3	7	19	41	x	mod 45
0	6	17	26	33	y	mod 45

For $v = 53$, the point set is $Z_{51} \cup \{x, y\}$ and the 459 blocks are

0	1	3	7	25	39	mod 51
0	1	5	8	35	41	mod 51
0	1	6	14	21	23	mod 51
0	1	10	29	32	49	mod 51
0	1	11	15	43	46	mod 51
0	2	6	18	27	44	mod 51
0	4	17	22	30	41	mod 51
0	2	12	23	39	x	mod 51
0	5	16	23	31	y	mod 51

The foregoing can be summarized as follows.

Theorem 2.5. *If* $v \in \{41, 47, 53, 59, 62, 71\}$, *then* $\sigma(v, 6, 5) = \phi(v, 6, 5)$.

3. Conclusion

It has been shown in Theorems 1.1 and 2.5 that if $v \neq 8$ is an integer greater than six, then there is a maximum $(v, 6, 5)$-packing which contains $\lfloor v \lfloor 5(v-1)/5 \rfloor /6 \rfloor$ blocks. Thus the packing problem with block size six and index five has been solved completely. As a consequence of the present result, we can claim that an IB$(6, 5; v, 2)$ exists if and only if $v \geq 11$ and $v \equiv 2 \pmod 3$ (see Lemma 2.1 [1]); this result may be useful for other combinatorial designs.

References

[1] A.M. Assaf, A. Hartman, and N. Shalaby, Packing designs with block size 6 and index 5, *Discrete Math.* **103** (1992) 121–128.

[2] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975) 225–369.

[3] H.F. MacNeish, Euler squares, *Ann.Math.* **23** (1922) 221–227.

[4] J. Schoenheim, On maximal systems of k-tuples, *Stud. Sci. Math. Hungar.* **1** (1966) 363–368.

[5] R.M. Wilson, Constructions and uses of pairwise balanced designs, *Mathematical Centre Tracts* **55** (1974) 18–41.

[6] J. Yin, On the packing of pairs by quintuples with index 2, *Ars Combinatoria* **31** (1991) 287–301.

[7] J. Yin, Packing pairs by quintuples: the case v congruent to 0 (mod 4), *Applied Mathmatics—A Journal of Chinese Universities*, to appear.