ON 4-(10,5,6m) DESIGNS WITH REPEATED BLOCKS*

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Abstract

Recently, there has been substantial interest in the problem of spectrum of possible support sizes of different families of BIB designs. In this paper, we first prove some theorems concerning the spectrum of any t-design with $v = 2k$ and $k = t+1$, and then we study the spectrum of the case 4-(10,5,6m) in more detail.

1. INTRODUCTION

A $t-(v,k,\lambda)$ design (or a t-design) is a collection of k-subsets (blocks) of a v-set V such that every t-subset of V appears in exactly λ blocks. (It is not required that the blocks to be distinct.)

Necessary conditions for existence of a $t-(v,k,\lambda)$ design are known to be that

$$\lambda_i = \lambda \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i} \text{ is an integer}, \quad 0 \le i \le t.$$

Following the usual notation in the literature, we put $b = \lambda_0$ and $r = \lambda_1$.

The set of all distinct blocks of a t-design is called the ***support*** of the design and its cardinality is denoted by b^*.

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A (v,k,t) *trade* of volume s consists of two disjoint collections of blocks T_1 and T_2 each consisting of s blocks, such that for every t-subset of V, the number of blocks containing this subset is the same in T_1 as in T_2. A (v,k,t) trade is denoted by (T_1,T_2).

In a (v,k,t) trade both collections of blocks must cover the same set of elements which is called the *foundation* of the trade. Hwang [6] has shown that the minimum foundation size of a (v,k,t) trade is $k+t+1$ and for $v > k+t+1$, the minimum volume of a (v,k,t) trade is 2^t. The trades with foundation size $k+t+1$ and volume 2^t are called *minimal trades*.

The frequency of a k-subset B of V in a design D is denoted by $\#_D(B)$; and if there is no ambiguity, we simply write $\#(B)$. If B is not a block of D then $\#(B)=0$. Similarly, we denote the frequency of B in a trade (T_1,T_2) by $\#(B)$, but if $B \in T_2$ then $\#(B)$ has a minus sign.

A $t-(v,k,\lambda)$ design ((v,k,t) trade) with $v=2k$ will be called *self-complementary* if for every k-subset B, $\#(B)=\#(V-B)$. A t-design with $v=2k$ is called *quasi-complementary* if for every block B, $\#(B)+\#(V-B)$ is constant; and a (v,k,t) trade with $v=2k$ is called *quasi-complementary* if $\#(B)=-\#(V-B)$.

A t-design is called *simple* if for every k-subset B, $\#(B)$ =0 or 1; and is called *trivial* if for all k-subsets B, $\#(B)=1$.

In this paper, following Hedayat and Li [5], we adopt vector representation of designs and trades. In this connection we first order all the k-subsets of V lexicographically and we let f_i to be the frequency of the i^{th} k-subset in the design. Therefore, every t-design is identified with a $\binom{v}{k}$-dimensional vector $(f_1, f_2, \cdots, f_{\binom{v}{k}})$. Likewise, a (v,k,t) trade is represented by a $\binom{v}{k}$-vector in which the frequencies of blocks of T_2 appears with minus signs.

It is known [3] that the (v,k,t) trades form a $\mathbb{Z}$-module with dimension $\binom{v}{k}-\binom{v}{t}$. Graham, Li and Li [2], obtained a basis for this module in terms of polynomials. The elements of the basis they produce are minimal trades. They considered the following polynomial

$$\varphi(x_1,\cdots,x_v)=(x_1-x_2)(x_3-x_4)\cdots(x_{2t+1}-x_{2t+2})x_{2t+3}\cdots x_{k+t+1}. \qquad (1)$$

If we multiply the factors out and identify each x_i with $i \in V$, then the resulting form is a (v,k,t) trade. It is shown that all minimal trades for given v, k and t are isomorphic [2] and [6].

In this paper, first we show that if $v=2k$ and $k=t+1$, then every (v,k,t) trade or every t-design is self-complementary if t is odd and is quasi-complementary if t is even. From these facts we drive some useful conclusions about the support of a t-design.

Next we study the special case of $v = 10$, $k = 5$ and $t = 4$. In this respect we show that designs with $b^* < 152$ are nonexistent. Using the trade-off methodology, we construct 88 4-designs with various support sizes. Finally, we make some remarks about existing and nonexisting various support sizes.

As mentioned in [4], the construction of t-designs becomes more complicated when the set of λ_i's are relatively prime. In the case of 4-designs, 4-(10,5,6m) designs are examples of such designs with smallest v. Since the only simple design in this case is the trivial design, the usual techniques of construction of designs with various support sizes can not be applied and one has to use other techniques such as the trade-off.

2. ON TRADES AND DESIGNS WITH $v = 2k$ AND $k = t + 1$

Theorem 1. Every $(2t + 2, t + 1, t)$ trade is self-complementary if t is odd, and is quasi-complementary if t is even.

Proof. Let $v = 2t + 2$ and $k = t + 1$. Then the polynomial (1) will be of the following form:

$$\varphi(x_1, x_2, \cdots, x_v) = (x_1 - x_2)(x_3 - x_4)\cdots(x_{2t+1} - x_{2t+2}).$$

Note that there are always $t + 1$ factors in (1). By a simple induction on t, it follows that the trade obtained from this polynomial has the claimed property.

Since all minimal trades are isomorphic, they all have this property. Now the theorem follows from the fact that the minimal trades form a basis for $\mathbb{Z}$-module of (v, k, t) trades. □

Theorem 2. Every t-design with $v = 2k$ and $k = t + 1$ is self-complementary if t is odd, and is quasi-complementary if t is even.

Proof. Suppose D_1 and D_2 are two $t - (v, k, \lambda)$ designs, then $D_1 - D_2$ is a (v, k, t) trade [1]. Let D be a $t - (v, k, \lambda)$ design and D' be the trivial $t - (v, k, \lambda')$ design, then $T = \lambda' D - \lambda D'$ is a (v, k, t) trade. Thus $D = (1/\lambda')(\lambda D' + T)$. If $v = 2k$, $k = t + 1$ and t is odd, then from Theorem 1, it follows that T is self-complementary and likewise D. If t is even, then T is quasi-complementary and for every block B in D,

$$\#(B) + \#(V - B) = 2\lambda/\lambda'. \tag{2}$$

Hence D is quasi-complementary. □

Corollary 1. For every odd t, the support size of any $t-(2t+2, t+1, \lambda)$ design is even.

Corollary 2. For t even, if a nontrivial simple $t-(2t+2, t+1, \lambda)$ design exists, then $\lambda = \lambda'/2 = (t+2)/2$.

Remark. In connection with Corollary 2, it is interesting to note that for $t = 6$, there exists a simple nontrivial 6-design with $\lambda = (t+2)/2$ [7]. For the case $t = 4$, there is no such design. For $t > 6$, the problem of existence of such designs is open.

3. STUDY OF 4-(10,5,6m) DESIGNS WITH REPEATED BLOCKS

From the necessary conditions for the existence of a t-design, it follows that for $v = 10$, $k = 5$, $t = 4$ we have

$$\lambda_0 = 42\lambda_4, \quad \lambda_1 = 21\lambda_4, \quad \lambda_2 = (28/3)\lambda_4, \quad \lambda_3 = (7/2)\lambda_4.$$

Hence, $\lambda_4 = 6m$. Since the trivial design has 252 blocks with $\lambda_4 = 6$, therefore the trivial design is the only simple design in this case, and all of the nontrivial designs are with repeated blocks, and to produce such designs one has to reduce the support size of the trivial design. There are at least two problems of interest in cases like this: (1) what are the possible b^*'s? and (2) for a given b^*, what is the minimum b?

Providing a complete solution to these problems via the existing techniques seems rather difficult. In the remaining parts of the article we offer some partial solutions.

3.1 A SET OF POSSIBLE b^*'S

In Table 1 we have demonstrated 88 designs with various support sizes. A few words about the method of construction are in order. In [5], a method of construction of t-designs with repeated blocks called "the method of trade-off" was introduced. Hedayat and Li and subsequently different authors, have used this method to construct BIB-designs. In short, the method is based on the fact that if D is a $t-(v, k, \lambda)$ design and T is a (v, k, t) trade, then $T + D$ is a design if and only if $D+T$ has non-negative frequencies. In practice, one usually chooses T to be a minimal trade, since to produce such a trade is very simple [6]. The resulting design, $D+T$, may have larger or smaller support size than D. By applying this method over and over with the same process, designs with desired b^*'s might be obtained.

As far as we know, this is the first time that the trade-off methodology is being applied to construct t-designs with $t > 2$. Specifically, we do the following: take a design D and let $B \in D$; form all the minimal trades containing B; add each of these trades to D; then the designs with new b^*'s are recorded (the b^*'s of the new designs differ at most by 16 from the b^* of the old design, since $t = 4$); the design with smallest b^* is chosen as a starting design and the process continues.

Here, we note that our initial designs are the multiples of the trivial design. A computer program based on the above algorithm was written and an extensive search was carried out to construct the designs of Table 1.

In Table 1, for every $152 \leq b^* \leq 252$, except for b^*=153, 154, 155, 158, 159, 160, 161, 166, 167, 169, a design is given. In the remaining part of this section nonexistence of designs with $b^* < 152$ is established. The existence or nonexistence of designs with remaining values of b^*'s is under investigation.

Lemma 1. The support size of every 4-(10,5,6) design is even.

Proof. For a given 4-(10,5,6) design D, let n_0, n_1 and n_2 denote the number of blocks in D of the form $1abcd$ with frequencies 0, 1 and 2, respectively. By the quasi-complementarity of D, $b^* = n_0 + 2n_1 + n_2$. Note that

$$n_0 + n_1 + n_2 = \lambda_1 = 126.$$

Hence $b^* = 126 + n_1$. Suppose n_1 is odd, e.g. $n_1 = 2s + 1$. Now we observe that the total number of quadruples of the form $1xyz$ in D is $4(2s+1) + 8n_2 = 4 \pmod 8$, while in every 4-(10,5,6) design there are $504 = 0 \pmod 8$ quadruples of the form $1xyz$, and hence a contradiction. □

3.2 THE MINIMUM SUPPORT SIZE

In this section we would like to show that $b^*_{\min} = 152$. To achieve this, we take a closer look at the block structure of a 4-(10,5,6m) design D.

If every block appears in D, then b^*=252. For otherwise, if $\#(B) = 0$ for some block B, then because of the quasi-complementarity of D, $\#(V - B) = 12m/6 = 2m$. With no loss of generality, suppose $B = 12345$ and $\#(12345) = 2m$. Now, with regards to the parameters of D, we can partition the blocks of D into the following classes:

classes	frequencies	blocks
I	$2m$	12345
II	$4m$	1234- 1235- 1245- 1345- 2345-
III	$11m$	123- - 124- - 125- - 134- - 135- - 145- - 234- - 235- - 245- - 345- -
IV	$9m$	12- - - 13- - - 14- - - 15- - - 23- - - 24- - - 25- - - 34- - - 35- - - 45- - -
V	$6m$	1- - - - 2- - - - 3- - - - 4- - - - 5- - - -

In this table, the frequencies column says that , for example, there are $4m$ blocks of the form $1234x$, where $x \in \{6,7,8,9,0\}$. The blocks of the form $1234x$ are a subclass of class II, and so on.

Since the maximum frequency of a block in D is $2m$, each subclass of classes II,$\cdots$,V have at least 2, 6, 5, and 3 distinct blocks, respectively. Therefore, D has at least 1+10+60+50+15=136 distinct blocks. This is a rough lower bound for $b^*_{\min}$.

A closer look at the block structure of classes II and III reveals that this number can be improved to 152, as shown by the following argument.

Lemma 2. The following situation is impossible:

In class II, only the subclass 1234- has 3 distinct blocks and the other subclasses of this class have 2 distinct blocks. At the same time in class III, only the subclasses 123- -, 124- -, 134- - and 234- - have 6 distinct blocks each and the rest of the subclasses of this class have 7 distinct blocks each.

Proof. Suppose the above situation occurs. Since the maximum frequency of a block is $2m$, hence the subclass 123- - contains exactly one block, say $123ab$ with $a, b \in \{6,7,8,9,0\}$, such that $0 < \#(123ab) < 2m$. Since the quadruples $123a$ and $123b$ must occur $\lambda_4 = 6m$ times, the subclass 1234- contains the blocks $1234a$ and $1234b$ with frequencies not equal to $2m$ and the other block of this subclass has frequency of $2m$. By considering the blocks with frequencies not equal to $2m$ in subclasses 124- -, 134- - and 234- - it follows that: the blocks with frequencies not equal to $2m$ in subclasses 123- -, 124- -, 134- - and 234- - are as follows

$$123ab \quad 124ab \quad 134ab \quad 234ab$$

and the blocks with frequencies not equal to $2m$ in subclass 1234- are of the form $1234a$ and $1234b$.

Now consider the subclass 125- -. The blocks with frequencies$\neq 2m$ are $125cd$, $125ce$, $125de$. Since otherwise a quadruple $125x$ exists such that it has the frequency$\neq 2m$ in these 3 blocks while the other blocks of this design which contain $125x$ each have frequencies of $2m$ and this is contradictory to $\lambda_4 = 6m$.

Now we consider the quadruples

$$12cd, \quad 12ce, \quad 12de. \tag{3}$$

These quadruples must occur in subclass 12- - - of class IV with frequencies$\neq 2m$. But this subclass is the complement of the subclass 345- - of class III. Without loss of generality, we can assume that these blocks with frequencies$\neq 2m$, in the subclass 345- - are of the following form:

$$34567, \quad 34568, \quad 34578.$$

Hence the blocks with frequencies$\neq 2m$ in subclass 12- - - are as follows:

$$12890, \quad 12790, \quad 12690.$$

Thus the six quadruples

$$12x9, \quad 12x0, \quad x \in \{6,7,8\}$$

appear in subclass 12- - - with frequencies$\neq 2m$. Whereas in subclass 125- - only 3 quadruples (3) occur with frequencies$\neq 2m$ and in the rest of the blocks of D, all the quadruples with the form $12xy$, $x, y \in \{6,7,8,9,0\}$ appear with frequencies $2m$, and this is in contradiction with $\lambda_4 = 6m$. □

Theorem 3. In 4-(10,5,6m) designs, $b^*_{\min} = 152$.

Proof. Table 1 contains a design with b^*=152. Thus it suffices to prove that for every 4-(10,5,6m) design, $b^* \geq 152$.

If each subclass of class III has at least 7 distinct blocks, then by quasi-complementarity, $b^* \geq 136 + 10 + 10 = 156$.

Suppose in calss III at least one subclass, e.g. 123- -, has exactly 6 distinct blocks. Hence in this subclass there exists a block, e.g. $B = 12367$, such that $0 < \#(B) < 2m$. Since each one of the 2 quadruples 1236 and 1237 must occur in $2m$ blocks, hence in one of the subclasses 1234- or 1235- (take 1234-), there exists a block $B' = 12346$ such that $0 < \#(B') < 2m$. Thus at least 3 distinct blocks exist in subclass 1234-. Now, if in class III every one of the subclasses (with the possible exceptions 123- -, 124- -, 134- -, 234- -) has more than 6 distinct blocks, then by Lemma 2, $b^* \geq 152$.

Now suppose that another subclass of III, beside 123- -, 124- -, 134- -, 234- -, e.g. 125- -, has excatly 6 distinct blocks. Therefore, by the above reasoninig, in class II at least one of the subclasses 1235- or 1245- must contain at least 3 distinct blocks. Suppose 1245- be such a subclass having 3 distinct blocks. In this way, the quadruples 124a and 124b appear with frequencies$\neq 2m$. In order these quadruples attain $6m$ frequency, subclasses 1234-, 1245-, 124- -, besides those which were counted, must have 3 distinct blocks with frequencies$= 2m$. If each of the 3 subclasses 135- -, 235- - and 345- - has at least 7 distinct blocks, thus the design D will have at least 136+8+8=152 distinct blocks.

If at least one of these 3 subclasses, namely 135- -, 235- - and 345- -, contains exactly 6 distinct blocks, then by repeating the above argument, it follows that $b^* \geq 152$. □

4. CONCLUDING REMARKS

For a given triple (v, k, t), the problem of the spectrum consists of two questions:

(i) what is the set of all feasible b^*'s of $t-(v,k)$ designs, and in particular, what is $b^*_{\min}$?

(ii) For a feasible b^*, what is the set of all b's such that a $t-(v,k)$ design with b blocks and b^* distinct blocks exists and, in particular, for a given feasible b^*, what is the minimum of such b's?

Concerning the problem of spectrum, here we list the solved and unsolved problems for the case 4-(10,5):

1) $152 \leq b^* \leq 252$, $b^*_{\min} = 152$.

2) By Table 1, only the existence of designs with b^* =153, 154, 157,$\cdots$, 161, 163, 165, 166, 167, and 169 remains open.

3) For those b^*'s for which a design with $b = 252$ is given in Table 1, the question (ii) is completely answered: the set of possible b's is $252m$, for $m = 1, 2, \cdots$.

4) For odd b^*'s, $b \geq 504$ (by Lemma 1). Hence for all odd b^*'s in Table 1, except for $b^* = 173$, the minimum b is equal to 504. The problem of existence of designs with odd b^*'s and $b = 252m$, for odd m remains open.

5) Since (10,5,4) trades with volume s, for $1 < s < 16$ [6] and $16 < s < 24$ [8] do not exist, hence designs with $b = 252$, $236 < b^* < 252$ and $228 < b^* < 236$ do not exist. Therefore, these designs with possible minimum b (i.e., b=504) appear in Table 1.

5) In Table 1 designs with $b^* = 4 \pmod 6$ and $b^* < 228$ are given with $b = 504$. The existence of these designs with $b = 252$ remains open.

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