On Uniform Subset Graphs

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Abstract. Certain graphs whose vertices are some collection of subsets of a fixed n-set, with edges determined by set intersection in some way, have long been conjectured to be Hamiltonian. We are particularly concerned with graphs whose vertex set consists in all subsets of a fixed size k, with edges determined by empty intersection, on the one hand, and with bigraphs whose vertices are all subsets of either size k or size $n-k$, with adjacency determined by set inclusion, on the other. In this note, we verify the conjecture for some classes of these graphs. In particular, we show how to derive a Hamiltonian cycle in such a bigraph from a Hamiltonian path in a quotient of a related graph of the first kind (based on empty intersection). We also use a recent generalization of the Chvátal-Erdös theorem to show that certain of these bigraphs are indeed Hamiltonian.

1. Introduction.

All graphs in this note are finite simple graphs unless otherwise specified, there being a few cases where loops or multiple edges are useful. Supporting definitions and theorems may be found in [7] or [3]. Notation generally follows [3].

Graphs based on families of subsets appear in the literature in many contexts, such as Kneser graphs [10] or in the study of Johnson schemes [12] to mention two. These examples and others are described in [4]. There Chen and Lih call a triple (n, k, t) of integers admissible if $0 \leq t < k < n$ and $n \geq 2k - t$, with strict inequality when $t = 0$. For every admissible triple they define the uniform subset graph $G(n, k, t)$ to have as vertices all k-sets of a fixed n-set, which for convenience we take to be the integers from 1 to n. An edge xy between x and y exists if $|x \cap y| = t$. The restriction to admissible triples guarantees that G will be connected. In fact, Chen and Lih show that G is regular of degree $\binom{k}{t}\binom{n-k}{k-t}$, which is also its connectivity. They also show that each G has an edge transitive automorphism group. Their big conjecture is that except for $G(5,2,0)$ and $G(5,3,1)$, each of which is a copy of the Petersen graph, all $G(n, k, t)$ are Hamiltonian. Earlier writers have conjectured that every $G(2k+1, k, 0)$, $k > 2$, is Hamiltonian. Using the easily verified fact that $G(6,3,1)$ is Hamiltonian, they reduce their conjecture to the simpler form: Every $G(n, k, 0)$ is Hamiltonian for $n > 2k$, excepting, of course, $G(5,2,0)$. They also verify the conjecture for certain infinite families of these graphs. In particular, they display functions $e(k)$ and $f(k)$ such that $G(n, k, 0)$ is Hamiltonian for all $n \geq e(k)$, and $G(n, k, 1)$ is Hamiltonian for all $n \geq f(k)$.

In [13] the author considers a related family of bipartite graphs $H(n,k,t)$ defined as follows. For each admissible triple (n,k,t) let vertex sets S and T respectively, be all k-sets and all $(n-k)$-sets of a fixed n-set, with xy' an edge exactly when $|x \cap y'| = k - t$. (We use unprimed letters for elements of S and primed letters for elements of T throughout.) These graphs are all conjectured to be Hamiltonian, and some cases verified, in [13]. The connection between each $H(n,k,t)$ and the corresponding $G(n,k,t)$ is also explained there. In fact, any graph G has an associated bipartite graph (bigraph), denoted by $VV(G)$ in [8], and by $K_2 \wedge G$ in [7]. The construction is a special case of the conjunction of two graphs, due to Miller [11]. In [13] it is shown that for every admissible triple $H(n,k,t) = VV(G(n,k,t))$, from which it follows that if $G(n,k,t)$ is Hamiltonian and $\binom{n}{k}$ is odd, then $H(n,k,t)$ is Hamiltonian. It is also shown that each $H(n,k,t)$ has an edge transitive automorphism group. In this note we extend to $H(n,k,t)$ most of Chen and Lih's theorems about $G(n,k,t)$ and show how certain properties of $G(n,k,t)$ may be used to generate a Hamiltonian cycle in $H(n,k,t)$.

2. Induction.

One of Chen and Lih's most useful results in an induction theorem which asserts that if both $G(n,k,t)$ and $G(n,k+1,t+1)$ are Hamiltonian then so is $G(n+1,k+1,t+1)$. It is this result, combined with a natural isomorphism via complements between $G(n,k,t)$ and $G(n,n-k,n-2k-t)$, that leads to the simplification of the general conjecture mentioned above. We will prove an analogous theorem for $H(n,k,t)$'s at the same time correcting an error (which seems to be more notational than conceptual) in the proof in [4]. In both situations the key lemma is an inequality about the maximum cycle length (circumference) of each graph. We denote the circumference of $G(n,k,t)$ by $c(n,k,t)$, and that of $H(n,k,t)$ by $C(n,k,t)$.

Theorem 1. *For any admissible triples, with $p = c$ or $p = C$ throughout:*

(a) $p(n,k,t) \le p(n+1,k,t)$.
(b) $p(n,k,t) \le p(n+1,k+1,t+1)$.
(c) $p(n,k,t) = p(n,n-k,n-2k+t)$.
(d) $p(n,k,t) + p(n,k+1,t+1) \le p(n+1,k+1,t+1)$.

Proof: Parts (a), (b), and (c), are proved as in [4]. For (a) $G(n,k,t)$ and $H(n,k,t)$ may be embedded, respectively, in $G(n+1,k,t)$ and $H(n+1,k,t)$. For G the mapping is given by $f(x) = x$ for $x \in V(G)$. For each H the same function is used for $x \in S$, extended by $f(x') = x' \cup \{n+1\}$ for $x' \in T$.

For (b) use the function given by $g(x) = x \cup \{n+1\}$ for $x \in V(G)$ to embed $G(n,k,t)$ in $G(n+1,k+1,t+1)$. Use the same function on S, extended to T by $g(x') = x'$, for H.

For (c) taking complementary sets establishes the isomorphism, as already observed about G's. For the H's taking complements essentially interchanges S and T.

(d1) This is the case that requires some correction, for $p = c$, which is to say, for the G's. In $G(n, k+1, t+1)$ find a cycle of maximum length and embed it as C_1 in $G(n+1, k+1, t+1)$ using f as in (a). For any edge $xy \in C_1$ choose element $a_0 \in x \cap y$, $a_1 \in (x-y)$ and $b_1 \in (y-x)$. Now, let $v = y \cup \{a_1, n+1\} - \{a_0, b_1\}$ and $u = x \cup \{n+1\} - \{a_1\}$. Then $|v \cap x| = |u \cap y| = |u \cap v| = t+1$. Hence, the edges xy, xv, yu, and uv all exist. By the edge transitivity of $G(n, k, t)$ choose a cycle of maximum length whose image C_2 under g (from (b)) uses uv. Note that C_1 and C_2 have no vertices in common. Replace xy and uv by xv and yu to change $C_1 \cup C_2$ into a cycle of length $c(n, k, t) + c(n, k+1, t+1)$ in $G(n+1, k+1, t+1)$.

(d2) For $p = C$ partition the vertex set $S \cup T$ of $H(n+1, k+1, t+1)$ into four sets with $S_1 \cup S_2 = S$ and $T_1 \cup T_2 = T$ so that vertices in S_2 and T_1 are the sets that contain the element $n+1$, while vertices in S_1 and T_2 do not. As in (d1) let C_1 be the image under f of a maximum length cycle in $h(n, k+1, t+1)$. All of its vertices are in $S_1 \cup T_1$. For any edge xy' of C_1, x is a $(k+1)$-set in S_1 and y' is an $(n-k)$-set in T_1, with $n+1 \in (y'-x)$ and $|x \cap y'| = k-t$. Now choose elements $a \in (x-y')$ and b not in $x \cup y'$. Let $u = x \cup \{n+1\} - \{a\}$ and $v' = y' \cup \{b\} - \{n+1\}$ so that $u \in S_2$ and $v' \in T_2$. From $|u \cap v'| = |x \cap v'| = |y' \cap u| = k-t$ all edges uv', xv', and uy' exist. As in (d1) find a cycle C_2 which is the image under g of a maximum length cycle in $H(n, k, t)$ that uses uv'. Replace xy' and uv' by xv' and $y'u$ to convert $C_1 \cup C_2$ to a cycle, as in case (d1), to complete the proof. ∎

Theorem 2. *If $H(n, k, t)$ and $H(n, k+1, t+1)$ are Hamiltonian then so is $H(n+1, k+1, t+1)$.*

Proof: The images under f and g, respectively, of the vertex sets of $H(n, k, t)$ and $H(n, k+1, t+1)$ have disjoint union equal to the vertex set of $H(n+1, k+1, t+1)$. Part (d2) of Theorem 1 proves the theorem. ∎

Theorem 3. *Fix $k > 0$. If $H(n, k, 0)$ is Hamiltonian for all $n \geq n_0$ and if $H(n_0, k+r, r)$ is Hamiltonian for $r = 0, 1, \ldots, n_0 - 2k$, then $H(n, k+r, r)$ is Hamiltonian for all $n \geq n_0$ and $r = 0, 1, \ldots, n-2k$.*

Proof: Follows by induction from Theorem 2 just as in [4]. ∎

Theorem 4. *$H(n, k, t)$ is Hamiltonian for every admissible triple (n, k, t) if and only if $H(n, k, 0)$ is Hamiltonian for every $n > 2k > 0$.*

Proof: One direction is trivial. Converely, if $H(2k+1, k, 0)$ is Hamiltonian, then so is $H(2k+1, k+1, 1)$ by Theorem 1(c). Theorem 3, with $n_0 = 2k+1$, completes the proof. ∎

3. Special Cases.

The problem of showing that for each k and every $n > 2k$, $H(n,k,0)$ is Hamiltonian seems to be a hard one, of greatest difficulty when $n = 2k+1$. For $k = 1$ and $n \geq 3$, $H(n,1,0)$ is $K_{n,n}$ with a perfect matching removed, and certainly Hamiltonian. By Theorem 3 $H(n,k,k-1)$ is Hamiltonian for every admissible $(n,k,k-1)$.

For $k = 2$, we will show shortly that $H(n,2,0)$ is Hamiltonian for all $n \geq 8$. The case $n = 5$ is well-known, appearing in both [6] and [13]. A technique for handling many such cases, including $n = 5, 6$, and 7, among others, is discussed in the next section. From these results it follows that every (admissible) $H(n,k,k-2)$ is Hamiltonian.

For larger k less is known. But it is a general fact that the problem becomes easier, for each k, as n grows. This is a consequence of the next theorem, whose proof depends on variations of two well-known theorems. One, By Chvátal and Erdös, asserts the existence of a Hamiltonian cycle in a graph if its independence number is no larger than its connectivity. The other is a special case of a theorem of Hilton and Milner [9] which we use to obtain an independence result about $H(n,k,0)$. The first variation is found in [13], the second in [14]. Some definitions are required. An arbitrary bigraph H with vertex set $S \cup T$ is balanced if $|S| = |T|$. The cross-independence number of any bigraph is the size of a largest independent set of vertices U such that neither $U \cap S$ nor $U \cap T$ is empty. The theorem, in [13], is that for a balanced bigraph if its cross-independence number is no larger than its connectivity, then the graph is Hamiltonian. The special case of Theorem 2 of [9, p. 370] (see also [14]) obtained by setting $p = 1$ and using the complements of our B_j's as the B_j's in [9] implies that the cross-independence number of $H(n,k,0)$ is $1 + \binom{n}{k} - \binom{n-k}{k}$. Combining these results with one more definition leads to the next theorem. For each k let $h(k) = \min\left\{n > 2k\colon \binom{n}{k} < 2\binom{n-k}{k}\right\}$.

Theorem 5. *For $k > 0$, if $n \geq h(k)$ then $H(n,k,0)$ is Hamiltonian.*

Proof: Each $H(n,k,0)$ is connected and regular of degree $\binom{n-k}{k}$, and has an edge transitive automorphism group. Hence, just as in [4], its connectivity is also $\binom{n-k}{k}$. Combining this with the theorems just quoted yields the result that $H(n,k,0)$ is Hamiltonian whenever $\binom{n}{k} < 2\binom{n-k}{k}$. But the ratio $b(n,k) = \binom{n}{k} / \binom{n-k}{k}$ decreases monotonically to 1 as n increases. ∎

The values of $h(k)$ for $k = 1$ to 10, respectively, are 3, 8, 16, 27, 41, 58, 78, 101, 126, and 154. A slightly weaker form of the theorem, with an explicit limit on n is given next.

Theorem 6. *$H(n,k,0)$ is Hamiltonian for $n \geq (3k^2 + k + 2)/2$.*

Proof: It is enough to show that $b(n,k) < 2$ for $n = (3k^2 + k + 2)/2$. For this value of n the inequality reduces to $b_k < 2$ with

$$b_k = \frac{(3k^2+k+2)(3k^2+k)\ldots(3k^2-k+4)}{(3k^2-k+2)(3k^2-k)\ldots(3k^2-3k+4)}.$$

One way to proceed is to observe that

$$b_k < \left(\frac{3k^2-k+4}{3k^2-3k+4}\right)^k = \left(1+\frac{2k}{3k^2-3k+4}\right)^k = \left(1+\frac{a_k}{k}\right)^k = c_k.$$

where $a_k = \frac{2k^2}{3k^2-3k+4}$. One may calculate directly that $c_k < 2$ for small values of k. Further, a_k is monotone decreasing to 2/3. Hence, for all $k \geq 25$

$$c_k < \left(1+\frac{1}{k}(a_{25})\right)^k = d_k,$$

with $a_{25} = 625/902$. But d_k is monotone increasing, for $k \geq 25$, to $e^{(a_{25})} < 2$. Hence, $b_k < 2$ for $k \geq 25$. Direct verification of $b_k < 2$ for $1 \leq k \leq 24$ finishes the proof. ∎

From this theorem and Theorem 3 it follows that every admissible $H(n,k,k-3)$ is Hamiltonian once it is verified that $H(n,3,0)$ is Hamiltonian for $n = 7, 8, \ldots, 15$.

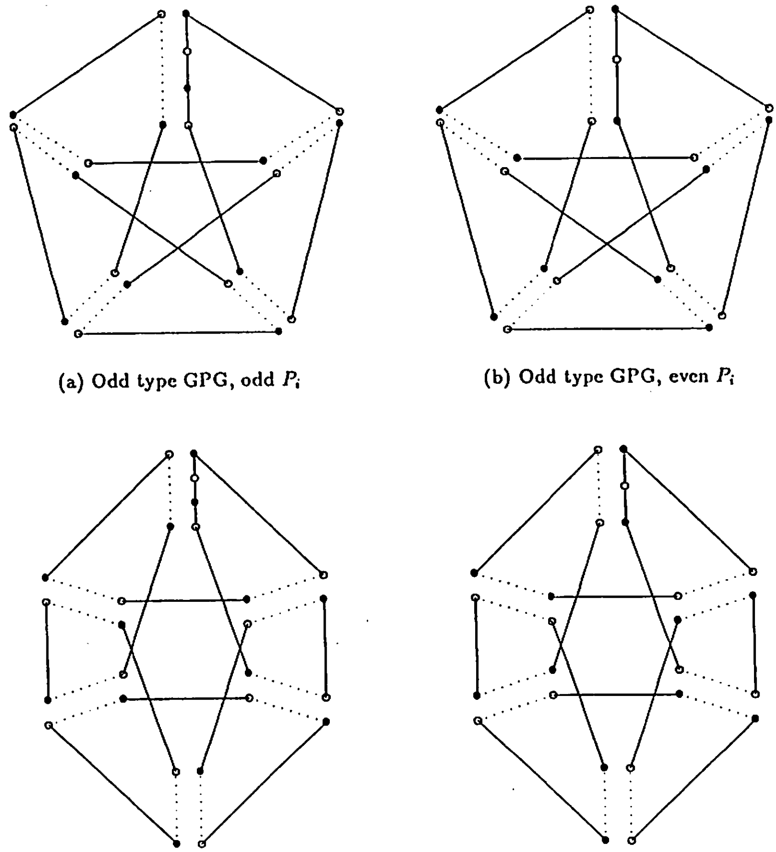
4. Generalized Petersen graphs.

In this section we extend and simplify some of the results found independently by Dejter [5] and some of his students [6]. The simplification comes primarily from exploiting the connections between G and $VV(G)$ for $G = G(n,k,0)$. The improvements lie in removing restrictions on n used in [5] and in phrasing results for wider application. We begin with a new class of graphs.

A generalized Petersen graph (GPG) may be of either odd or even type. A GPG of odd type n is any graph all of whose vertices fall on one of two disjoint odd length cycles, C_1: $x_1x_2\ldots x_nx_1$ and C_2: $y_1y_3\ldots y_ny_2y_4\ldots y_{n-1}y_1$, or on one of n disjoint paths P_i: $x_i\ldots y_i$, $1 \leq i \leq n$, which have only their endpoints on the cycles. The classical Petersen graph is a GPG of type 5 in which each P_i is a single edge. A GPG of even type n is any graph all of whose vertices fall either on an even length cycle C_1: $x_1x_2\ldots x_nx_1$, or on either of two cycles each of length $n/2$, C_2: $y_1y_3\ldots y_{n-1}y_1$, or C_3: $y_2y_4\ldots y_ny_2$, or on one of n disjoint paths P_i: $x_i\ldots y_i$ having only their endpoints on the cycles. As with the odd type, the cycles are disjoint. A GPG of type 6 with 12 vertices may be constructed from a cycle on 6 vertices and two copies of K_3 by joining the vertices of each K_3 to alternating vertices of the cycle. Roughly speaking, a GPG is spanned by an n-gon surrounding an n-star with the vertices of the star and the n-gon joined by paths. There is no restriction on the existence of other edges. The important point is that the cycles and paths span the graph.

Theorem 7. *Let G be any GPG. If every path P_i of the definition has even length, or if every such path has odd length, then $VV(G)$ is Hamiltonian.*

Proof: There are four cases, depending on whether G is of odd or even type and whether the P_i's are of odd or even length. Hamiltonian cycles are shown for $VV(G)$ in Figure 1 using solid dots for vertices in S and open dots for vertices in T. Graphs of type $n = 5$ and $n = 6$ with all odd P_i and all even P_i are shown in the four diagrams. The constructions shown are clearly valid for larger values of n. ∎



(a) Odd type GPG, odd P_i

(b) Odd type GPG, even P_i

(c) Even type GPG, odd P_i

(d) Even type GPG, even P_i

Figure 1: Examples of GPG's

Theorem 8. *Let G be a GPG of even order $n = 4s$. Then G is Hamiltonian.*

Proof: The pattern shown in Figure 2 for $n = 8$ extends readily to any n if $n \equiv 0 \pmod 4$. ∎

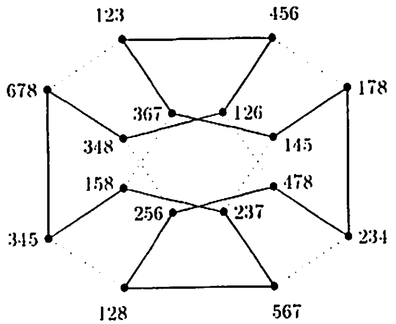


Figure 2: Theorem 8, with labels for $G(10,3,0)/(8)$

We conjecture that for every GPG G, $VV(G)$ is Hamiltonian. The only difference here is that some paths may be odd and some even. We have verified the conjecture for a number of cases, but the only ones we need for the present application are those given in the theorems.

Virtually everyone who has worked on the problem of finding a Hamiltonian cycle in some $G(n,k,t)$ or $H(n,k,t)$ has used one or another quotient graph induced by permuting the ground set (the integers from 1 to n). In [6] one even finds a quotient of a quotient put to effective use. For our purposes we need only permutations given by a single cycle: $(1,2,3\ldots s)$ for $2 \leq s \leq n$. We denote the corresponding permutation by (s) and the quotient graph by $G(n,k,t)/(s)$ or $H(n,k,t)/(s)$. It is sometimes useful to treat these quotients as labelled digraphs. In that case a particular member of each orbit is chosen to be the representative element, denoted by $x \cdot 0$ for vertex x. Any other element of that orbit is identified as $x \cdot r$, where r is the smallest power of the permutation (s) that carries $x \cdot 0$ into $x \cdot r$. An arc from x to y in the quotient graph is labelled with integer j if $x \cdot 0$ is adjacent to $y \cdot j$ in the original graph. Since $x \cdot 0$ may be adjacent to more than one element of the orbit y, or even to an element of its own orbit, the quotient digraph may have arcs or loops with multiple labels. An arc from x to y with label j has a companion from y to x with label $s - j$. Only one of each such pair is shown in the figures.

The next theorem is a variation of Theorem 10 of [5].

Theorem 9. *For $n > 2k \geq 4$, let $s = n$ or $n-1$, depending on whether n is odd or even. Suppose that s and k are relatively prime and that there is a Hamiltonian path in $G(n,k,0)/(s)$ from x to y, where $x \cdot 0$ is adjacent to $x \cdot u$, and $y \cdot 0$ is adjacent to $y \cdot v$ with $v = 2u < s$. Then $G(n,k,0)$ is a GPG of type s and $H(n,k,0)$ is Hamiltonian.*

Proof: The restrictions on k guarantee that every orbit has size s. The orbits x and y provide the cycles C_1 and C_2 while the Hamiltonian path from x to y provides the s paths required to show that $G(n,k,0)$ is a GPG of odd type s. Since every path has the same length, Theorem 7 shows that $H(n,k,0)$ is Hamiltonian. ∎

The requirements on x and y are not restrictive, for we may let x have as its representative $x \cdot 0 = \{1,2,\ldots,k\}$, and let y have as its representative $y \cdot 0 = \{1,3,5,\ldots,2k-1\}$. If n is odd and if we form the quotient $G(n,k,0)/(n)$ then $x \cdot 0$ is adjacent to $x \cdot u$ for $u = (n-1)/2$, and $y \cdot 0$ is adjacent to $y \cdot v$ for $v = n-1$. When n is even, using the quotient $G(n,k,0)/(n-1)$, $x \cdot 0$ is adjacent to $x \cdot u$ for $u = (n-2)/2$, and $y \cdot 0$ is adjacent to $y \cdot v$ for $v = n-2$. Thus, finding a Hamiltonian path for this x and y is always sufficient.

A few simple examples will clarify the construction. The most classic of course is $G(5,2,0)/(5)$ which has only the two adjacent vertices x and y, shown in Figure 3(a). (Each vertex is identified by its representative element.) Each of $G(6,2,0)/(5)$ and $G(7,2,0)/(7)$ is an oriented version of K_3, with some loops, as shown in Figures 3(b) and (c). These examples complete our verification that every $H(n,2,0)$ is Hamiltonian.

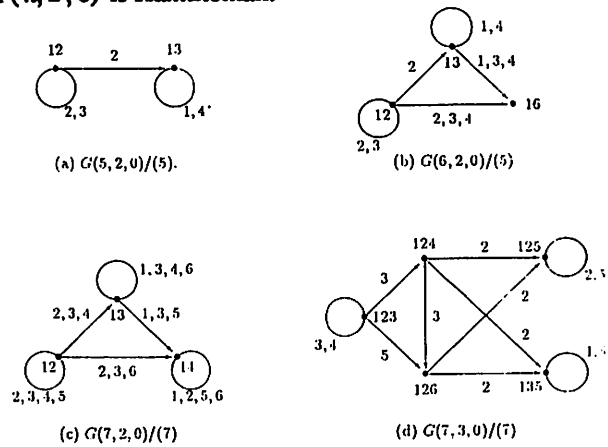


Figure 3: Application of Theorem 9 to $H(n,2,0)$.

For $k = 3$, $G(7,3,0)/(7)$ is in Figure 3(d), where the existence of the desired Hamiltonian path is clear. In fact, one may use a path from x with $x \cdot 0 = 123$ to y with $y \cdot 0 = 135$, relying on the arcs from $x \cdot 0$ to $x \cdot 3$ and $y \cdot 0$ to $y \cdot 6$. Or one may use a path that relies on the arcs from $z \cdot 0 = \{125\}$ to $z \cdot 2$ and $x \cdot 0$ to $x \cdot 4$, or one that relies on the arcs from $y \cdot 0$ to $y \cdot 1$ and $z \cdot 0$ to $z \cdot 2$. A similar construction of a Hamiltonian path is relatively easy to develop in each $G(n,3,0)/(s)$ for the cases $n = 11$ or 12 $(s = 11)$, and $n = 13$ or 14 $(s = 13)$. In fact, the construction used in the theorem works quite well for $G(9,3,0)/(7)$ and for $G(15,3,0)/(13)$.

That is, it is not really necessary for s to be n or $n-1$. Having $s > 2k$, $s > n-k$ and s odd and relatively prime to k is sufficient. Only $H(10,3,0)$ is yet to be considered, to show that every $H(n,3,0)$ is Hamiltonian. It then will follow from Theorem 3 that every admissible $H(n,k,k-3)$ is Hamiltonian.

The graph $G(10,3,0)$ may be realized as a GPG of order 8. The vertices of the quotient $G(10,3,0)/(8)$ are of two kinds. There are 13 vertices representing orbits of size 8 and two of size 4. These two may be joined into a single "orbit" whose elements $z \cdot 0$ to $z \cdot 7$ are $159, 269, 379, 489, 15A, 26A, 37A$, and $48A$. The cycle C_1 is given by the orbit of 123 under $(8)^3$. The cycles C_2 and C_3 are obtained by splitting the orbit of 126 into the orbits under $(8)^3$ of 367 and 126, respectively. Paths P_i comes from repeated application of $(8)^3$ to P_1 where

$$P_1: 123, 459, 12A, 359, 47A, 235, 19A, 247, 135, 467, 125, 489, 367.$$

Figure 2 is labelled to match this example. It follows from Theorem 7 and Theorem 8 that both $G(10,3,0)$ and $H(10,3,0)$ are Hamiltonian.

We conjecture that every $G(n,k,0)$ may be realized as a GPG, possibly with some even and some odd length P_i's. We may remark that a Hamiltonian cycle in $G(2k+1,k,0)/(2k-1)$, if its total length is relatively prime to $2k+1$, may be used to generate a Hamiltonian cycle in $G(2k+1,k,0)$. In any event, it is clear that finding a Hamiltonian path in $G(n,k,0)$ is an effective way to generate a Hamiltonian cycle in many of these graphs.

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