On λ-packings of pairs by quintuples: $\lambda \equiv 0 \pmod 4$

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Abstract. A λ-packing of pairs by quintuples of a v-set V is a family of 5-subsets of V (called blocks) such that every 2-subset of V occurs in at most λ blocks. The packing number is defined to be the maximum number of blocks in such a λ packing. These numbers are determined here for $\lambda \equiv 0 \pmod 4$ and all integers $v \geq 5$ with the exceptions of $(v,\lambda) \in \{(22,16),(22,36),(27,16)\}$.

1. Introduction

Let V by a v-set (of points). A λ-packing of pairs by quintuples of V (briefly packing) is a family of 5-subsets of V (called blocks) such that every 2-subset of V occurs in at most λ blocks. The packing number $D_\lambda(v)$ is defined to be the maximum number of blocks in such a packing. The packing problem is to determine the packing number.

Schoenheim [7] has shown that

$$D_\lambda(v) \leq \left\lfloor \frac{v}{5} \left\lfloor \frac{\lambda(v-1)}{4} \right\rfloor \right\rfloor = B_\lambda(v) \tag{1.1}$$

where $\lfloor x \rfloor$ is the floor of x. The values of $D_2(v)$ for all v have been determined by Yin [9],[10] with 11 possible exceptions of v. Recently, Assaf and Hartman have studied in [1] the packing number $D_4(v)$. They prove the following.

Theorem 1.1. *If v is an integer and $v \geq 5$, then*

$$D_4(v) = \begin{cases} B_4(v) & \text{when } v \not\equiv 3 \pmod 5 \text{ and } v \neq 7, \\ B_4(v) - 1 & \text{when } v \equiv 3 \pmod 5 \text{ or } v = 7. \end{cases}$$

The following result is contained in [5].

Theorem 1.2. *If $v \equiv 0$ or $1 \pmod 5$ and $v \geq 5$, then $D_\lambda(v) = B_\lambda(v)$ for all $\lambda \equiv 0 \pmod 4$.*

It is our purpose here to determine $D_\lambda(v)$ for all $\lambda \equiv 0 \pmod 4$ and all integers $v \geq 5$ with the possible exceptions of $(v,\lambda) \in \{(22,16),(22,36),(27,16)\}$. We shall sometimes refer to a λ-packing of pairs by quintuples of a v-set V as a $(v,5,\lambda)$ packing. A $(v,5,\lambda)$ packing with $D_\lambda(v)$ blocks will be called a maximum packing.

2. Preliminaries

In order to state our results we need several other types of designs. The related definitions can be found in [2],[9]. In what follows, we say that a GDD $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ with index λ is a (K, λ)-GDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The type of a GDD $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ is defined to be the multiset $\{|G| : G \in \mathcal{G}\}$, which is usually denoted exponentially by $1^j 2^k \ldots$ if there are j groups of size 1, k groups of size 2, etc. When $K = \{k\}$, a (K, λ)-GDD is written as a (k, λ)-GDD. A (k, λ)-GDD of type n^k is called a transversal design, denoted by TD (k, λ, n). We also refer to a BIBD with parameters (v, k, λ) as a B $(k, \lambda; v)$. By $(v, w; K, \lambda)$-IPBD we mean an incomplete PBD of order v with a hole of size w, block sizes from K and index λ. Write $(v, w; k, \lambda)$-IPBD for $K = \{k\}$ and define

$$\mathrm{IP}_\lambda(w) = \{v: a(v, w; 5, \lambda) - \mathrm{IPBD\ exists}\}.$$

We now list some of those results which will be used later.

Lemma 2.1. ([5]) *There exists a $B(5, \lambda; v)$ for all integers $v \geq 5$ which satisfy $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ with the exception of $v = 15$ and $\lambda = 2$.*

Lemma 2.2. ([4]) *There exists a $(n, 9; 5, 1)$-IPBD if $n \equiv 19$ or $17 \pmod{20}$, $n \geq 37$ and $n \neq 49$. There exists a $(n, 13; 5, 1)$-IPBD if $n \equiv 13 \pmod{20}$ and $n \geq 53$.*

Lemma 2.3. ([5]) *There exist the following designs:*

(1) *a $(5, 2)$-GDD of type 5^7;*
(2) *a $(5, 1)$-GDD of type 5^9;*
(3) *a $(6, 2)$-GDD of type 5^7; and*
(4) *a $TD(7, \lambda, r)$ for all positibe integers r and all integers $\lambda \geq 2$.*

Constructions by filling in the groups of GDDs provide us with the following useful results.

Lemma 2.4. *Let t, a, b and w be non-negative integers satisfying $0 \leq a, b \leq t$ and $t \geq 1$. Let $\lambda = 8, 12$ or 16. Then $25t + 5a + 5b + w \in IP_\lambda(5b + w)$ if $\{5t + w, 5a + w\} \subseteq IP_\lambda(w)$; and $25t + w \in IP_\lambda(5t + w)$ if $5t + w \in IP_\lambda(w)$ and $a = b = 0$.*

Proof: From Lemma 2.3, we have a $\mathrm{TD}(7, \frac{\lambda}{4}, t)$. In such a TD, we delete $t - a$ points from one group and $t - b$ points from another group. We then give weight 5 to every point of the resulting design and use Wilson's Fundamental Construction for GDDs ([8]). The required input designs $(5, 4)$-GDDs of type 5^5 and 5^7 exist by Lemma 2.3, and a $(5, 4)$-GDD of type 5^6 can be constructed using a $B(5, 4; 6)$ and a $\mathrm{TD}(5, 1, 5)$. Therefore there exists a $(5, \lambda)$-GDD of type $(5t)^5 (5a)^1 (5b)^1$. Adding w new points to this GDD gives the conclusion.

Lemma 2.5. *Let t, a, b and c be non-negative integers satisfying $0 \le a, b+c \le t$ and $t \ge 1$. Then $25t + 5a + 5b + 2c + 2 \in IP_{16}(5b + 2c + 2)$ if $\{5t + 2, 5a + 2\} \subseteq IP_{16}(2)$.*

Proof: Delete 3 points from one group of a TD$(6,2,5)$ and a $(6,2)$-GDD of type 5^7 (see Lemma 2.3) respectively. This produces $(\{5,6\},2)$-GDDs of type $5^5 2^1$ and $5^6 2^1$. Since both a $B(5,4;5)$ and a $B(5,4;6)$ exist by Lemma 2.1, we can know that a $(5,8)$-GDD of type $5^5 2^1$ or $5^6 2^1$ exists. From the proof of Lemma 2.4, we have also a $(5,8)$-GDD of type 5^5 or 5^6 or 5^7. So the conclusion holds by using the Fundamental Construction and the fact that a TD$(7,2,t)$ exists for all positive integers t.

The following useful construction was shown by Yin in [9].

Lemma 2.6. *Let e and m by positive integers satisfying $e \equiv 0 \pmod m$ and let $q \ge 0$. Suppose that the following designs exist:*

(1) *a $(u+e+q, e+q; K, \lambda)$-IPBD; and*
(2) *a $(u+q, q; K, (m-1)\lambda)$-IPBD.*

Then there exists a $(u+w, w; K, m\lambda)$-IPBD where $w = q + \frac{e}{m}$.

We shall also make use of the following result.

Lemma 2.7. *Let $\lambda \equiv 0 \pmod 4$ and $v \ge 9$. Then $v \in IP_\lambda(2)$ if $v \equiv 2$ or $4 \pmod 5$ and $v \in IP_\lambda(3)$ if $v \equiv 3 \pmod 5$.*

Proof: Careful inspection of the proof Theorem 1.3, Lemmas 2.2, 2.3 and Corollary 4.3 in [1] yields the conclusion for $\lambda = 4$ and $v \ne 43, 68$. A $(43,3;5,4)$-IPBD can be constructed by applying Lemma 2.6 with $K = \{5\}$, $e = 4$, $q = 1$, $u = 40$, and $\lambda = m = 2$. The conditions are satisfied because of Lemma 2.1. Adding 3 infinite points to a resolvable $B(5,1;65)$ (see [3]) guarantees that $68 \in IP_4(3)$ since a $B(5,4;6)$ exists by Lemma 2.1. We then get the required result by taking $\frac{\lambda}{4}$ copies of a $(v, w; 5, 4)$-IPBD with $w = 2$ or 3.

Finally, we mention the following two lemmas.

Lemma 2.8. *If there exists a $B(5,\lambda; v)$, $D_{\lambda'}(v) = B_{\lambda'}(v) - \epsilon$ and $D_{\lambda'+\lambda}(v) \le B_{\lambda'+\lambda}(v) - \epsilon$, then $D_{\lambda'+\lambda}(v) = B_{\lambda+\lambda'}(v) - \epsilon$ where $\epsilon = 0$ or 1.*

Proof: Construct a $(v, 5, \lambda')$ packing with $B_{\lambda'}(v) - \epsilon$ blocks on a v-set V. Adjoin its blocks to a $B(5, \lambda; v)$ also defined on V. This produces a $(v, 5, \lambda + \lambda')$ packing with $B_{\lambda+\lambda'}(v) - \epsilon$ blocks and the conclusion holds.

Lemma 2.9. *Let $\lambda \equiv 0 \pmod 4$ and $u \ge 5$. Suppose that $v \in IP_\lambda(u)$ and one of the following two holds:*

(1) $v \equiv u \equiv 3 \pmod 5$;
(2) $v, u \equiv 2$ *or* $4 \pmod 5$.

Then $D_\lambda(v) = B_\lambda(v)$ *if* $D_\lambda(u) = B_\lambda(u)$.

Proof: This is because that the number of pairs (counting multiplicities) which occur less than λ times in the blocks of a maximum $(v, 5, \lambda)$ packing is the same as that of a maximum $(u, 5, \lambda)$ packing.

3. Packing number $D_\lambda(v)$ for $v \equiv 3 \pmod 5$

We first note that $D_\lambda(v) \leq B_\lambda(v) - 1$ when $\lambda \equiv 4 \pmod{20}$ and $\lambda \equiv 3 \pmod 5$. The proof is similar to the case for $\lambda = 4$ and $\lambda \equiv 3 \pmod 5$ (see [1]). In view of Lemma 2.1, Lemma 2.8 and Theorem 1.1, it is sufficient to determine $D_\lambda(v) = B_\lambda(v)$ for $\lambda = 8, 12$ and 16.

Lemma 3.1. *If* $v \equiv 3 \pmod 5$ *and* $v \geq 5$, *then* $D_8(v) = B_8(v)$.

Proof: When $v \geq 13$, it is shown in Lemma 2.7 that $\{v+1, v-1\} \subseteq IP_4(2)$. Define $I(n) = \{1, 2, \ldots, n\}$. Let $(I(v-1), \{1,2\}, \mathcal{A})$ be a $(v-1, 2; 5, 4)$-IPBD. Let $(I(v+1), \{v, v+1\}, \mathcal{B})$ be a $(v+1, 2; 5, 4)$-IPBD. Let $\mathcal{D}$ denote the configuration obtained from $\mathcal{B}$ by replacing symbol $v+1$ by v wherever it occurs. It is easy to check that $\mathcal{A} \cup \mathcal{D}$ forms an 8-packing on $I(v)$ with $B_8(v)$ blocks. For $v = 8$, let $V = Z_6 \cup \{x, y\}$. Develop under the action of Z_6 the following base blocks to obtain an 8-packing on V with $B_8(8)$ blocks.

x	0	1	2	3	
x	0	2	3	5	(orbit length 3)
y	0	1	2	3	
y	0	2	3	5	(orbit length 3)
x	y	0	2	4	(orbit length 2)
x	y	0	2	4	(orbit length 2)

The conclusion then follows from (1.1).

Lemma 3.2. *If* $v \in \{8, 13, 18\}$, *then* $B_{12}(v) = D_{12}(v)$.

Proof: From (1.1), we construct a $(v, 5, 12)$ packing with $B_{12}(v)$ blocks as follows, for each $v \in \{8, 13, 18\}$, to yield the result.

For $v = 8$, let the point set be $Z_4 \cup \{A, B, C, D\}$. Then the blocks are

A	B	1	2	3	(three times)
A	C	0	1	3	(three times)
A	D	0	1	2	(three times)
A	B	C	0	2	(three times)
A	B	D	0	3	(three times)
A	C	D	2	3	(three times)
A	B	C	D	1	(three times)
B	C	0	1	2	mod 4
B	D	0	1	2	mod 4
D	C	0	1	2	mod 4

For $v = 13$, let the point-set be $V = (Z_5 \times Z_2) \cup \{x, y, z\}$. Let $\mathcal{A}$ consist of the following 75 blocks:

x	(0,0) (0,1) (2,1) (3,1)	mod $(5, -)$	(taken twice)
x	(0,0) (1,0) (4,0) (0,1)	mod $(5, -)$	(taken twice)
x	(1,0) (4,0) (2,1) (3,1)	mod $(5, -)$	
y	(0,0) (0,1) (2,1) (3,1)	mod $(5, -)$	(taken twice)
y	(0,0) (1,0) (4,0) (0,1)	mod $(5, -)$	(taken twice)
y	(1,0) (4,0) (2,1) (3,1)	mod $(5, -)$	
z	(1,0) (4,0) (2,1) (3,1)	mod $(5, -)$	(taken 3 times)
z	(0,0) (0,1) (2,1) (3,1)	mod $(5, -)$	
z	(0,0) (1,0) (4,0) (0,1)	mod $(5, -)$	

It is readily checked that each pairset of V not contained in $F = \{x, y, z\}$ occurs exactly in 10 blocks of $\mathcal{A}$, whereas no pairset of F is contained in any block of $\mathcal{A}$. Now construct a $(5,2)$-GDD of type 2^5 $(Z_5 \times Z_2, \mathcal{G}, \mathcal{B})$ by Lemma 2.3 and define $\mathcal{D} = \{F \cup G: G \in \mathcal{G}\}$. To the required set of blocks, take two copies of the blocks in $\mathcal{D}$, denoted by $\mathcal{C}$. It follows that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ forms a 12-packing on V with $B_{12}(13)$ blocks.

For $v = 18$, let the point-set be $V = Z_{15} \cup \{\infty_i : i \in Z_3\}$. Delete one point in a $B(4, 1; 16)$ (see [5]) to produce a $(4, 1)$-GDD of type 3^5. We label the GDD as $(Z_{15}, \mathcal{G}, \mathcal{A})$. Define $\mathcal{B}_i = \{\{\infty_i\} \cup A : A \in \mathcal{A}\}$ and $\mathcal{D}_i = \{\{\infty_i, \infty_{i+1}\} \cup G : G \in \mathcal{G}\}$ for each $i \in Z_3$. Take two copies of all blocks in $\mathcal{B}_i \cup \mathcal{D}_i$ for each $i \in Z_3$. These blocks together with the blocks of a $B(5, 6; 15)$ on Z_{15} form a 12-packing on V with $B_{12}(18)$ blocks.

Lemma 3.3. *If* $v \in \{23, 33, 43, 48\}$, *then* $D_{12}(v) = B_{12}(v)$.

Proof: Take 3 copies of all blocks in a $(33, 8; 5, 4)$-IPBD which exists by Theorem 4.2 in [1]. This gives that $33 \in IP_{12}(8)$. From Lemma 2.1 and Lemma 2.2, we can use Lemma 2.6 with $K = \{5\}$ and $(u, e, q, m, \lambda) = (40, 4, 1, 2, 1)$ to get $43 \in IP_2(3)$. We also apply Lemma 2.6 with $K = \{5\}$ and $(u, e, q, m, \lambda) = (40, 10, 3, 2, 2)$ to obtain $48 \in IP_4(8)$, and hence $48 \in IP_{12}(8)$. This guarantees that $D_{12}(v) = B_{12}(v)$ for $v = 33$ or 48 because of Lemma 3.2 and Lemma 2.9. For $v = 23$, we have a $B(5, 10; 23)$ by Lemma 2.1. Applying Lemma 2.6 with $K = \{5\}$ and $(u, e, q, m, \lambda) = (20, 4, 1, 2, 1)$ we have also a $(23, 3; 5, 2)$-IPBD. This gives rise to a 12-packing with $B_{12}(23)$ blocks and hence $D_{12}(23) = B_{12}(23)$. For $v = 43$, the proof is similar.

Lemma 3.4. $D_{12}(28) = B_{12}(28)$.

Proof: Let $V = Z_{25} \cup \{x, y, z\}$. We construct a 12-packing on V with $B_{12}(28)$ blocks below.

We first start with a TD$(6, 1, 5)$ and delete two points from one group to obtain a $(\{5, 6\}, 1)$-GDD of type $5^5 3^1$. This gives rise to a $(28, 3; \{5, 6\}, 1)$-IPBD.

Let us label its point-set as $V = Z_{25} \cup \{x, y, z\}$ and the hole as $\{x, y, z\}$. We then easily construct a $(28, 3; \{5, 6\}, 3)$-IPBD $(V, \{x, y, z\}, \mathcal{A})$ in such a way that $\mathcal{A}$ contains the special blocks: $B_1 = \{x, 0, 1, 2, 3, 4\}$, $B_2 = \{y, 0, 1, 2, 3, 4\}$, $B_3 = \{z, 0, 1, 2, 3, 4\}$. To the required set of blocks, we remove B_1, B_2, and B_3 from $\mathcal{A}$ and replace them with the blocks: $\{x, y, z, 0, 1\}$, $\{x, y, z, 0, 2\}$, $\{x, y, z, 0, 3\}$, $\{x, y, z, 0, 4\}$, $\{x, y, z, 1, 2\}$, $\{x, y, z, 1, 3\}$, $\{x, y, z, 1, 4\}$, $\{x, y, z, 2, 4\}$, $\{x, y, z, 2, 3\}$, $\{x, y, z, 3, 4\}$, and $\{0, 1, 2, 3, 4\}$ (taken 11 times). We then put a $B(5, 4; |A|)$ on each block $A \in \mathcal{A} \backslash \{B_1, B_2, B_3\}$, where we make use of the $B(5, 4; 5)$ and $B(5, 4; 6)$ in Lemma 2.1. The result is a 12-packing on V with $B_{12}(28)$ blocks.

Lemma 3.5. $D_{12}(38) = B_{12}(38)$.

Proof: Start with a TD$(6, 1, 7)$ and delete 6 points from one group to get a $(\{5, 6\}, 1$-GDD of type $7^5 1^1$. Let the point-set be $Z_{35} \cup \{x\}$ and the group of size one be $\{x\}$. On each block A of the GDD except for one distinguised block, say $B_1 = \{x, 0, 1, 2, 3, 4\}$, we construct a $B(5, 4; |A|)$. Then we adjoin two new points y, z to the GDD and put a $(9, 2; 5, 4)$-IPBD on each group of size 7 together with points y, z in such a way that the hole is $\{y, z\}$. Here $9 \in IP_4(2)$ follows from Lemma 2.7. Copy the resulting configuration three times and make the permutation (x, y, z). This gives a configuration $\mathcal{D}$ which is based on $V = Z_{35} \cup \{x, y, z\}$ and satisfies the following properties:

(1) $\mathcal{B} = \{B_0 = \{x, y, z\}, B_1 = \{x, 0, 1, 2, 3, 4\}, B_2 = \{y, 0, 1, 2, 3, 4\}, B_3 = \{z, 0, 1, 2, 3, 4\}\} \subset \mathcal{D}$,
(2) $|A| = 5$ for any $A \in \mathcal{D} \backslash \mathcal{B}$,
(3) no pair of points of V which lies in $\{x, y, z\}$ or $\{0, 1, 2, 3, 4\}$ occurs in any block of $\mathcal{D} \backslash \mathcal{B}$,
(4) each pair of points of V not contained in $\{x, y, z, 0, 1, 2, 3, 4\}$ occurs in exactly 12 blocks of $\mathcal{D} \backslash \mathcal{B}$, and
(5) each pair of points in which one point lies in $\{x, y, z\}$ and the other lies in $\{0, 1, 2, 3, 4\}$ occurs in exactly 8 blocks of $\mathcal{D} \backslash \mathcal{B}$.

Now we replace those blocks in $\mathcal{B}$ with the blocks as in Lemma 3.4. The result is a 12-packing on V with $B_{12}(38)$ blocks and the conclusion holds.

Lemma 3.6. *If* $v \equiv 3 \pmod 5$ *and* $v \geq 5$, *then* $D_{12}(v) = B_{12}(v)$.

Proof: It is shown in Lemmas 3.2–3.5 that the conclusion holds for $v < 53$. Noticing Lemma 2.7 and Lemma 2.9, we apply Lemma 2.4 with $t \geq 2$ and $w = 3$ to obtain the result for $v = 25t + 5a + 5b + 3 \geq 53$, where (a, b) is taken from $\{(0,0), (0,1), (0,2), (2,1), (2,2)\}$ when $t = 2$, and from $\{(0,1), (0,2), (0,3), (2,2), (2,3), (3,3)\}$ when $t \geq 3$.

Lemma 3.7. *If* $v \equiv 3 \pmod 5$ *and* $v \geq 5$, *then* $D_{16}(v) = B_{16}(v)$.

Proof: For these values of v, a 16-packing with $B_{16}(v)$ blocks can be constructed by taking two copies of the blocks of a maximum $(v, 5, 8)$ packing.

The foregoing can be summarized as follows.

Theorem 3.8. *If* $v \equiv 3 \pmod 5$, $v \geq 5$ *and* $\lambda \equiv 0 \pmod 4$, *then*

$$D_\lambda(v) = \begin{cases} B_\lambda(v) & \text{when } \lambda \not\equiv 4 \pmod{20}, \\ B_\lambda(v) - 1 & \text{when } \lambda \equiv 4 \pmod{20}. \end{cases}$$

4. Packing numbers $D_\lambda(v)$ for $v \equiv 2$ or $4 \pmod 5$

It is easy to show the following.

Lemma 4.0. $D_{24}(7) = B_{24}(7) = 50$.

Lemma 4.1. *If* $v \equiv 2$ *or* $4 \pmod 5$ *and* $v \geq 5$, *then* $D_8(v) = B_8(v)$.

Proof: Take two copies of the blocks of a maximum $(v, 5, 4)$ packing to obtain the result for $v \geq 9$. When $v = 7$, let the point-set be $Z_4 \cup \{x, y, z\}$. Then the blocks are:

x	y	0	1	2	mod 4
y	z	0	1	2	mod 4
x	z	0	1	2	mod 4
x	y	z	0	1	mod 4

Lemma 4.2. *If* $v \equiv 2$ *or* $4 \pmod 5$ *and* $v \geq 5$, *then* $D_{12}(v) = B_{12}(v) - 1$.

Proof: Assume that there exists a $(v, 5, 12)$ packing on V with $B_{12}(v)$ blocks. Let Y_x be the numbers of blocks containing x ($x \in V$). It is easily verified that the degree of vertex x ($x \in V$) in the non-occurence graph must be $12(v-1) - 4Y_x$ which is divisible by 4. However the number of pairs of points of V which occur in less than 12 blocks (counting multiplicities) is $6v(v-1) - 10B_{12}(v) = 2$ when $v \equiv 2$ or $4 \pmod 5$. This is a contradiction. We then have $D_{12}(v) \leq B_{12}(v) - 1$. On the other hand, a 12-packing with $B_{12}(v) - 1$ blocks can be constructed by taking three copies of the blocks of a maximum $(v, 5, 4)$ packing when $v \geq 9$. A $(7, 5, 12)$ packing with $B_{12}(7) - 1$ blocks follows from a $B(5, 10; 7)$ and a maximum $(7, 5, 2)$ packing ([9]). Therefore the conclusion holds.

Lemma 4.3. *If* $v \in \{7, 9, 12, 17\}$, *then* $D_{16}(v) = B_{16}(v)$.

Proof: We construct a 16-packing with $B_{16}(v)$ blocks for these values of v to yield the result.

For $v = 7$, let the point-set be $Z_4 \cup \{x, y, z\}$. Then the blocks are:

x	y	0	1	2	mod 4	(taken twice)
x	z	0	1	2	mod 4	(taken twice)
y	z	0	1	2	mod 4	(taken twice)
x	0	1	2	3		
y	0	1	2	3		
z	0	1	2	3		
x	y	z	0	1		
x	y	z	0	2		
x	y	z	0	3		
x	y	z	1	2		
x	y	z	1	3		
x	y	z	2	3		

For $v = 9$, let the point-set be $Z_6 \cup \{x, y, z\}$. Then the blocks can be obtained by developing under the action of Z_6 the following base blocks:

x	0	1	4	5		x	y	0	1	3
y	0	1	4	5		y	z	0	1	3
z	0	1	4	5		x	z	0	1	2
x	y	z	0	2						
x	0	1	3	4		(orbit length 3)				
y	0	1	3	4		(orbit length 3)				
z	0	1	3	4		(orbit length 3)				
x	y	0	2	4		(orbit length 2)				
y	z	0	2	4		(orbit length 2)				
x	z	0	2	4		(orbit length 2)				

For $v = 12$, let the point-set be $Z_6 \cup \{A, B, C\} \cup \{x, y, z\}$. It was shown in [5] that a $B(4, 3; 9)$ and a $B(3, 1; 9)$ both exist. Put a $B(4, 3; 9)$ with block set $\mathcal{A}$ on $Z_6 \cup \{A, B, C\}$ and a $B(3, 1; 9)$ with block set $\mathcal{B}$ on $Z_6 \cup \{A, B, C\}$ in such a way that $\{A, B, C\} \in \mathcal{B}$. Define

$S_1 = \{\{x\} \cup E: E \in \mathcal{A}\} \cup \{\{y\} \cup E: E \in \mathcal{A}\} \cup \{\{z\} \cup E: E \in \mathcal{A}\}$ and

$S_2 = \{\{x, y\} \cup F: F \in \mathcal{F}\} \cup \{\{y, z\} \cup F: F \in \mathcal{F}\} \cup \{\{x, z\} \cup F: F \in \mathcal{F}\}$,

where $\mathcal{F} = \mathcal{B} \setminus \{\{A, B, C\}\}$. Then the required family of blocks follows from the 87 blocks in $S_1 \cup S_2$ and the following 18 blocks:

A	B	C	0	2	mod 6					
x	y	z	A	B		A	1	2	4	5
x	y	z	B	C		B	2	3	4	5
x	y	z	C	A		C	0	1	2	5
A	0	2	3	5		A	0	1	3	4
B	0	1	2	3		B	0	1	4	5
C	0	3	4	5		C	1	2	3	4

For $v = 17$, let the point-set be $Z_{14} \cup \{x, y, z\}$. The blocks then are obtained by developing under the action of Z_{14} the following base blocks:

0	1	2	4	6		y	0	1	5	7
0	2	4	5	8		y	0	3	7	11
0	1	4	6	9		y	0	5	11	13
0	3	4	5	8		z	0	3	7	8
x	0	1	2	5		z	0	3	6	10
x	0	2	4	7		z	0	4	5	6
x	0	2	3	8		x	y	z	0	1
x	0	2	7	9		(orbit length 7)				
y	0	2	7	9		(orbit length 7)				
z	0	1	7	8		(orbit length 7)				

Lemma 4.4. *If $v \in \{14, 19, 24\}$, then $D_{16}(v) = B_{16}(v)$.*

Proof: For given values of v, let $V = Z_{v-1} \cup \{\infty\}$. We first construct a $(v - 1, 5, 12)$ packing of Z_{v-1} with $B_{12}(v-1)$ blocks. Then we construct a configuration $\mathcal{D}$ on V such that each pairset of V containing the point ∞ occurs in exactly 16 blocks in $\mathcal{D}$ and each other pairset of V occurs in exactly 4 blocks in $\mathcal{D}$. This gives rise to the result, where $\mathcal{D}$ consists of the following blocks:

For $v = 14$:	∞	0	1	3	9	mod 13	(taken 4 times)
For $v = 19$:	0	1	5	7	10	mod 18	
	∞	0	2	4	7	mod 18	
	∞	0	5	6	8	mod 18	
	∞	0	1	4	10	mod 18	
	∞	0	4	10	11	mod 18	
For $v = 24$:	0	1	4	6	13	mod 23	(taken twice)
	∞	0	2	5	8	mod 23	
	∞	0	4	12	13	mod 23	
	∞	0	7	8	12	mod 23	
	∞	0	8	10	17	mod 23	

Lemma 4.5. $D_{16}(34) = B_{16}(34)$.

Proof: Let $V = Z_{27} \cup \{\infty_i : 0 \le i \le 6\}$. A $(34, 7; 5, 16)$-IPBD can be constructed by developing under the action of Z_{27} the following base blocks:

∞_0	0	10	11	13	(taken 4 times)				
∞_1	0	12	17	21	(taken 4 times)				
∞_2	0	1	3	7	∞_5	0	1	7	16
∞_2	0	3	8	14	∞_5	0	3	14	18
∞_2	0	5	7	8	∞_5	0	5	8	15
∞_2	0	2	8	9	∞_5	0	2	9	13
∞_3	0	2	6	15	∞_6	0	1	3	16
∞_3	0	5	11	15	∞_6	0	3	8	18
∞_3	0	2	3	10	∞_6	0	5	7	15
∞_3	0	6	7	11	∞_6	0	2	8	13
∞_4	0	3	7	16	0	1	3	7	16
∞_4	0	8	14	18	0	3	8	14	18
∞_4	0	7	8	15	0	5	7	8	15
∞_4	0	8	9	13	0	2	8	9	13

So the result follows from Lemma 2.9 and Lemma 4.3.

Lemma 4.6. *If* $v \equiv 2 \pmod 5$, $v \geq 5$ *and* $v \neq 22, 27$, *then* $B_{16}(v) = D_{16}(v)$.

Proof: By Lemma 4.3, it is sufficient to establish the lemma for $v \geq 32$. As we did in the proof of Lemma 3.6, we can apply Lemma 2.4 with $w = 2$ to give the conclusion for $v \geq 52$. For $32 \leq v \leq 47$, we use Lemma 2.9 by a suitable IPBD. First note that $37 \in IP_{16}(9)$ by Lemma 2.2. We readily obtain that $42 \in IP_{16}(7)$ by deleting two points from one group in a $TD(6,1,7)$ and adding two infinite points to each group of the resulting GDD. Taking $K = \{5\}$ and $(u,e,q,m,\lambda) = (40,12,1,2,8), (25,2,6,2,8)$ in Lemma 2.6, we also obtain that $\{32,47\} \subset IP_{16}(7)$, where we make use of a $(53,13;5,8)$-IPBD in Lemma 2.2 and a $(33,8;5,8)$-IPBD in Theorem 4.2 in [1]. The required fact that $31 \in IP_8(6)$ follows from a $B(6,1;31)$ and a $B(5,8;6)$.

Lemma 4.7. *If* $v \equiv 4 \pmod 5$ *and* $v \geq 5$, *then* $D_{16}(v) = B_{16}(v)$.

Proof: It is shown in Lemmas 4.3–4.5 that $D_{16}(v) = B_{16}(v)$ if $v = 9, 14, 19, 24$, or 34. It is known that $\{29,39,49,79\} \subset IP_2(7)$ (see Lemma 4.3 [6]), and hence $\{29,39,49,79\} \subset IP_{16}(7)$. We can also obtain that $44 \in IP_{16}(9)$ using a $TD(6,1,7)$ and the fact that $9 \in IP_{16}(2)$ in Lemma 2.7. Since a $B(5,16;10)$ exists from Lemma 2.1, adjoining one infinite point to a $(5,16)$-GDD of type $9^5 8^1$ which comes from a $TD(6,1,9)$, a $(54,9;5,16)$-IPBD is obtained. So the conclusion holds for $v \in \{29,39,44,49,54,79\}$ by making use of Lemma 2.9. The remaining cases for $9 \leq v \leq 79$ are covered by applying Lemma 2.7 and Lemma 2.5 with $t = 2$, $(a,b,c) = (0,1,1), (0,2,1), (2,1,1)$ and $(2,2,1)$.

We also apply Lemma 2.7 and Lemma 2.5 with $t \geq 3$. This guarantees that $D_{16}(v) = B_{16}(v)$ whenever $v \geq 84$ and the proof is complete.

We wish to remark that $D_\lambda(v) \leq B_\lambda(v) - 1$ when $\lambda \equiv 12 \pmod{20}$ and $v \equiv 2$ or $4 \pmod{5}$. The proof is similar to that of Lemma 4.2. The foregoing can be summarized as follows.

Theorem 4.8. *If* $v \equiv 2$ *or* $4 \pmod{5}$ *and* $v \geq 5$; $\lambda \equiv 0 \pmod{4}$, *then*

$$D_\lambda(v) = \begin{cases} B_\lambda(v) & \text{when } \lambda \equiv 4 \pmod{20} \text{ and } (v,\lambda) \neq (7,4), \\ B_\lambda(v) - 1 & \text{when } \lambda \equiv 12 \pmod{20} \text{ or } (v,\lambda) = (7,4), \\ B_\lambda(v) & \text{when } \lambda \equiv 0 \text{ or } 8 \pmod{20}, \\ B_\lambda(v) & \text{when } \lambda \equiv 16 \pmod{20} \text{ and } v \neq 22 \text{ or } 27. \end{cases}$$

5. Miscellaneous Values

The purpose of this section is to handle the remaining cases for $v = 22, 27$ and $\lambda \equiv 16 \pmod{20}$.

Lemma 5.1. $D_{56}(22) = B_{56}(22)$.

Proof: First note that Hanani [5] has constructed directly a $TD(8,2,4)$. By the appropriate deletion of points from this TD, we readily obtain a $(\{5,6,7\},2)$-GDD of type $3^6 4^1$. Using Lemma 2.1, we then get a $(5,40)$-GDD of type $3^6 4^1$. Let us label the GDD as $(\mathcal{X}, \mathcal{G}, \mathcal{A})$, where $|\mathcal{X}| = 22$, $\mathcal{G} = \{G_i : 1 \leq i \leq 6\} \cup \{Z_4\}$. On each set $G_i \cup G_j$, $1 \leq i < j \leq 6$, we construct a $B(5,4;6)$ with block set $\mathcal{B}_{ij}$. On $\mathcal{X} \backslash Z_4$, we construct an $(18,5,12)$ maximum packing with block family $\mathcal{B}$. For $1 \leq i \leq 6$ and $0 \leq t \leq 3$, we construct a $B(5,4;6)$ with block set $\mathcal{A}_{it}$ on the set $G_i \cup \{t, t+1, t+2\}$, where $t+1$ and $t+2$ are taken module 4. Let $\mathcal{C}$ consist of the following blocks:

$$\begin{array}{ll} G_1 \cup \{0,1\} & G_4 \cup \{0,1\} \\ G_1 \cup \{2,3\} & G_4 \cup \{2,3\} \\ G_2 \cup \{0,2\} & G_5 \cup \{0,2\} \\ G_2 \cup \{1,3\} & G_5 \cup \{1,3\} \\ G_3 \cup \{0,3\} & G_6 \cup \{0,3\} \\ G_3 \cup \{1,2\} & G_6 \cup \{1,2\} \end{array}$$

Let $\mathcal{D} = \{G_i \cup \{a,b\} : (a,b) = (0,1), (0,2), (0,3), (1,2), (1,3)$ or $(2,3)$, $i = 1,2,\ldots,6\}$. It is then a straightforward verification that

$$\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \left(\bigcup_{1 \leq i < j \leq 6} \mathcal{B}_{ij} \right) \cup \left[\bigcup_{1 \leq i \leq 6} \left(\bigcup_{0 \leq t \leq 3} \mathcal{A}_{it} \right) \right]$$

is a 56-packing on $\mathcal{X}$ with $B_{56}(22)$ blocks.

Lemma 5.2. $D_{36}(27) = B_{36}(27)$.

Proof: It is well known that a resolvable $B(5,1;25)$ exists. Delete one point from the design. This gives rise to a $(5,1)$-GDD of type 4^6, say $(V,\mathcal{G},\mathcal{A})$, in which the set of blocks can be partitioned into holey parallel class, each of which is a partition of $V\backslash G$ for some $G \in \mathcal{G}$. Write $\mathcal{G} = \{G_1, G_2, \ldots, G_6\}$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_6$ denote these holey parallel classes, where $\mathcal{A}_i$ partitions $V\backslash G_i$ $(1 \leq i \leq 6)$. To the required result, we use this GDD together with three new points x, y and z and proceed as follows.

(1) For $1 \leq i \leq 5$, on sets $\{x,y\} \cup G_i$, $\{y,z\} \cup G_i$ and $\{x,z\} \cup G_i$, we separately construct a $B(5,4;6)$.
(2) For each $A \in \mathcal{A}_i$ $(1 \leq i \leq 6)$, on sets $A \cup \{x\}$, $A \cup \{y\}$ and $A \cup \{z\}$, we separately construct a $B(5,4;6)$.
(3) For $1 \leq i \leq 5$, on sets $\{x\} \cup G_i$, $\{y\} \cup G_i$ and $\{z\} \cup G_i$, we separately construct a $B(5,8;5)$.
(4) On $G_6 \cup \{x,y,z\}$, we construct a $(7,5,16)$ maximum packing.
(5) For each $A \in \mathcal{A}_6$, we construct a $B(5,5;9)$ on $A \cup G_6$, and then copy each block in $\mathcal{A}_i$ $(1 \leq i \leq 5)$ 5 times.
(6) Construct a $(5,5)$-GDD of type 4^5 with group set $\{G_1, G_2, \ldots, G_5\}$.
(7) Construct a$(5,14)$-GDD of type 4^6 with group set $\{G_1, G_2, \ldots, G_6\}$.

Thus, a 36-packing with $B_{36}(27)$ blocks on $V \cup \{x,y,z\}$ is formed when the total collection of blocks from the above systems is taken.

Combining the results of Lemmas 2.1, 2.8, 5.1, and 5.2, we have established the following.

Theorem 5.3. $D_{20m+16}(22) = B_{20m+16}(22)$ *and* $D_{20n+16}(27) = B_{20n+16}(27)$ *where* $m \geq 2$ *and* $n \geq 1$.

6. Concluding Remarks

Summarizing the previous results, we have already determined $D_\lambda(v)$ for all positive integers $\lambda \equiv 0 \pmod 4$ and $v \geq 5$ with exceptions of $(v,\lambda) = (22,16)$, $(22,36)$ and $(27,16)$. The author would like to thank the referee for pointing out the result for $\lambda = 8, 12, 16$, with a few possible exceptions, has been obtained (in a paper to appear in J.C.T.) by A. M. Assaf and N. Shalaby independently.

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