The Separation Number

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Abstract. We obtain bounds for the separation number of a graph in terms of simpler parameters. With the aid of these bounds, we determine the separation number for various special graphs, in particular multiples of small graphs. This leads to concepts like robustness and asymptotic separation number.

1. Introduction and summary.

Let $G = (V(G), E(G))$ be a graph on n points and let f be a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$. Define

$$M(G,f) = \begin{cases} n \text{ if } E(G) = \emptyset, \\ \min\{|f(x) - f(y)| \,|\, xy \in E(G)\} \text{ otherwise.} \end{cases}$$

Also define

$$\mu(G) = \max_f M(G,f)$$

where the maximum is taken over all *labelings* of G, that is, over all bijections from $V(G)$ to $\{1, 2, \ldots, n\}$. We call $\mu(G)$ the *separation number* of G. Our purpose is to study μ.

In Section 2, we present bounds for the separation number. In Section 3, we consider special graphs. This leads to the concept of multiplicativity, to be investigated in Section 4. Section 5 is on the asymptotic separation number. We conclude, in Section 6, with two tables.

The problem of determining $\mu(G)$ for a given graph can be reformulated as follows. Consider the graph $H_{n,i}$ with point set $\{1, 2, \ldots, n\}$ in which p and q are adjacent if and only if $|p - q| \geq i$. In $H_{n,i}$, we choose the labeling $f(p) = p$ for all p. It is easily seen that $\mu(H_{n,i}) = i$. In fact, a graph G on n points has $\mu(G) \geq i$ if and only if G is isomorphic to a subgraph of $H_{n,i}$. Or, since $H_{n,i} \cong \overline{P_n^{i-1}}$:

$$(*) \qquad \mu(G) \geq i \Leftrightarrow G \text{ is isomorphic to a subgraph of } \overline{P_n^{i-1}}.$$

However, this is not very useful as a way to determine μ.

In [1], it is shown that determining μ is NP-complete as is the question whether $\mu = 1$.

The separation number has received much less attention than the bandwidth β. One may compare (*) with the analogue for β:

$$\beta(G) \leq i \Leftrightarrow G \text{ is isomorphic to a subgraph of } P_n^i.$$

Notation: In the sequel we denote by $n(G)$ or n the number of points of G, by $\delta(G)$ or δ the smallest degree, by $\Delta(G)$ or Δ the largest degree, and by $\alpha(G)$ or α the independence number. The *join* of G and H is denoted by $G + H$.

When the union of graphs is formed, the point sets are assumed to be pairwise disjoint.

Finally, **N** is the set of positive integers.

2. Bounds for the separation number.

We start with upper bounds for $\mu(G)$.

Proposition 2.1. *If $\delta > 0$, then $\mu \leq [\frac{n-\delta+1}{2}]$. If $\delta = 0$, then $\mu \leq [\frac{n+\omega_1}{2}]$, where ω_1 is the number of trivial components.*

Proof: Let $\delta > 0$. If n is odd, let $n = 2m-1$. If n is even, let $n = 2m$. Consider the point P with label m in an arbitrary labeling. Since the number of neighbours of P is at least δ, the upper bound follows from a counting-argument.

Let $\delta = 0$. Take an arbitrary labeling. Let x be the label assigned to a point with positive degree for which $|x - \frac{n}{2}|$ is minimum. Then at least $2|x - \frac{n}{2}|$ points have degree 0, that is $2|x - \frac{n}{2}| \leq \omega_1$. There are two cases: $x \leq \frac{n}{2}$ and $x > \frac{n}{2}$. Let $x \leq \frac{n}{2}$. Then $n - 2x \leq \omega_1$ or equivalently $n - x \leq \frac{n+\omega_1}{2}$ and, hence, $\mu \leq n - x \leq \frac{n+\omega_1}{2}$ (or else $\mu \leq x - 1 \leq \frac{n-2}{2}$). The case $x > \frac{n}{2}$ is similar. ∎

Proposition 2.2. $\mu \leq n - \Delta$.

Proof: Take an arbitrary labeling. Let v be a point with degree Δ and let L be its label. Let $L_1, \ldots, L_\Delta$ be the labels of the neighbours of v. If both $L_i < L$ and $L_j > L$ occur, then $\mu \leq [\frac{n-\Delta+1}{2}] \leq n - \Delta$. If all $L_i < L$ or all $L_i > L$, then also $\mu \leq n - \Delta$. ∎

Proposition 2.3. *If G contains K_q $(q \geq 2)$ as a subgraph, then $\mu \leq \left[\frac{n-1}{q-1}\right]$.*

Proof: Let the labels of the points of a K_q be $L_1, \ldots, L_q$ with $L_1 < \ldots < L_q$. If a value x for μ is to be achieved, then $L_1 \geq 1, L_2 \geq 1 + x, L_3 \geq 1 + 2x, \ldots, L_q \geq 1 + (q-1)x$. But $L_q \leq n$, hence, $1 + (q-1)x \leq n$ or $x \leq \left[\frac{n-1}{q-1}\right]$. ∎

Proposition 2.4. *If each point of G belongs to some K_q (q fixed), then $\mu \leq \left[\frac{n}{q}\right]$.*

Proof: Suppose the label $L_1 = \left[\frac{n}{q}\right] + 1$ is assigned to v_1 and let $v_2, \ldots, v_q$ be in the same K_q as v_1. If some v_j $(j = 2, \ldots, q)$ has a label lower than L_1, we are through. Let v_j have label L_j ($j = 2, \ldots, q$) with $L_1 < L_2 < \ldots < L_q$. If μ were at least $\left[\frac{n}{q}\right] + 1$, then $L_j \geq j\left(\left[\frac{n}{q}\right] + 1\right)$ for $j = 1, \ldots, q$. In particular $L_q \geq q\left[\frac{n}{q}\right] + q$. But $L_q \leq n$, and we have a contradiction. Hence $\mu \leq \left[\frac{n}{q}\right]$. ∎

Proposition 2.5. $\mu \leq \alpha$.

Proof: If $E(G) = \emptyset$, then $\mu = n$. If $E(G) \neq \emptyset$, then $\alpha \leq n - 1$, so the numbers $1, \ldots, \alpha + 1$ are labels. Any assignment of these labels is certain to choose adjacent points. Hence, $\mu \leq \alpha$. ∎

The following "meta-result" implies that, in a certain sense, none of the Propositions 2.1 to 2.5 is superfluous.

Proposition 2.6. *For each $i \in \{1, 2, 3, 4, 5\}$ there are infinitely many graphs for which Proposition 2.i yields a smaller upper bound than the other four.*

Proof: Let $G_1 = C_{2m}$, $G_2 = K_{2m,1}$, $G_3 = K_1 \cup K_q$ ($q \geq 2$), $G_4 = P^2_{6m-2}$, $G_5 = K_q \cup (q - 1)\ K_{q+1}$ ($q \geq 3$). The table below gives the values of the upper bound $2.j$ for the graph G_i.

	2.1	2.2	2.3	2.4	2.5
G_1	$m-1$	$2m-2$	$2m-1$	m	m
G_2	m	1	$2m$	m	$2m$
G_3	$1+\left[\frac{q}{2}\right]$	2	1	$q+1$	2
G_4	$3m-2$	$6m-6$	$3m-2$	$2m-1$	$2m$
G_5	$\left[\frac{q^2+1}{2}\right]$	q^2-1	$q+1$	$q+1$	q

∎

Next we mention a lower bound.

Proposition 2.7. $\mu(pG) \geq p\mu(G)$.

Proof: Let $V(G) = \{v_1, \ldots, v_q\}$ and suppose the labeling f achieves the value $\mu(G)$. Let the points of pG be called (i, v_j) with $i = 1, \ldots, p$ and $j = 1, \ldots, q$ in the obvious manner. Now assign the label $i + p(f_j - 1)$ to (i, v_j). This achieves the value $p\mu(G)$ in pG. ∎

In the sequel, lower bounds for μ will usually be obtained by (ad hoc) constructions.

Proposition 2.8. $\mu(G) \geq 2$ *if and only if $\overline{G}$ has a Hamilton path.*

Proof: Let $\mu(G) \geq 2$. For $i = 1, \ldots, n - 1$, the points with labels i and $i + 1$ are non-adjacent in an optimal labeling of G, hence, they are adjacent in $\overline{G}$, so $(1, 2, \ldots, n)$ is a Hamilton path $\overline{G}$.

Conversely, take a Hamilton path in $\overline{G}$, assign labels $1, 2, \ldots, n$ along the path, and a labeling with absolute differences ≥ 2 between neighbours in G is obtained. ∎

Corollary 2.9. $\mu(G + \dot{H}) = 1$.

Proof: The result follows at once from Proposition 2.8, since a join has a disconnected complement. Second proof: suppose without loss of generality that label 1 is assigned to a point of G. Let x be the smallest label assigned to a point of H. Then $x - 1$ is a label of a point in G, etc. ∎

3. Special graphs.

In this section we determine the separation number for some special graphs, *viz.* for certain unions of well-known graphs.

Proposition 3.1. *Let G be the union of complete graphs G_i $(i = 1, 2, \ldots, m)$ where $G_i \cong K_{q_i}$ with $q_1 \geq q_2 \geq \cdots \geq q_m$, $\sum_1^m q_i = n$. Let h be the maximum index with $q_1 = \ldots = q_h$. If $E(G) \neq \emptyset$, then*

$$\mu(G) = \left[\frac{n-h}{q_1 - 1}\right].$$

Proof: Let $q = q_1$. Suppose $\mu(G) \geq u = \left[\frac{n-h}{q-1}\right] + 1$. There exists an $i \in \{1, 2, \ldots, h\}$ such that the smallest label in G_i is at least h. The other labels in this G_i are at least $h + u, h + 2u, \ldots, h + (q-1)u$. The last-mentioned number can be written as $h + q - 1 + (q-1)\left[\frac{n-h}{q-1}\right]$ and since $\left[\frac{A}{B}\right] \geq \frac{A}{B} - \frac{B-1}{B}$ for all natural numbers A and B, we have $h + q - 1 + (q-1)\left[\frac{n-h}{q-1}\right] \geq n + 1$, which is a contradiction. Hence, $\mu(G) \leq \left[\frac{n-h}{q-1}\right]$.

To see that the value $v = \left[\frac{n-h}{q-1}\right]$ can be achieved, consider the following pseudo-algol statements.

(1) $L := 1$;
(2) *for* $j := 1$ *until* m *do*
(3) *for* $i := 1$ *until* q_j *do*
(4) *begin* assign label L to an unlabelled point of G_j;
(5) $L := L + v$;
(6) *if* $L > n$ *then* $L :=$ the smallest unassigned label
(7) *end*.

It is trivial that each point of G has a label after the execution of these statements. It is easily verified that the result is a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$. We now show that this labeling achieves the value v, except in the case $q = 3$.

When no "overflow" occurs in line 5, the difference between the last label and the new one is simply v. When an overflow occurs, the last label will be at least $n - v + 1$, and the new label will be at most v. The difference $n - 2v + 1$ is $\geq v$

if and only if $v \leq \frac{n+1}{3}$, and since $v = \left[\frac{n-h}{q-1}\right]$, this inequality is certainly satisfied if $q \geq 4$.

The case $q = 2$ causes no problems: after the K_2's have been labelled, only K_1's remain, if anything.

Now let $q = 3$; hence, $v = \left[\frac{n-h}{2}\right]$. If $n - h$ is even, then the unassigned labels after the labeling of the K_3's form 2 sequences of consecutive integers, and there is no problem. But if $n - h$ is odd, the number n will be an unassigned label after the K_3's have been labelled, which will cause a problem. (See the example below). The solution is simple: the label n should be assigned to some K_1. It is easy to see that now the value v is achieved. ∎

Example: Take the following sequence of q's: $3,3,2,2,2,1$. Then $n = 13$, $q = 3$, $h = 2$, $v = 5$. The labels are applied as follows (according to the original algorithm): $1,6,11$ to some K_3; $2,7,12$ to the other K_3; 3 and 8 to a K_2; 13 and 4 to another K_2; 9 and 5 (with difference 4 only) to the third K_2; and 10 to the K_1.

Corollary 3.2. $\mu(pK_q) = p \ (q \geq 2)$.

Proposition 3.3.

(a) $\mu(P_n) = \left[\frac{n}{2}\right] \ (n \geq 2)$.

(b) *Let G be the union of paths P_i with $i = q_j > 0$ ($j = 1, \ldots, m$) and $\sum_1^m q_i = n$. Then $\mu(G) = \left[\frac{n}{2}\right]$.*

Proof:

(a) Let the points of P_n be $v_1, \ldots, v_n$ in the obvious order. Assign label i to v_{2i} and label $i + \left[\frac{n}{2}\right]$ to v_{2i-1}, for all possible i. All absolute differences are $\geq \left[\frac{n}{2}\right]$, hence, $\mu \geq \left[\frac{n}{2}\right]$. The converse inequality follows from Proposition 2.1.

(b) $\mu \geq \left[\frac{n}{2}\right]$, since G is a spanning subgraph of P_n. The other half follows from Proposition 2.1.

∎

Proposition 3.4.

(a) $\mu(C_n) = \left[\frac{n-1}{2}\right]$.

(b) *If q is even, then $\mu(kC_q) = \frac{kq}{2} - 1$.*

(c) *If q is odd, then $\mu(kC_q) = k\frac{q-1}{2}$.*

Proof:

(a) Analogous to the proof of Proposition 3.3(a).

(b) Proposition 2.1 gives $\mu \leq \left[\frac{kq-1}{2}\right] = \frac{kq}{2} - 1$. Let $q = 2m$. In the first copy of C_q we assign labels alternatingly from $\{1, 2, \ldots, m\}$ and from $\{km +$

$1, km+2, \ldots, km+m\}$, in the natural order. In the other copies of C_q, these values are increased by an appropriate multiple of m. This gives $\mu \geq km - 1 = \frac{kq}{2} - 1$.

(c) Let $q = 2m+1$. Let the labels in the ith C_q be $L_i^{(1)} < L_i^{(2)} < \ldots < L_i^{(2m+1)}$. Among any $(m+1)$-tuple of labels in the ith C_q, there will be at least 2 labels assigned to adjacent points. We apply this fact to the $(m+1)$-tuple $L_i^{(m+1)}, \ldots, L_i^{(2m+1)}$. We infer that $\mu \leq L_i^{(2m+1)} - L_i^{(m+1)}$ for $i = 1, \ldots, k$. Now number the C_q's such that the median labels are in increasing order: $L_1^{(m+1)} < L_2^{(m+1)} < \ldots < L_k^{(m+1)}$. The number of labels less than $L_k^{(m+1)}$ is at least $km + k - 1$. Hence, $L_k^{(m+1)} \geq km + k$. Also $L_k^{(2m+1)} \leq kq = 2km + k$. Hence $\mu \leq km = k\frac{q-1}{2}$. The other half of part (c) follows from part (a) and Proposition 2.7.

∎

The results of this section suggest the following definition.

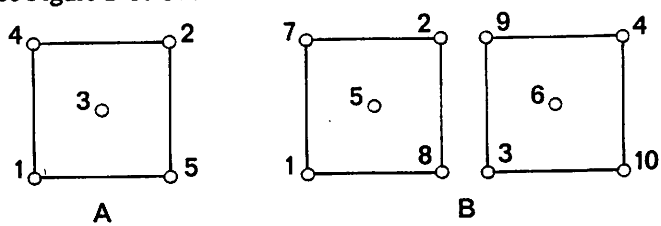
Definition: The graph G is *multiplicative* if $\mu(kG) = k\mu(G)$ for all $k \in \mathbb{N}$.

We have seen that K_n, P_{2n} and C_{2n+1} are multiplicative. It will be seen later that all graphs $H_{n,i}$ are multiplicative, too. All these graphs have a unique optimal labeling, so the question arises whether that is what makes these graphs multiplicative. Let us first give a precise definition.

Definition: An optimal labeling f of a graph G is *unique* if for each optimal labeling g of G, $g^{-1}f$ is an automorphism of G. A graph is *uolic* if it has a unique optimal labeling.

The following example shows that there exist uolic graphs that are not multiplicative.

Example: Take $G = K_1 \cup C_4$. It is easily seen that $\mu(G) = 2$ and that G is uolic; its optimal labeling is given in Figure A below. The graph $2G$ has a labeling with $M = 5$; see Figure B below.



Hence, $\mu(2G) \geq 5 > 2\mu(G)$ and G is not multiplicative.

Still, uniqueness of the optimal labeling seems to be an interesting property that deserves further investigation. We remark that in the example $\mu(2G) = 5$ and that $2G$ is uolic!

In the next section we present two sufficient conditions for multiplicativity.

4. Multiplicativity and robustness.

The proof of Proposition 3.4c can be generalized in two seemingly different directions. We start with the simplest possibility.

Proposition 4.1. *If $\alpha(G) = \mu(G)$, then G is multiplicative.*

Proof: Let $G_1, \ldots, G_k$ be copies of G, let $n = |V(G)|$ and let $L_i^{(1)} < \ldots < L_i^{(n)}$ be the labels in G_i $(i = 1, \ldots, k)$. For each $(\alpha + 1)$-tuple of labels in G_i there exist two "adjacent labels". Take the $(\alpha+1)$-tuple $L_i^{(1)}, \ldots, L_i^{(\alpha+1)}$. We conclude that

$$(*) \qquad \mu(kG) \leq L_i^{(\alpha+1)} - L_i^{(1)} \text{ for } i = 1, \ldots, k.$$

Choose the numbering of the components such that $L_1^{(\alpha+1)} < L_2^{(\alpha+1)} < \ldots < L_k^{(\alpha+1)}$. In (*) we now choose $i = 1$:

$$\mu(kG) \leq L_1^{(\alpha+1)} - L_1^{(1)} \leq L_1^{(\alpha+1)} - 1.$$

Since there are $(k-1)(n-\alpha) + (n-\alpha-1)$ labels known to be larger than $L_1^{(\alpha+1)}$, we conclude that $L_1^{(\alpha+1)} \leq k\alpha + 1$, and, hence, $\mu(kG) \leq k\alpha(G) = k\mu(G)$. From Proposition 2.6 we know that $\mu(kG) \geq k\mu(G)$, so G is multiplicative. ∎

Alternatively, the proof of Proposition 3.4c suggests the following definitions.

Definition: A *pseudo-labeling* of a graph G is an injection from $V(G)$ to $\mathrm{N} = \{1, 2, \ldots\}$. Let

$$M(G, f) = \begin{cases} \infty \text{ if } E(G) = \emptyset, \\ \min\{|f(x) - f(y)| \, |xy \in E(G)\} \text{ otherwise,} \end{cases}$$

where f is a pseudo-labeling. A pseudo-labeling f^* is *optimal* if $M(G, f^*) \geq M(G, f)$ for each pseudo-labeling f the values of which form a permutation of the values of f^*.

Definition: A graph is robust if for each optimal labeling f and for each increasing function ϕ: $\mathrm{N} \to \mathrm{N}$, the composition ϕf is an optimal pseudo-labeling.

We shall see presently that robustness is sufficient for multiplicativity. However, the technique we used to prove Proposition 3.4c is not quite strong enough to obtain that result. The proof can be given indirectly.

Proposition 4.2. *If G is robust, then $\alpha(G) = \mu(G)$, and conversely.*

Before we can give the proof, we require a definition.

Definition: Let G be a graph with $E(G) \neq \emptyset$, let f be a pseudo-labeling of G, and let M be the minimum of $|f(x) - f(y)|$ over all $xy \in E(G)$. Then a *bottleneck-pair* (with respect to f) is a pair $\{u, v\}$ of labels of adjacent points with $|u - v| = M$.

Note that if the above mentioned pseudo-labeling is an increasing transformation of an optimal labeling, there may be bottleneck-pairs $\{\phi i, \phi j\}$ with $|i-j| > \mu$. An example is given by $G = P_2 \cup P_3$, shown below with an optimal labeling; $\mu = 2$.

Now take $\phi(i) = i$ for $i = 1, 2, 3, 4$ and $\phi(5) = 7$. The pair $\{1, 4\}$ becomes a bottleneck-pair, but the ϕ-originals of 1 and 4 (*viz.* 1 and 4) have absolute difference 3.

Proof of Proposition 4.2: We prove the contraposition.

Suppose $\alpha \neq \mu$. Then $\alpha > \mu$, hence, there exists an independent set of $\mu + 1$ points in G. Take an arbitrary optimal labeling f of G, and let $\{i, j\}$ be a bottleneck-pair w.r.t. f. Now choose an increasing function ϕ: $\mathbf{N} \to \mathbf{N}$ such that $\{\phi i, \phi j\}$ becomes the *unique* bottleneck-pair w.r.t. ϕf and such that $|\phi p - \phi q| > |\phi i - \phi j|$ for all labels p and q unless both p and q are in $\{i, i+1, \ldots, j\}$. (Such a function ϕ is easy to give. Assume without loss of generality that $i < j$, so that $j = i + \mu$. Now take, for example,

$$\phi(k) = \begin{cases} (\mu+1)k \text{ for } k = 1, \ldots, i-1, \\ (\mu+1)i + k - i \text{ for } k = i, \ldots, j, \\ (\mu+1)(k-\mu) + \mu \text{ for } k = j+1, \ldots, n. \end{cases}$$

Our claim is easily verified now). Since there exists an independent set of $\mu + 1$ points, the pair $\{\phi i, \phi j\}$ and all pairs $\{\phi p, \phi q\}$ with $i \leq p < q \leq j$ can be avoided as labels of adjacent points in a suitable permutation of the labels. Hence G is not robust.

Conversely, suppose G is not robust. Then there exists an optimal labeling f and an increasing ϕ such that ϕf is a non-optimal pseudo-labeling. In fact, ϕ can even be chosen such that ϕf is non-optimal *and* such that some $\{\phi i, \phi j\}$, where without loss of generality $i < j$, becomes the unique bottleneck-pair w.r.t. ϕf. (Use a perturbation technique.) Since the pseudo-labeling is non-optimal, the pair $\{\phi i, \phi j\}$ can be eliminated by a suitable permutation of the labels. This means that $\{\phi i, \phi j\}$ and all pairs $\{p, q\}$ with $\phi i \leq p < q \leq \phi j$ can be avoided as labels of adjacent points. Hence, the points that have, after the permutation, the labels $\phi i, \phi(i+1), \ldots, \phi j$ form an independent set. From $j - i \geq \mu$ we then have $\alpha \geq j - i + 1 \geq \mu + 1$, so $\alpha \neq \mu$. ∎

We now present two useful applications of Proposition 4.2.

Corollary 4.3. *If G is robust and $\mu(G+e) = \mu(G)$, where e is a line, then $G+e$ is robust.*

Proof: $\mu(G+e) \leq \alpha(G+e)$. On the other hand, $\mu(G+e) = \mu(G) = \alpha(G) \geq \alpha(G+e)$, hence $\mu(G+e) = \alpha(G+e)$ and $G+e$ is robust. ∎

Corollary 4.4. *If G is not robust and $m \in \mathbb{N}$, then $G \cup K_m$ is not robust.*

Proof: Take an optimal labeling f of $G \cup K_m$. If $\mu(G \cup K_m) \geq 2 + \mu(G)$, then delete the points of K_m from $G \cup K_m$ and apply a non-increasing transformation to the remainder, thus, obtaining a labeling g of G. If $\{p, q\}$ is a bottleneck-pair w.r.t. f, then the set $\{p, p+1, \ldots, q\}$ contains at most one point of K_m. This implies that g achieves a value $\geq 1 + \mu(G)$ for the minimum of the absolute differences of adjacent labels. Hence $\mu(G \cup K_m) \leq 1 + \mu(G) < 1 + \alpha(G) = \alpha(G \cup K_m)$, hence $G \cup K_m$ is not robust. ∎

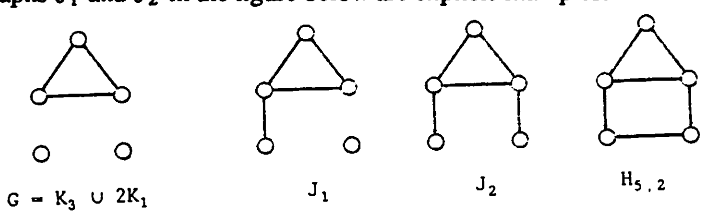
Multiples of multiplicative graphs are trivially multiplicative. We now prove the analogous result for robustness.

Proposition 4.3. *If G is robust, then mG is robust for all $m \in \mathbb{N}$.*

Proof: Let G be robust. Then G is multiplicative, hence, $\mu(mG) = m\mu(G) = m\alpha(G) = \alpha(mG)$, hence, mG is robust. ∎

Robustness is not a necessary condition for multiplicativity, as is shown by the following examples. If G is a union of complete graphs, then $\mu(G) = \left\lceil \frac{n-h}{q_1-1} \right\rceil$, where we use the result and the notation of Proposition 3.1. Since mG is also a union of complete graphs, we have $\mu(mG) = \left\lceil \frac{mn-mh}{q_1-1} \right\rceil$. Now suppose $q_1 - 1$ divides $n - h$. Then $\mu(mG) = m\mu(G)$ for all m and G is multiplicative. An explicit example with $\alpha \neq \mu$ is $K_3 \cup 2K_1$.

A second type of example can be constructed as follows. Let G be a non-robust multiplicative graph on n points with $\mu(G) = i$. Then G is a proper spanning subgraph of $H_{n,i}$. It is easily seen that all graphs J with $G \subset J \subset H_{n,i}$ are multiplicative. Hence, when $\alpha(J) > \alpha(H_{n,i})$, J is non-robust but multiplicative. The graphs J_1 and J_2 in the figure below are explicit examples.



The third type is related to the second. Again, let G be a non-robust multiplicative graph on n points with $\mu(G) = i$. Trivially, all graphs mG are multiplicative, and, hence, so are all graphs J with $mG \subset J \subset mH_{n,i}$. To obtain a non-robust example, take $G = K_3 \cup 2K_1$, $m = 3$, and $J = G \cup J_1 \cup J_2$ where J_1 and J_2 are the same as in the previous example.

5. The asymptotic separation number.

For all $m \in \mathbb{N}$ and for all graphs G we have $m\mu(G) \leq \mu(mG) \leq m\alpha(G)$ from

Proposition 2.7 and Proposition 2.5, respectively. Hence,

$$\mu(G) \leq \liminf_{m\to\infty} \frac{\mu(mG)}{m} \leq \limsup_{m\to\infty} \frac{\mu(mG)}{m} \leq \alpha(G).$$

So the following definition seems natural.

Definition: The *asymptotic separation number* of a graph G, denoted by $\tilde{\mu}(G)$ is $\lim_{m\to\infty} \frac{\mu(mG)}{m}$ if it exists.

The existence of $\tilde{\mu}$ easily follows from the following result.

Proposition 5.1. *$\mu(mG)$ is non-decreasing in m for all G.*

Proof: Let $|V(G)| = n$. Choose an optimal labeling of mG. Fix a copy G^0 of G in mG. Denote by G^* the $(m+1)$-st (new) copy of G and let f be an isomorphism from G^0 to G^*. Now if point i of G^0 has the label L_i, we assign the number $i + \frac{1}{2}$ to the point $f(i)$ of G^* $(i = 1, 2, \ldots, n)$. Finally, we replace the numbers $1, 2, \ldots, mn, L_1 + \frac{1}{2}, \ldots, L_n + \frac{1}{2}$ by the numbers $1, 2, \ldots, mn + n$, preserving the relative order. We conclude that $\mu((m+1)G) \geq \mu(mG)$. ∎

Proposition 5.2. *$\tilde{\mu}(G)$ exists for all G and is equal to* $\sup_m \frac{\mu(mG)}{m}$.

Proof: It is clear that $\tilde{\mu}(G)$ if it exists, cannot be smaller than $s = \sup_m \frac{\mu(mG)}{m}$. Also $s \leq \alpha(G) < \infty$.

Let ϵ be positive. Choose m_0 so large that $s - \frac{\mu(m_0G)}{m_0} < \epsilon$. Then $s - \frac{\mu(mG)}{m} < \epsilon$ for all multiples m of m_0. If m is not a multiple of m_0, say $m = qm_0 + r$ with $0 < r < m_0$ then applying Proposition 5.1:

$$\frac{\mu(mG)}{m} \geq \frac{\mu(qm_0G)}{m} = \frac{\mu(qm_0G)}{qm_0} \cdot \frac{qm_0}{m} \geq \frac{\mu(m_0G)}{m_0} \cdot \frac{m-r}{m}.$$

Now

$$\frac{\mu(m_0G)}{m_0} > s - \epsilon, \text{ and } \frac{m-r}{m} > \frac{m-m_0}{m} > 1 - \frac{1}{q}.$$

By choosing q sufficiently large we find that $\frac{\mu(mG)}{m} > s - 2\epsilon$ for all m sufficiently large. ∎

Proposition 5.3. *G is multiplicative if and only if $\tilde{\mu} = \mu$.*

Proof: If G is not multiplicative then $\mu(m_0G) > m_0\mu(G)$ for some m_0, hence, $\tilde{\mu}(G) = \sup_m \frac{\mu(mG)}{m} \geq \frac{\mu(m_0G)}{m_0} > \mu(G)$. The other half is trivial. ∎

Proposition 5.4. *If G is bipartite, then $\tilde{\mu}(G) = \frac{|V|+\omega_1}{2}$, where ω_1 is the number of trivial components of G.*

Proof: Choose a bipartition S, T of the non-isolated points of G. Take m copies of G: $G_1, \ldots, G_m$ with corresponding point-classes S_i, T_i in G_i $(i = 1, \ldots, m)$. (Hence, $|S_1| = \ldots = |S_m|$, $|T_1| = \ldots = |T_m|$, and the only lines in G_i are lines between S_i and T_i).

Starting with label 1, the labels are assigned in increasing order to the points of S_1, then T_2, then $S_3, \ldots$, ending with S_m or T_m, depending on the parity of m. Let A be the largest label assigned so far. The next $m\omega_1$ labels, $A+1, A+2, \ldots, A+m\omega_1$, are assigned to the trivial components of mG. Finally, the labels $A+m\omega_1+1, A+m\omega_1+2, \ldots$ are assigned to the points of $T_1, S_2, T_3, S_4, \ldots$ until all points have a label.

The minimum absolute value between labels of adjacent points is now $A + m\omega_1 + 1 - |S_1|$. Since $A = |S_1| + |T_2| + \ldots + |S_m|$ or $A = |S_1| + |T_2| + \ldots + |T_m|$, a lower bound for A is $(|V| - \omega_1)\left(\frac{m}{2} - 1\right)$. It follows that $\tilde{\mu}(G) = \lim_{m\to\infty} \frac{\mu(mG)}{m} \geq \frac{|V|+\omega_1}{2}$. The reverse inequality follows at once from Proposition 2.1. ∎

The case $\omega_1 = 0$ can be generalized to k-partite graphs, as follows.

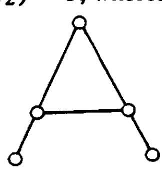
Proposition 5.5. *Let $\chi(G) = c$, $|V(G)| = n$. If each point of G belongs to some K_c, then $\tilde{\mu}(G) = \frac{n}{c}$.*

Proof: Colour the points of G with c colours; let $n_1, n_2, \ldots, n_c$ be the number of points in the respective colour classes. Now take m copies of G: $G_1, \ldots, G_m$, and colour each G_i in the same way as G. As in the proof of Proposition 5.4, we assign the labels in increasing order, to the points of: the first colour class of G_1, the second colour class of $G_2, \ldots$, the c-th colour class of G_c, the first colour class of G_{c+1}, the second colour class of G_{c+2}, etc., until the points of 1 colour class of G_m have been coloured. Then we continue with the second colour class of G_1, etc. But for "border effects" we find $\mu(mG) \geq \frac{mn}{c}$, and, hence, $\tilde{\mu}(G) \geq \frac{n}{c}$. The reverse inequality follows from Proposition 2.4. ∎

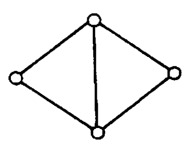
Corollary 5.6. *Let $\chi(G) = c$, $|V(G)| = n$. Then $\tilde{\mu}(G) \geq \frac{n}{c}$.*

Proof: This has in fact been proved above. ∎

Example: For the graph G_1 we have $\mu = 2$, $\frac{n}{\chi} = \frac{5}{3}$, hence, the "old" lower bound μ is better. But $\mu(G_2) = 1$, whereas $\frac{n}{\chi} = \frac{4}{3} > 1$.



G_1



G_2

Proposition 5.7. *Let G be a union of complete graphs, with the same notation as in Proposition 3.1. Then $\tilde{\mu}(G) = \frac{n-h}{q_1-1}$.*

Proof: The result is trivial since a multiple of such a graph is again a union of complete graphs. ∎

So far the $\tilde{\mu}$-values for some special graphs. We now apply Proposition 5.5 to prove a useful result.

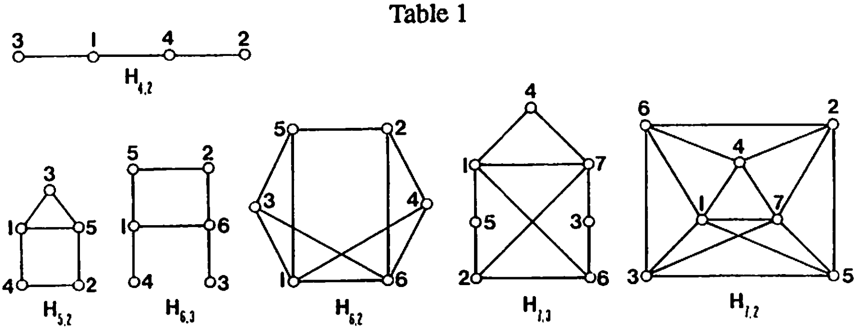
Proposition 5.8. *If $G \not\cong K_n$ and $\mu(G) = 1$, then G is not multiplicative.*

Proof: First let $G = K_n - e$. According to Proposition 5.5, $\tilde{\mu}(G) = \frac{n}{n-1}$. Hence, $\mu(mG) > m$ for m sufficiently large. Now let $G \not\cong K_n$ be an arbitrary graph on n points with $\mu(G) = 1$. Then G is a subgraph of $K_n - e$ and, hence, $\mu(mG) > m$ for the above m. So G is not multiplicative. ∎

6. Tables.

In Table 1 we present drawings of some graphs $H_{n,i}$. For $i = 1$, $H_{n,i} \cong K_n$ and for $2i > n > 1$, $H_{n,i} \cong H_{n-1,i-1} \cup K_1$. The remaining cases with $n \leq 7$ are shown below.

Table 1



In Table 2 we have collected some information on the 4- and 5-point-graphs. The meaning of the column headings is as follows: n is the number of points, m is the number of lines, # is the number that Harary assigns to the graph (see table on p. 215 ff. of [2]), α is the independence number, μ is the separation number, R = robust, U = uolic, M = multiplicative (and we have indicated by a digit whether the graph in question has the property (1) or not (0)). In the last column, without heading, H means: the graph is of the form $H_{n,i}$, and J means: the graph is a join.

Remark: The graph G (with n points and separation number μ) is uolic if and only if there exists a unique subgraph of $H_{n,\mu}$ isomorphic to G. This provides a quick way to see whether G is uolic (if $n \leq 5$).

Table 2

n	m	#	α	μ	R	U	M	
4	0	-	4	4	1	1	1	H
	1	-	3	3	1	1	1	H
	2	1	3	2	0	0	0	
		2	2	2	1	1	1	
	3	1	2	1	0	0	0	
		2	2	2	1	1	1	H
		3	3	1	0	0	0	J
	4	1	2	1	0	0	0	J
		2	2	1	0	0	0	J
	5	-	2	1	0	0	0	J
	6	-	1	1	1	1	1	H
5	0	-	5	5	1	1	1	H
	1	-	4	4	1	1	1	H
	2	1	4	3	0	0	0	
		2	3	3	1	1	1	
	3	1	3	3	1	1	1	H
		2	3	2	0	1	1	
		3	4	2	0	0	0	
		4	3	2	0	0	0	
	4	1	3	2	0	1	0	
		2	3	2	0	0	1	
		3	4	1	0	0	0	J
		4	3	2	0	0	0	
		5	2	2	1	1	1	
		6	3	2	0	0	0	
	5	1	3	1	0	0	0	
		2	3	2	0	1	1	
		3	3	2	0	0	0	
		4	2	2	1	0	1	
		5	3	1	0	0	0	J
		6	2	2	1	1	1	
	6	1	2	1	0	0	0	J
		2	2	1	0	0	0	J
		3	2	1	0	0	0	
		4	2	2	1	1	1	H
		5	2	1	0	0	0	J
		6	2	1	0	0	0	
	7	1	2	1	0	0	0	J
		2	3	1	0	0	0	J
		3	2	1	0	0	0	J
		4	2	1	0	0	0	J
	8	1	2	1	0	0	0	J
		2	2	1	0	0	0	J
	9	-	2	1	0	0	0	J
	10	-	1	1	1	1	1	H

References

1. J. Leung, O. Vornberger, and J. Witthoff, *On some variants of the bandwidth minimization problem*, SIAM Journal on Computing 13, 650–666.
2. F. Harary, "Graph Theory", Addison Wesley, Reading MA., 1969.