On Symmetric Block Designs (45, 12, 3) with Automorphisms of Order 5

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Dedicated to Professor Zvonimir Janko on his 60th birthday

1. Introduction and preliminaries.

The aim of this paper is to prove the following

Theorem. *Let $\mathcal{D}$ be a symmetric block design* $(45, 12, 3)$ *admitting an automorphism of order* 5. *Then $\mathcal{D}$ is isomorphic to one of the 13 designs listed in the Table 1. The generators and orders of their automorphism groups are listed in the Table 2. All of the 13 designs from the Table 1 are mutually non-isomorphic. However, there are dual isomorphisms among them which are presented in the Table 3.*

This result was obtained by means of combinatorial and group theoretical methods and with help of a computer.

At the beginning we recall some basic definitions and facts related with symmetric block designs (see for example [1], [4]). We assume all sets under consideration to be finite.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an incidence structure with point set $\mathcal{P}$, line set $\mathcal{B}$ and incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$. For $P \in \mathcal{P}$, $x \in \mathcal{B}$ denote $\langle P \rangle = \{y \in \mathcal{B} \mid (P, y) \in I\}$, $\langle x \rangle = \{Q \in \mathcal{P} \mid (Q, x) \in I\}$, $|P| = |\langle P \rangle|$, $|x| = |\langle x \rangle|$.

Definition 1: A *symmetric block design* (v, k, λ), $v, k, \lambda \in \mathbf{N}$, $n = k - \lambda > 0$, is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ such that:

(i) $|\mathcal{P}| = |\mathcal{B}| = v = k(k-1)/\lambda + 1$;

(ii) $|x| = |P| = k$ for all $x \in \mathcal{B}$, $P \in \mathcal{P}$;

(iii) $|\langle x \rangle \cap \langle y \rangle| = |\langle P \rangle \cap \langle Q \rangle| = \lambda$, for all $x, y \in \mathcal{B}$, $P, Q \in \mathcal{P}$ with $x \neq y$, $P \neq Q$.

The difference $n = k - \lambda$ is called the *order* of $\mathcal{D}$.

For two such designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$ an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and lines onto lines preserving the incidences. Similarly, dual isomorphisms are such bijections which map points onto lines and lines onto points and preserve the incidences. An isomorphism from $\mathcal{D}$ onto $\mathcal{D}$ is an automorphism of $\mathcal{D}$. In the following we shall use the term *design* for symmetric block designs.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a (v, k, λ)-design and G an automorphism group of $\mathcal{D}$, that is $G \leq \operatorname{Aut} \mathcal{D}$. For $x \in \mathcal{B}, P \in \mathcal{P}, g \in G$ we denote by xg, Pg the g-images of x and P, and with $xG = \{xg \mid g \in G\}, PG = \{Pg \mid g \in G\}$ the G-orbits of x and P, respectively. By a known result the number of point orbits equals the number of line orbits. Denoting this number by t and the corresponding orbits by $\mathcal{B}_i, \mathcal{P}_r, 1 \leq i, r \leq t$, we have:

$$\mathcal{B} = \bigsqcup_{i=1}^{t} \mathcal{B}_i, \quad \mathcal{P} = \bigsqcup_{r=1}^{t} \mathcal{P}_r, \tag{1}$$

Denote $|\mathcal{B}_i| = \Omega_i, |\mathcal{P}_r| = \omega_r$. From (1) and the Definition 1(i), it follows immediately that

$$\sum_i \Omega_i = \sum_r \omega_r = v, \tag{2}$$

Let $x \in \mathcal{B}_i, P \in \mathcal{P}_r$. Then $|\langle x \rangle \cap \mathcal{P}_r| = |\langle x \rangle g \cap \mathcal{P}_r g| = |\langle xg \rangle \cap \mathcal{P}_r|$, for all $g \in G$. As $xG = \mathcal{B}_i$, we see that $|\langle x \rangle \cap \mathcal{P}_r| = \gamma_{ir}$ depends on $\mathcal{B}_i$ and $\mathcal{P}_r$ only. Similarly, $|\langle P \rangle \cap \mathcal{B}_i| = \Gamma_{ir}$ does not depend on the choice of P. From (1) and the Definition 1(ii), it follows that

$$\sum_r \gamma_{ir} = \sum_i \Gamma_{ir} = k. \tag{3}$$

The introduced cardinalities satisfy some important relations (see [1], [3], [5]):

Lemma 1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a (v, k, λ)-design, $G \leq \operatorname{Aut} \mathcal{D}$, and $\mathcal{B}_i$, $\mathcal{B}_j \subseteq \mathcal{B}$, $\mathcal{P}_r, \mathcal{P}_s \subseteq \mathcal{P}$ some G-orbits of lines and points, respectively. With other notation as above it follows that:*

(i) $\Omega_i \gamma_{ir} = \omega_r \Gamma_{ir}$;

(ii) $\sum_r \gamma_{ir} \Gamma_{jr} = \lambda\, \Omega_j + \delta_{ij} n$; $\sum_i \Gamma_{ir} \gamma_{is} = \lambda\, \omega_s + \delta_{rs} n$, *where δ_{ij}, δ_{rs} are the Kronecker symbols, and the indices i, r run over the set $\{1, 2, \dots, t\}$. Because of (i), we can rewrite (ii) as:*

(iii) $\sum_r \frac{\Omega_i}{\omega_r} \gamma_{ir} \gamma_{jr} = \lambda\, \Omega_j + \delta_{ij} n$; $\sum_i \frac{\Omega_i}{\omega_r} \gamma_{ir} \gamma_{is} = \lambda\, \omega_s + \delta_{rs} n$.

Definition 2: We denote

$$[\mathcal{B}_i, \mathcal{B}_j] = \sum_r \gamma_{ir} \Gamma_{jr} \text{ and } [\mathcal{P}_r, \mathcal{P}_s] = \sum_i \Gamma_{ir} \gamma_{is}$$

and call these expressions the *orbit products*.

Definition 3: Given a (v, k, λ)-design $\mathcal{D}$ and $G \leq \operatorname{Aut} \mathcal{D}$, the matrices (γ_{ir}) and (Γ_{ir}) are called the *orbital structures* of $\mathcal{D}$ *with respect* to G, for lines and points, respectively.

Obviously, the orbital structures of $\mathcal{D}$ are uniquely determined up to the order of rows and columns.

Let $P_r \in \mathcal{P}$ and $\mathcal{P}_r = P_r G$. We denote the points of $\mathcal{P}_r$ by $P_{r_0} = P_r, P_{r_1}, \ldots, P_{r_{\omega_r - 1}}$ or, abbreviated in a customary manner, as $r_0, r_1, \ldots, r_{\omega_r - 1}$. Thus, $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r - 1}\}$. Now, for each orbit $\mathcal{P}_r$ the automorphism group G is represented as a permutation group on the indices $0, \ldots, \omega_r - 1$. The same holds for the line orbits.

As an important step for what follows, we introduce canonical forms for lines and designs.

Definition 4: Suppose there is a given ordering among point orbits. Let $x \in \mathcal{B}$ and $\langle x \rangle = \bigsqcup_{r=1} (\langle x \rangle \cap \mathcal{P}_r)$. The unique sequence $\tilde{x}$ of points of $\langle x \rangle$ lexicographically ordered in terms of point orbits and in terms of their indices within the same orbit will be called the *canonical form* of x. In the following we shall identify $\tilde{x}$ with x.

Let x, y be two lines with the same orbital structure and $\tilde{x}, \tilde{y}$ their canonical forms. Then x *precedes* y *canonically* if $\tilde{x}$ precedes $\tilde{y}$ lexicographically in terms of indices.

Suppose, there are given orderings among point orbits and among line orbits. Then there is a unique sequence $\tilde{\mathcal{D}}$ consisting of canonical lines $\tilde{\mathcal{D}}(i)$ of $\mathcal{D}$ such that:

(a) $\tilde{\mathcal{D}}(i) \in \mathcal{B}_i$, for $1 \leq i \leq t$;
(b) $\tilde{\mathcal{D}}(i)$ precedes other lines in $\mathcal{B}_i$ canonically.

Because $\mathcal{B}_i = \tilde{\mathcal{D}}(i)G$, $\tilde{\mathcal{D}}$ and $\mathcal{D}$ are in a one-to-one correspondence. The sequence $\tilde{\mathcal{D}}$ will be called the *canonical form* of $\mathcal{D}$ (with respect to $\{\mathcal{B}_i\}, \{\mathcal{P}_r\}, G$).

In the following we shall identify $\tilde{\mathcal{D}}$ with $\mathcal{D}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two designs with the same orbital structure. Then $\mathcal{D}_1$ *precedes* $\mathcal{D}_2$ *canonically* if $\tilde{\mathcal{D}}_1$ precedes $\tilde{\mathcal{D}}_2$, lexicographically in terms of the canonical precedence of their lines.

The concept of G-isomorphisms of designs will be very important for our construction of designs (see [3]).

Definition 5: Let $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}, \mathcal{B}, I_2)$ be two designs and $G \leq \text{Aut } \mathcal{D}_1 \cap \text{Aut } \mathcal{D}_2 \leq S \equiv S(\mathcal{P}) \times S(\mathcal{B})$, where $S(S)$ denotes the symmetric group on the set S. A bijection $\alpha \in S$ is a *G-isomorphism* of $\mathcal{D}_1$ onto $\mathcal{D}_2$ if:

(i) α is an isomorphism of $\mathcal{D}_1$ onto $\mathcal{D}_2$; and
(ii) there is an automorphism τ: $G \to G$ such that for each $P, Q \in \mathcal{P}$ and each $g \in G$: $(P\alpha)(g\tau) = Q\alpha \Longleftrightarrow Pg = Q$.

If $I_1 = I_2 \subseteq \mathcal{P} \times \mathcal{B}$, α is a *G-automorphism* of $\mathcal{D}$. It follows that

Lemma 2. *The condition (ii) in the Definition 5 is equivalent to $\alpha \in N_s(G)$, that is α normalizes G in S.*

Proof: From $(P\alpha)\ (g\tau) = Q\alpha$ and $Pg = Q$ it follows that $P\alpha\,(g\tau) = Pg\alpha$. Hence, $\alpha(g\tau) = g\alpha$, and, therefore, $g\tau = \alpha^{-1}\, g\alpha$, for all $g \in G$. Thus, $\alpha^{-1}G\,\alpha = G\tau = G$ and so $\alpha \in N_s(G)$. The converse is now trivial. ∎

An immediate consequence of Definition 5 and Lemma 2 is

Lemma 3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a design and $G \leq Aut\,\mathcal{D}$. Let $S = S(\mathcal{P}) \times S(\mathcal{B})$ and $\alpha \in N_s(G)$. Then $\mathcal{D}$ is G-isomorphic to the design $\mathcal{D}\alpha = (\mathcal{P}, \mathcal{B}, I_\alpha)$, for which $(P\alpha, x\alpha) \in L_\alpha \Longleftrightarrow (P, x) \in I$.*

2. Construction of designs by means of automorphisms.

Here we sketch an algorithm for constructing all designs with parameters (v, k, λ) admitting a given automorphism group, as presented in [3]. We shall use the previously introduced notation.

Algorithm. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a (v, k, λ)-design and $G \leq Aut\,\mathcal{D}$, $S \equiv S(\mathcal{P}) \times S(\mathcal{B})$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_t$ and $\mathcal{B}_1, \ldots, \mathcal{B}_t$ be the G-orbits of points and lines in a given order and $\Omega_i = |\mathcal{B}_i|$, $\omega_r = |\mathcal{P}_r|$ for $1 \leq i,\ r \leq t$. We build at first the possible orbital structures $\Gamma = (\gamma_{ir})$ and after that the designs themselves by "indexing" the points (see [5]).*

Step 1: By Definition 2 $[\mathcal{B}_i, \mathcal{B}_i] = \sum_r \gamma_{ir}\Gamma_{ir} = \lambda\Omega_i + n = \sum_r \frac{\Omega_i}{\omega_r}\gamma_{ir}^2$, using Lemma 1 (ii) and (iii), for each i, $1 \leq i \leq t$. Recall that the solution vectors $(\gamma_{ir})_i$, the outer index denoting the fixed index, are called the line orbital structures of the line orbits $\mathcal{B}_i$ (see Definition 3). For two line orbital structures $(\gamma'_{ir})_i$ and $(\gamma''_{ir})_j$ we say that they are of the same type, if there is some $\alpha \in N_s(G)$ such that $(\mathcal{B}_i)\alpha = \mathcal{B}_j$ and $\gamma'_{ir} = \gamma''_{jr\alpha}$, where $(\mathcal{P}_r)\alpha = \mathcal{P}_{r\alpha}$, for all r, $1 \leq r \leq t$. As the type representative, we choose that line orbital structure, which is the first in the reverse lexicographical order with respect to the usual ordering in $\mathbf{N}$. We order the types in the order of their representatives ordered by the same principle. We apply the same ordering to the line orbital structures within the same type.

After finding all possible types for line orbits, we build the partial orbital structures. A jth *layer partial orbital structure* is any matrix $\Delta(j) = (\gamma_{ir})$, $1 \leq i \leq j$, $1 \leq r \leq t$, satisfying the row conditions from Lemma 1 and not violating the column conditions from the same lemma. We order them in terms of the above ordering of line orbital structures. Denote the set of all "essential" partial structures of the jth layer by $\Delta^{(j)}$.

We construct these in the following way:

1) $\Delta^{(1)}$ consists of all type representatives for the first orbit.
2) We construct $\Delta^{(j)}$ from $\Delta^{(j-1)}$ by joining to each $\Delta(j-1)$ as the next row all possible line orbital structures for $\mathcal{B}_j$ such that the obtained matrix $\Delta(j)$ is a jth layer partial orbital structure. We include such a matrix $\Delta(j)$ into the layer $\Delta^{(j)}$, if it cannot be eliminated by finding some $\alpha \in N_s(G)$ such that $\Delta(j)\alpha \prec \Delta(j)$, in terms of the precedence of partial structures

considered as parts of the whole structures. Since $\Delta\alpha$ is G-isomorphic with Δ, we eliminate in this way a lot of isomorphic designs, retaining only those among them which are first in terms of the defined precedence.

At the end of this procedure $\Delta^{(t)}$ will be the set of all possible orbital structures for the given problem.

Step 2: Next, we deal with each orbital structure separately. Thus, let $\Delta = (\gamma_{ir})$ be the structure under consideration. For each row in Δ we construct orbit by orbit all possible canonical lines. In this way we build the set $\Delta^{(j)}$ of jth layer partial designs $\mathcal{D}(j)$, for each j, $0 \leq j \leq t$.

Let $\mathcal{D}^{(0)}$ be the null set. We construct $\mathcal{D}^{(j)}$ from $\mathcal{D}^{(j-1)}$, $1 \leq j \leq t$, in the following way.

To each $\mathcal{D}(j-1) \in \mathcal{D}^{(j-1)}$ we join all possible canonical lines for the orbit $\mathcal{B}_j$, which satisfy the design conditions from Definition 1 and the given orbit conditions for $\mathcal{B}_j$. In such a way we obtain jth layer partial designs. We include such a partial design $\mathcal{D}(j)$ into $\mathcal{D}^{(j)}$, if it cannot be eliminated by finding an $\alpha \in N_s(G)$ with $\Delta\alpha = \Delta$, such that $\mathcal{D}(j)\alpha \prec \mathcal{D}(j)$ in terms of the precedence of partial designs considered as parts of the assumed whole designs.

At the end of this procedure, $\mathcal{D}^{(t)}$ will be the set of all possible designs with the orbital structure Δ, admitting the given automorphism group G. Because of the above eliminations they appear in the canonical form.

Carrying out this construction for all orbital structures Δ, we get all the required designs. It can happen, that among the obtained designs there are isomorphic or dually isomorphic ones. The identification and elimination of such designs is the last step in solving the given problem.

3. Automorphisms of order 5 acting on (45, 12, 3)-designs.

Let $\mathcal{D}$ be a (45, 12, 3)-design and $\rho \in \operatorname{Aut}\mathcal{D}$, $|\rho| = 5$. We shall determine first the action of ρ on $\mathcal{P}$ and $\mathcal{B}$. For this purpose, we use the following lemma proved in [3].

Lemma 4. *Let $\mathcal{D}$ be a symmetric (v, k, λ)-design, $\rho \in Aut\,\mathcal{D}$ and $|\rho| = p > \lambda$, p a prime. Denote by f the number of points in $\mathcal{P}$ fixed by ρ. If there exists a ρ-fixed line consisting of fixed points only, then $k \leq f \leq v-(k-\lambda)p$. Otherwise, if each ρ-fixed line contains at least τ full nontrivial orbits of ρ, then $(\tau p+1)f \leq v$.*

Applying this lemma to our case, we get

Lemma 5. *Let $\mathcal{D}$ be a $(45, 12, 3)$-design and $\rho \in Aut\,\mathcal{D}$, $|\rho| = 5$. Then ρ acts fixed point freely on both sets $\mathcal{P}$ and $\mathcal{B}$.*

Proof: Since $v-(k-\lambda)p = 45-9\cdot 5 = 0$, each ρ-fixed line contains by Lemma 4 a nontrivial ρ-orbit and $(5+1)f \leq 45$. It follows that $f \in \{0, 5\}$. If $f = 5$, then $\tau = 2$ and so $(2 \cdot 5 + 1) f \leq 45$, which implies $f < 5$ and, therefore, $f = 0$. ∎

Thus, all the ρ-orbits on $\mathcal{P}$ and $\mathcal{B}$ are of the length 5, and there are nine such orbits on each of them.

4. Proof of the theorem.

Next, we determine the orbital structures of $\mathcal{D}$ with respect to ρ. We use the notation of paragraph 1.

Lemma 6. *Let $\mathcal{D}$ be a $(45, 12, 3)$-design and $\rho \in Aut\ \mathcal{D}$, $|\rho| = 5$. Then there are up to isomorphism 21 possible orbital structures (γ_{ij}) for $\mathcal{D}$ with respect to ρ, namely:*

1.

4	1	1	1	1	1	1	1	1
1	4	1	1	1	1	1	1	1
1	1	4	1	1	1	1	1	1
1	1	1	4	1	1	1	1	1
1	1	1	1	4	1	1	1	1
1	1	1	1	1	4	1	1	1
1	1	1	1	1	1	4	1	1
1	1	1	1	1	1	1	4	1
1	1	1	1	1	1	1	1	4

2.

4	1	1	1	1	1	1	1	1
1	4	1	1	1	1	1	1	1
1	1	4	1	1	1	1	1	1
1	1	1	4	1	1	1	1	1
1	1	1	1	4	1	1	1	1
1	1	1	1	1	4	1	1	1
1	1	1	1	1	1	3	3	0
1	1	1	1	1	1	3	0	3
1	1	1	1	1	1	0	3	3

3.

4	1	1	1	1	1	1	1	1
1	4	1	1	1	1	1	1	1
1	1	4	1	1	1	1	1	1
1	1	1	3	3	1	1	1	0
1	1	1	3	0	1	1	1	3
1	1	1	1	1	3	3	0	1
1	1	1	1	1	3	0	3	1
1	1	1	1	1	0	3	3	1
1	1	1	0	3	1	1	1	3

4.

4	1	1	1	1	1	1	1	1
1	4	1	1	1	1	1	1	1
1	1	4	1	1	1	1	1	1
1	1	1	3	2	2	2	0	0
1	1	1	2	3	0	0	2	2
1	1	1	2	0	3	0	2	2
1	1	1	2	0	0	3	2	2
1	1	1	0	2	2	2	3	0
1	1	1	0	2	2	2	0	3

5.

4	1	1	1	1	1	1	1	1
1	4	1	1	1	1	1	1	1
1	1	3	3	1	1	1	1	0
1	1	2	1	2	2	0	0	3
1	1	2	0	3	0	2	2	1
1	1	2	0	0	3	2	2	1
1	1	1	2	0	0	2	2	3
1	1	0	2	2	2	3	0	1
1	1	0	2	2	2	0	3	1

6.

4	1	1	1	1	1	1	1	1
1	3	3	1	1	1	1	1	0
1	1	1	3	3	1	1	0	1
1	2	1	2	0	2	0	1	3
1	2	0	2	1	0	2	3	1
1	2	0	0	2	3	2	1	1
1	1	2	0	2	0	2	1	3
1	0	2	2	0	2	3	1	1
1	0	2	1	2	2	0	3	1

7.

4	1	1	1	1	1	1	1	1
1	3	2	2	2	1	1	0	0
1	2	3	0	0	1	1	2	2
1	2	0	2	1	2	0	3	1
1	2	0	1	2	0	2	1	3
1	1	1	2	0	3	2	0	2
1	1	1	0	2	2	3	2	0
1	0	2	3	1	0	2	2	1
1	0	2	1	3	2	0	1	2

8.

3	3	1	1	1	1	1	1	0
3	0	1	1	1	1	1	1	3
1	1	3	3	1	1	1	0	1
1	1	3	0	1	1	1	3	1
1	1	1	1	3	3	0	1	1
1	1	1	1	3	0	3	1	1
1	1	1	1	0	3	3	1	1
1	1	0	3	1	1	1	3	1
0	3	1	1	1	1	1	1	3

9.

3	3	1	1	1	1	1	1	0
3	0	1	1	1	1	1	1	3
0	3	1	1	1	1	1	1	3
1	1	3	2	2	2	0	0	1
1	1	2	3	0	0	2	2	1
1	1	2	0	3	0	2	2	1
1	1	2	0	0	3	2	2	1
1	1	0	2	2	2	3	0	1
1	1	0	2	2	2	0	3	1

10.

3	3	1	1	1	1	1	1	0
1	1	3	3	1	1	1	0	1
1	1	1	1	3	3	0	1	1
2	1	2	0	2	0	1	1	3
2	0	2	1	0	2	1	3	1
2	0	0	2	2	1	3	1	1
1	2	0	2	0	2	1	1	3
0	2	2	0	1	2	3	1	1
0	2	1	2	2	0	1	3	1

11.

3	3	1	1	1	1	1	1	0
2	1	2	2	1	1	0	0	3
2	0	2	1	2	0	3	1	1
2	0	1	2	0	2	1	3	1
1	2	0	0	1	1	2	2	3
1	1	2	0	3	2	0	2	1
1	1	0	2	2	3	2	0	1
0	2	3	1	0	2	2	1	1
0	2	1	3	2	0	1	2	1

12.

3	2	2	2	1	1	1	0	0
2	3	0	0	1	1	1	2	2
2	0	3	0	1	1	1	2	2
2	0	0	1	3	2	2	1	1
1	1	1	3	2	0	0	2	2
1	1	1	2	0	2	2	3	0
1	1	1	2	0	2	2	0	3
0	2	2	1	2	3	0	1	1
0	2	2	1	2	0	3	1	1

13.

3	2	2	2	1	1	1	0	0
2	3	0	0	1	1	1	2	2
2	0	2	1	2	1	0	3	1
2	0	0	1	2	2	3	1	1
1	1	3	0	0	2	2	1	2
1	1	1	3	0	0	2	2	2
1	1	1	2	2	2	0	0	3
0	2	2	1	3	0	2	1	1
0	2	1	2	1	3	1	2	0

14.

3	2	2	2	1	1	1	0	0
2	2	1	1	1	0	0	3	2
2	1	1	0	2	2	1	0	3
2	1	0	1	0	2	3	2	1
1	1	2	0	3	1	2	2	0
1	0	2	2	1	3	0	2	1
1	0	1	3	2	0	2	1	2
0	3	0	2	2	2	1	1	1
0	2	3	1	0	1	2	1	2

15.

3	2	2	2	1	1	1	0	0
2	2	1	1	1	0	0	3	2
2	1	1	0	0	3	2	2	1
2	0	1	1	2	0	3	1	2
1	2	0	2	1	2	1	0	3
1	1	0	2	3	2	1	2	0
1	0	3	1	2	2	0	1	2
0	3	2	0	2	1	2	1	1
0	1	2	3	0	1	2	2	1

16.

3	2	2	2	1	1	1	0	0
2	2	1	1	1	0	0	3	2
2	0	2	0	1	3	1	2	1
2	0	1	1	2	0	3	1	2
1	3	0	0	2	2	2	1	1
1	1	1	2	2	2	0	0	3
1	1	0	3	0	2	2	2	1
0	2	3	1	0	1	2	1	2
0	1	2	2	3	1	1	2	0

17.

3	2	2	2	1	1	1	0	0
2	2	1	1	1	0	0	3	2
2	0	1	1	2	3	0	1	2
2	0	1	1	2	0	3	1	2
1	3	0	0	2	2	2	1	1
1	1	3	0	0	2	2	2	1
1	1	0	3	0	2	2	2	1
0	2	2	2	1	1	1	0	3
0	1	2	2	3	1	1	2	0

18.

3	2	2	2	1	1	1	0	0
2	2	1	0	2	1	0	3	1
2	1	1	1	0	0	3	2	2
2	1	0	1	2	2	1	0	3
1	1	0	2	1	3	2	2	0
1	0	3	0	2	2	2	1	1
1	0	2	3	1	1	0	2	2
0	3	2	1	0	2	1	1	2
0	2	1	2	3	0	2	1	1

19.

3	2	2	2	1	1	1	0	0
2	2	1	0	2	1	0	3	1
2	1	1	1	0	0	3	2	2
2	0	1	2	1	2	0	1	3
1	2	0	1	3	1	2	0	2
1	1	0	2	1	3	2	2	0
1	0	3	0	2	2	2	1	1
0	3	2	1	0	2	1	1	2
0	1	2	3	2	0	1	2	1

20.

3	2	2	2	1	1	1	0	0
2	2	0	0	3	1	1	2	1
2	1	1	1	0	3	0	2	2
2	1	1	1	0	0	3	2	2
1	1	2	2	2	0	0	1	3
1	0	3	0	2	2	2	1	1
1	0	0	3	2	2	2	1	1
0	3	1	1	1	2	2	0	2
0	2	2	2	1	1	1	3	0

21.

3	2	2	2	1	1	1	0	0
2	1	1	1	3	0	0	2	2
2	1	1	1	0	3	0	2	2
2	1	1	1	0	0	3	2	2
1	3	0	0	2	2	2	1	1
1	0	3	0	2	2	2	1	1
1	0	0	3	2	2	2	1	1
0	2	2	2	1	1	1	3	0
0	2	2	2	1	1	1	0	3

Proof: Such an orbital structure must satisfy the conditions (2), (3) and Lemma 1(iii). By Lemma 5 we have $t = 9$ and $\Omega_j = \omega_s = 5$, for all $j, s \in \{1, \dots, 9\}$. Also $n = 12 - 3 = 9$.

Thus we have:

(a) $\sum_{r=1}^{9} \gamma_{ir} = 12$;

(b) $[\mathcal{B}_i, \mathcal{B}_i] = \sum_{r=1}^{9} \gamma_{ir}^2 = 3 \cdot 5 + 9 = 24$, for $1 \leq i \leq 9$;

(c) $[\mathcal{B}_i, \mathcal{B}_j] = \sum_{r=1}^{9} \gamma_{ir}\gamma_{jr} = 3 \cdot 5 = 15$, for $1 \leq i < j \leq 9$.

Now $\gamma_{ir} \leq \omega_r = 5$, for all i, r. However, ρ does not fix any element and, thus, $\gamma_{ir} < 5$. One can easily see that there are up to order exactly four solutions $(\gamma_{ir})_i$ for (a) and (b), namely: 4 1 1 1 1 1 1 1 1, 3 3 1 1 1 1 1 1 0, 3 2 2 2 1 1 1 0 0, and 2 2 2 2 2 2 0 0 0.

Concerning (c) and applying Step 1 of our Algorithm, we can now construct the orbital structures for the given group and parameters. We obtained with the help of a computer as the only solutions up to isomorphism, the 21 orbital structures listed in the statement of the Lemma.

We proceed by indexing the obtained orbital structures, and, thus, construct the designs themselves. Refering to paragraph 2. we denote $\mathcal{P}_r = \{r_0, r_1, r_2, r_3, r_4\}$ and $\mathcal{B}_i = \{\bar{i}_0, \bar{i}_1, \bar{i}_2, \bar{i}_3, \bar{i}_4\}$ for $1 \leq i, r \leq 9$. Now, ρ acts on $\mathcal{P}_r$ as $r_a\rho = r_{a+1}$, and on $\mathcal{B}_i$ as $\bar{i}_a, \rho = \bar{i}_{a+1}$, for $a \in \{0, 1, 2, 3, 4\}$, the sums $a + 1$ being modulo 5.

In the following we need some additional notation.

Let S_i, $i = 1,\ldots,s$ be disjoint ordered sets of the same cardinality c, and $S_i(j)$, for $1 \leq j \leq c$, be the jth element of S_i. We define the *blockwise symmetric group* $S(S_1,\ldots,S_s)$ over the ordered sets $S_1,\ldots,S_s$ as the group $S(S_1,\ldots,S_s) = \{\pi'(\pi) \in S\left(\bigsqcup_{i=1}^{s} S_i\right) \mid \pi \in S(1,2,\ldots,s) \wedge \left(S_i(j)\pi'(\pi) = S_{i\pi}(j)\right)$, for all j, $1 \leq j \leq c$ and all i, $1 \leq i \leq s\}$ $S(S)$ denoting the symmetric group over the set S.

One can easily see that the normalizer $N_s(G)$ from Lemma 2 is the group

$$N_s(\langle\rho\rangle) = \langle\alpha_1,\ldots,\alpha_9,\ \mathcal{B}_1,\ldots,\mathcal{B}_9,\ S(\mathcal{P}_1,\ldots,\mathcal{P}_9),\ S(\mathcal{B}_1,\ldots,\mathcal{B}_9),\ \sigma\rangle,$$

where

$$\alpha_r = (r_0 r_1 r_2 r_3 r_4), \mathcal{B}_i = (\bar{i}_0 \bar{i}_1 \bar{i}_2 \bar{i}_3 \bar{i}_4\},$$

and

$$\sigma = \prod_{r=1}^{9}(r_1 r_2 r_4 r_3) \prod_{i=1}^{9}(\bar{i}_1 \bar{i}_2 \bar{i}_4 \bar{i}_3).$$

Applying Step 2 of our Algorithm, we get, again with the help of a computer, as the only possible solutions 13 mutually non-isomorphic designs. They are presented in the Table 1, after translating points of $\mathcal{P}$ by the rule $r_a \to (r-1)\cdot 5 + a$, thus, denoting the points of the designs by $0,\ldots,44$. Each of the line orbits $\mathcal{B}_1,\ldots,\mathcal{B}_9$ is represented by only one line; the others are obtained by applying the automorphism ρ, which acts on the points $0,\ldots,44$ as the permutation

$$i\rho = i - 4 \ \text{ if } \ 5 \mid i + 1, \ \ i\rho = i + 1 \ \ \text{otherwise.}$$

The numbers in square brackets in Table 1 denote the orbital structures from Lemma 6 from which the designs are obtained. We see immediately, that there are two designs for each of the structures 6 and 7, one design for each of the structures $1, 2, 4, 5, 10, 11, 13, 15$ and 21, and none for the others.

In the Table 2 the automorphism groups of designs 1. to 13. are represented in terms of their generators. These are presented in such a way that each generator is the lexicographically first automorphism of the design which is not contained in the group generated by previous generators. The generators are given as permutations of the points $0,\ldots,44$ in their natural ordering. Construction of such a basis of generators can be realized in a very effective manner and enables us to determine the orders of the automorphism groups in an easy way (see [2]). Moreover, we present for each design its point orbits function, denoting the mapping of each point to the first point in the natural ordering belonging to the same orbit.

The relations of dual isomorphism are given in the Table 3. With the exception of Design 5 and Design 6, which are dually isomorphic, all other designs are self-dual. The listed permutations are the dual isomorphic mappings from lines and points of the first design onto the points and lines of the second. The obtained results prove our Theorem. ∎

TABLE1.

1. [1]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	7	8	10	16	21	27	33	39	44
0	5	10	11	12	13	17	24	26	34	36	43
0	6	12	15	16	18	19	22	29	31	35	43
0	6	14	17	21	22	23	24	25	30	38	41
0	7	11	19	20	25	26	28	29	33	37	41
0	8	14	16	20	28	30	31	32	34	36	42
0	9	11	15	23	27	31	36	37	38	39	40
0	9	13	18	21	26	32	35	40	41	42	44

2. [2]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	7	8	11	16	22	29	30	38	44
0	6	10	11	12	14	18	20	26	32	39	43
0	6	13	15	16	17	19	24	26	33	37	40
0	7	10	19	20	22	23	24	28	31	36	41
0	9	11	16	23	25	26	27	28	34	35	42
0	9	13	15	21	27	30	31	32	36	38	39
0	5	12	18	21	29	31	33	34	40	41	42
0	8	14	17	21	25	35	36	37	41	43	44

3. [4]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	7	8	10	16	22	29	34	36	43
0	5	10	11	12	13	17	21	26	33	39	44
0	6	12	15	16	19	23	24	25	27	31	33
0	7	11	18	19	20	21	24	35	37	41	43
0	9	11	15	17	27	28	29	36	38	40	41
0	9	13	16	18	30	31	32	35	36	42	44
0	6	14	20	22	26	28	30	31	37	38	39
0	8	14	21	23	25	26	32	34	40	41	42

4. [5]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	7	8	9	11	16	20	27	33	39	41
0	6	10	11	12	16	18	19	23	29	32	35
0	6	13	14	16	22	24	25	28	40	41	42
0	5	12	14	20	21	24	31	32	36	38	43
0	7	10	13	26	27	28	31	34	35	36	44
0	6	11	15	17	30	34	37	38	41	43	44
0	8	15	19	21	22	26	29	30	31	33	42
0	9	17	18	21	23	25	26	36	37	39	40

5. [6]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	7	11	13	14	15	21	29	32	38
0	6	11	16	17	19	20	23	24	27	33	40
0	8	9	11	15	16	25	28	37	41	42	44
0	7	9	16	18	20	31	32	35	36	39	43
0	5	8	21	22	26	27	28	31	34	39	40
0	6	10	12	21	23	30	34	36	42	43	44
0	10	14	17	18	26	29	30	31	33	37	41
0	12	13	19	22	24	25	26	35	36	38	41

6. [6]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	8	10	11	14	16	22	29	32	38
0	6	13	17	18	19	21	22	24	25	31	40
0	7	9	10	16	18	26	27	36	40	43	44
0	7	8	15	17	21	30	34	36	38	39	41
0	5	6	20	23	26	27	28	31	34	39	42
0	9	11	13	20	21	32	33	35	41	42	44
0	11	12	15	19	26	29	30	31	33	37	43
0	12	14	16	23	24	25	28	35	36	37	41

7. [7]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	7	10	13	16	17	21	24	29	33
0	5	8	10	11	12	28	34	36	37	41	44
0	6	7	15	19	22	26	28	36	38	39	40
0	6	9	17	22	23	30	31	35	41	43	44
0	9	13	16	18	25	26	27	32	34	40	41
0	8	14	20	21	27	29	30	31	32	36	38
0	11	12	16	18	19	20	31	33	35	39	42
0	11	14	15	21	23	24	25	26	37	42	43

8. [7]

0	1	2	3	5	10	15	20	25	30	35	40
0	5	6	8	10	12	16	17	23	24	26	34
0	5	7	10	11	13	29	31	36	37	43	44
0	6	7	16	19	21	25	27	35	38	39	43
0	8	9	16	20	22	30	31	38	41	42	44
0	6	14	15	17	27	28	29	31	33	40	41
0	9	11	20	21	26	28	32	33	34	35	37
0	11	12	15	18	19	23	30	32	36	39	41
0	13	14	18	21	22	24	25	26	36	40	42

9. [10]

0	1	2	5	6	8	10	15	20	25	30	35
0	5	10	11	13	16	17	18	22	26	34	40
0	6	14	15	21	22	24	26	27	28	37	43
0	2	9	11	12	20	23	34	37	42	43	44
0	1	12	13	16	25	28	32	36	37	39	41
0	2	18	19	21	22	29	30	32	33	36	44
0	7	8	17	19	25	26	31	38	41	43	44
5	6	10	12	23	26	29	31	32	33	39	43
5	7	11	15	18	20	21	33	37	38	39	41

10. [11]

0	1	2	5	6	8	10	15	20	25	30	35
0	1	7	10	12	17	18	21	28	40	41	43
0	2	11	12	15	21	24	32	33	34	38	44
0	2	13	16	19	27	28	32	35	36	39	43
0	8	9	22	26	31	34	35	37	40	43	44
0	5	12	13	20	21	23	26	29	36	37	42
0	7	15	16	22	23	25	27	29	31	33	41
5	7	11	13	14	18	25	26	32	33	35	40
5	6	11	15	17	18	22	23	34	36	39	43

11. [13]

0	1	2	5	6	10	12	15	17	20	25	30
0	1	5	7	8	22	26	33	35	37	40	41
0	2	13	14	19	20	23	26	35	36	39	43
0	2	15	21	22	28	29	31	32	34	35	42
0	7	10	11	14	25	27	31	33	38	42	43
0	8	12	16	18	19	30	31	36	37	42	44
0	6	13	16	17	22	24	27	28	41	43	44
5	7	10	13	19	21	22	23	30	34	38	44
5	6	11	18	19	23	25	27	28	32	35	37

12. [15]

0	1	2	5	6	10	12	15	17	20	25	30
0	1	6	8	11	22	23	27	35	36	38	40
0	2	6	14	18	32	33	34	36	37	42	44
0	2	13	18	19	22	25	29	39	40	41	43
0	7	8	17	21	23	24	25	31	34	42	43
0	7	15	19	20	26	28	29	31	32	36	38
0	12	13	14	21	24	26	27	30	33	38	41
5	6	8	11	12	19	26	28	33	39	42	43
5	12	14	15	16	18	22	23	31	37	38	43

13. [21]

0	1	2	5	6	10	12	15	17	20	25	30
0	1	7	10	19	21	22	23	35	37	40	42
0	2	5	13	19	26	27	28	36	39	42	43
0	2	9	14	18	30	31	32	37	38	40	43
0	6	7	8	20	24	27	29	31	33	38	42
0	11	12	13	22	24	25	28	31	32	35	44
0	15	16	17	21	23	26	27	32	34	38	44
5	7	12	13	15	18	22	29	34	37	38	39
5	7	11	14	17	18	23	25	33	42	43	44

TABLE 2.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22
23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44

Remark: rho stands for the automorphism ϱ introduced in Paragraph 4.:

1 2 3 4 0 6 7 8 9 5 11 12 13 14 10 16 17 18 19 15 21 22 23
24 20 26 27 28 29 25 31 32 33 34 30 36 37 38 39 35 41 42 43 43 40

1. Automorphism group generators.

0 1 2 3 4 10 11 12 13 14 5 6 7 8 9 25 26 27 28 29 35 36 37
38 39 15 16 17 18 19 40 41 42 43 44 20 21 22 23 24 30 31 32 33 34

0 2 4 1 3 19 16 18 15 17 29 26 28 25 27 34 31 33 30 32 9 6 8
5 7 44 41 43 40 42 39 36 38 35 37 14 11 13 10 12 24 21 23 20 22

1 0 4 3 2 33 32 31 30 34 43 42 41 40 44 38 37 36 35 39 18 17 16
15 19 23 22 21 20 24 13 12 11 10 14 28 27 26 25 29 8 7 6 5 9

5 6 7 8 9 0 1 2 3 4 10 11 12 13 14 39 35 36 37 38 44 40 41
42 43 33 34 30 31 32 27 28 29 25 26 16 17 18 19 15 21 22 23 24 20

Order = 360.
Point orbits function.
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

2. Automorphism group generators.

```
 0  4  3  2  1 12 11 10 14 13  7  6  5  9  8 27 26 25 29 28 22 21 20
24 23 17 16 15 19 18 32 31 30 34 33 37 36 35 39 38 42 41 40 44 43

 1  0  4  3  2 13 12 11 10 14  8  7  6  5  9 28 27 26 25 29 23 22 21
20 24 18 17 16 15 19 33 32 31 30 34 38 37 36 35 39 43 42 41 40 44

20 22 24 21 23 16 18 15 17 19 25 27 29 26 28 12 14 11 13 10  2  4  1
 3  0  9  6  8  5  7 38 35 37 39 36 33 30 32 34 31 43 40 42 44 41
```

Order = 20.
Point orbits function.

```
 0  0  0  0  0  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  0  0  0
 0  0  5  5  5  5  5 30 30 30 30 30 30 30 30 30 30 40 40 40 40 40
```

3. Automorphism group generators.

```
 0  1  2  3  4 10 11 12 13 14  5  6  7  8  9 20 21 22 23 24 15 16 17
18 19 35 36 37 38 39 40 41 42 43 44 25 26 27 28 29 30 31 32 33 34
```

rho

```
 5  6  7  8  9  0  1  2  3  4 10 11 12 13 14 29 25 26 27 28 43 44 40
41 42 16 17 18 19 15 36 37 38 39 35 34 30 31 32 33 22 23 24 20 21
```

Order = 30.
Point orbits function.

```
 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 15 15 15 15 15 15 15 15
15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15
```

4. Automorphism group generators.

```
 0  1  2  3  4  5 26 32 38 24 10 31 27 23 39 15 36 22 28 34 20 41 17
13  9 25  6 12 18 44 30 11  7 43 19 35 16 42  8 14 40 21 37 33 29

 0  1  2  3  4 10 21 27 13 44  5 36 32  8 19 40 31 37 43 14 35  6 42
38 29 30 41  7 33 24 25 16 12 28 39 20 11 17 23 34 15 26 22 18  9

 0  1  2  3  4 15 16 37 43 34 40 41 22 18 29  5  6 27 13 24 30 31 12
28 19 35 36 17 23 14 20 21 42 38  9 25 26  7 33 44 10 11 32  8 39

 0  1  2  3  4 20  6 27 33 39 35 11 32 28 24 30 16 37 23 29  5 21 42
18 14 40 26  7 13 19 15 31 12  8 44 10 36 17 43  9 25 41 22 38 34

 0  1  2  4  3  9  6  7  8  5 44 11 17 13 40 19 41 12 43 15 39 21 22
38 25 24 31 27 28 30 29 26 37 33 35 34 36 32 23 20 14 16 42 18 10

 0  1  3  2  4  5  6  7  9  8 10 16 12 44 43 40 11 42 19 18 20 21 37
29 38 30 26 27 34 23 25 36 32 39 28 35 31 22 24 33 15 41 17 14 13

 0  2  1  3  4  5  6  8  7  9 15 11 43 42 14 10 41 18 17 44 20 36 28
37 24 25 26 33 22 34 35 31 38 27 29 30 21 23 32 39 40 16 13 12 19

 0  6 11 16 41 10  4  5 40 15 12  3  7 14 34 23  2  8 22 37 35 26 39
42 44 19 21 20 24 17 18 36 27 25 38 32 31 33 13 30 29  1  9 28 43
```

1 0 2 3 4 5 7 6 8 9 10 42 41 13 19 40 17 16 43 14 35 27 36
23 24 25 32 21 33 29 30 37 26 28 39 20 22 31 38 34 15 12 11 18 44

Order = 51840.
Point orbits function.
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

5. Automorphism group generators.

rho

Order = 5.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 10 10 10 10 10 15 15 15 15 15 20 20 20
20 20 25 25 25 25 25 30 30 30 30 30 35 35 35 35 35 40 40 40 40 40

6. Automorphism group generators.

rho

Order = 5.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 10 10 10 10 10 15 15 15 15 15 20 20 20
20 20 25 25 25 25 25 30 30 30 30 30 35 35 35 35 35 40 40 40 40 40

7. Automorphism group generators.

0 1 2 3 4 10 11 12 13 14 5 6 7 8 9 35 36 37 38 39 40 41 42
43 44 30 31 32 33 34 25 26 27 28 29 15 16 17 18 19 20 21 22 23 24

0 1 2 4 3 19 11 12 38 25 39 6 7 18 30 44 16 42 13 5 29 31 37
43 35 9 41 27 28 20 14 21 32 33 40 24 36 22 8 10 34 26 17 23 15

0 1 3 2 4 5 6 17 34 38 10 11 37 29 18 35 21 7 14 23 25 16 22
19 33 20 31 32 44 13 40 26 27 24 8 15 41 12 9 43 30 36 42 39 28

0 2 1 3 4 5 16 33 37 9 10 36 28 17 14 20 6 13 22 39 15 21 18
32 29 30 31 43 12 24 25 26 23 7 44 40 11 8 42 19 35 41 38 27 34

0 6 7 17 22 5 11 12 30 26 10 1 2 41 38 33 34 37 9 19 29 8 42
23 28 13 18 27 35 36 21 14 32 44 40 24 20 3 31 39 16 25 4 43 15

1 0 2 3 4 15 32 36 8 9 35 27 16 13 14 5 12 21 38 24 20 17 31
28 19 30 42 11 23 34 25 22 6 43 29 10 7 41 18 44 40 37 26 33 39

Order = 19440.
Point orbits function.
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

8. Automorphism group generators.

0 1 2 3 4 10 11 12 13 14 5 6 7 8 9 35 36 37 38 39 40 41 42
43 44 30 31 32 33 34 25 26 27 28 29 15 16 17 18 19 20 21 22 23 24

0 4 3 2 1 22 21 20 24 23 42 41 40 44 43 27 26 25 29 28 12 11 10
14 13 37 36 35 39 38 17 16 15 19 18 32 31 30 34 33 7 6 5 9 8

1 0 4 3 2 23 22 21 20 24 43 42 41 40 44 28 27 26 25 29 13 12 11
10 14 38 37 36 35 39 18 17 16 15 19 33 32 31 30 34 8 7 6 5 9

Order = 20.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 5 5 5 5 5 15 15 15 15 15 5 5 5
5 5 15 15 15 15 15 15 15 15 15 15 15 15 15 15 15 5 5 5 5 5

9. Automorphism group generators.

rho

15 16 17 18 19 14 10 11 12 13 21 22 23 24 20 25 26 27 28 29 9 5 6
7 8 4 0 1 2 3 44 40 41 42 43 33 34 30 31 32 37 38 39 35 36

Order = 15.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 5 5 5 5 5 0 0 0 0 0 5 5 5
5 5 0 0 0 0 0 30 30 30 30 30 30 30 30 30 30 30 30 30 30 30

10. Automorphism group generators.

rho

Order = 5.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 10 10 10 10 10 15 15 15 15 15 20 20 20
20 20 25 25 25 25 25 30 30 30 30 30 35 35 35 35 35 40 40 40 40 40

11. Automorphism group generators.

rho

Order = 5.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 10 10 10 10 10 15 15 15 15 15 20 20 20
20 20 25 25 25 25 25 30 30 30 30 30 35 35 35 35 35 40 40 40 40 40

12. Automorphism group generators.

rho

10 11 12 13 14 29 25 26 27 28 31 32 33 34 30 22 23 24 20 21 44 40 41
42 43 36 37 38 39 35 3 4 0 1 2 9 5 6 7 8 18 19 15 16 17

Order = 15.
Point orbits function.
0 0 0 0 0 5 5 5 5 5 0 0 0 0 0 15 15 15 15 15 15 15 15
15 15 5 5 5 5 5 0 0 0 0 0 5 5 5 5 5 15 15 15 15 15

13. Automorphism group generators.

rho

5 6 7 8 9 23 24 20 21 22 30 31 32 33 34 26 27 28 29 25 4 0 1
2 3 41 42 43 44 40 37 38 39 35 36 10 11 12 13 14 15 16 17 18 19

10 11 12 13 14 30 31 32 33 34 27 28 29 25 26 21 22 23 24 20 39 35 36
37 38 4 0 1 2 3 43 44 40 41 42 16 17 18 19 15 8 9 5 6 7

Order = 45.
Point orbits function.
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

TABLE 3 .

1. Selfdual

0 1 2 3 4 32 33 34 30 31 42 43 44 40 41 37 38 39 35 36 17 18 19
15 16 22 23 24 20 21 12 13 14 10 11 27 28 29 25 26 7 8 9 5 6

2 3 4 0 1 30 31 32 33 34 40 41 42 43 44 35 36 37 38 39 15 16 17
18 19 20 21 22 23 24 10 11 12 13 14 25 26 27 28 29 5 6 7 8 9

2. Selfdual

0 1 2 3 4 12 13 14 10 11 7 8 9 5 6 27 28 29 25 26 22 23 24
20 21 17 18 19 15 16 42 43 44 40 41 32 33 34 30 31 37 38 39 35 36

2 3 4 0 1 10 11 12 13 14 5 6 7 8 9 25 26 27 28 29 20 21 22
23 24 15 16 17 18 19 35 36 37 38 39 40 41 42 43 44 30 31 32 33 34

3. Selfdual

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 35 39 38 37 36 40 44 43 42 41

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 35 39 38 37 36 40 44 43 42 41

4. Selfdual

0 1 2 3 4 7 8 9 5 6 42 43 44 40 41 12 13 14 10 11 27 28 29
25 26 22 23 24 20 21 17 18 19 15 16 37 38 39 35 36 32 33 34 30 31

2 3 4 0 1 5 6 7 8 9 15 16 17 18 19 30 31 32 33 34 25 26 27
28 29 20 21 22 23 24 40 41 42 43 44 35 36 37 38 39 10 11 12 13 14

5. Dually isomorphic to 6.

0 4 3 2 1 35 39 38 37 36 40 44 43 42 41 20 24 23 22 21 10 14 13
12 11 25 29 28 27 26 15 19 18 17 16 30 34 33 32 31 5 9 8 7 6

0 4 3 2 1 40 44 43 42 41 20 24 23 22 21 30 34 33 32 31 15 19 18
17 16 25 29 28 27 26 35 39 38 37 36 5 9 8 7 6 10 14 13 12 11

6. Dually isomorphic to 5.

0 4 3 2 1 35 39 38 37 36 40 44 43 42 41 20 24 23 22 21 10 14 13
12 11 25 29 28 27 26 15 19 18 17 16 30 34 33 32 31 5 9 8 7 6

0 4 3 2 1 40 44 43 42 41 20 24 23 22 21 30 34 33 32 31 15 19 18
17 16 25 29 28 27 26 35 39 38 37 36 5 9 8 7 6 10 14 13 12 11

7. Selfdual

0 1 2 3 4 27 28 29 25 26 32 33 34 30 31 42 43 44 40 41 17 18 19
15 16 12 13 14 10 11 7 8 9 5 6 22 23 24 20 21 37 38 39 35 36

2 3 4 0 1 25 26 27 28 29 30 31 32 33 34 40 41 42 43 44 15 16 17
18 19 10 11 12 13 14 5 6 7 8 9 20 21 22 23 24 35 36 37 38 39

8. Selfdual

0 1 2 3 4 22 23 24 20 21 42 43 44 40 41 27 28 29 25 26 12 13 14
10 11 37 38 39 35 36 17 18 19 15 16 32 33 34 30 31 7 8 9 5 6

2 3 4 0 1 40 41 42 43 44 20 21 22 23 24 30 31 32 33 34 5 6 7
8 9 15 16 17 18 19 35 36 37 38 39 25 26 27 28 29 10 11 12 13 14

9. Selfdual

30 34 33 32 31 42 41 40 44 43 38 37 36 35 39 28 27 26 25 29 9 8 7
6 5 14 13 12 11 10 22 21 20 24 23 16 15 19 18 17 3 2 1 0 4

39 38 37 36 35 28 27 26 25 29 34 33 32 31 30 16 15 19 18 17 23 22 21
20 24 42 41 40 44 43 6 5 9 8 7 0 4 3 2 1 14 13 12 11 10

10. Selfdual

40 44 43 42 41 5 9 8 7 6 31 30 34 33 32 37 36 35 39 38 2 1 0
4 3 28 27 26 25 29 24 23 22 21 20 10 14 13 12 11 17 16 15 19 18

22 21 20 24 23 5 9 8 7 6 35 39 38 37 36 42 41 40 44 43 34 33 32
31 30 28 27 26 25 29 11 10 14 13 12 17 16 15 19 18 0 4 3 2 1

11. Selfdual

20 24 23 22 21 33 32 31 30 34 12 11 10 14 13 9 8 7 6 5 37 36 35
39 38 43 42 41 40 44 18 17 16 15 19 4 3 2 1 0 27 26 25 29 28

39 38 37 36 35 19 18 17 16 15 12 11 10 14 13 33 32 31 30 34 0 4 3
2 1 42 41 40 44 43 8 7 6 5 9 22 21 20 24 23 28 27 26 25 29

12. Selfdual

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 35 39 38 37 36 40 44 43 42 41

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 35 39 38 37 36 40 44 43 42 41

13. Selfdual

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 40 44 43 42 41 35 39 38 37 36

0 4 3 2 1 5 9 8 7 6 10 14 13 12 11 15 19 18 17 16 20 24 23
22 21 25 29 28 27 26 30 34 33 32 31 40 44 43 42 41 35 39 38 37 36

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