k-equitable Labellings of Cycles and Some Other Graphs

Zsuzsanna Szaniszló

Department of Mathematics
University of Nebraska-Lincoln
810 Oldfather Hall
Lincoln,NE 68588

Department of Mathematics
Kossuth University
4010 Debrecen, Hungary

Abstract. In this thesis we examine the k-equitability of certain graphs. We prove the following: The path on n vertices, P_n, is k-equitable for any natural number k. The cycle on k vertices, C_n, is k-equitable for any natural number k, if and only if all of the following conditions hold: $n \neq k$; if $k \equiv 2,3 \pmod 4$ then $n \neq k-1$; if $k \equiv 2,3 \pmod 4$ then $n \not\equiv k \pmod{2k}$. The only 2-equitable complete graphs are K_1, K_2, and K_3. The complete graph on n vertices, K_n, is not k-equitable for any natural number k for which $3 \leq k < n$. If $k \geq n$, then determining the k-equitability of K_n is equivalent to solving a well-known open combinatorial problem involving the notching of a metal bar. The star on $n+1$ vertices, S_n, is k-equitable for any natural number k. The complete bipartite graph $K_{2,n}$ is k-equitable for any natural number k if and only if $n \equiv k-1 \pmod k$, or $n \equiv 0, 1, \ldots, [k/2]-1 \pmod k$, or $n = [k/2]$ and k is odd.

1. Introduction.

The definition of k-equitable labellings was introduced by I. Cahit in [1] as a natural generalization of cordial labellings. In [1] and [3] Cahit proves results on 3-equitability of certain graphs. In this paper we find necessary and sufficient conditions for k-equitability of cycles and some other graphs for arbitrary natural number k.

2. Basic definitions and results.

By a graph we mean a finite undirected graph without loops and multiple edges.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$.

A labelling (or numbering) of G is a mapping f of the vertex set to the natural numbers N. If f is a labelling, $v_i, v_j \in V(G)$ and $e = (v_i, v_j) \in E(G)$ then $f(e) = |f(v_i) - f(v_j)|$.

Definition 2.1: Let G be a graph and f be a labelling such that $f: V(G) \to \{0, 1, \ldots, k-1\} \subseteq N$. Let $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges, respectively, with the label i. The labelling is called vertex-k-equitable if

$$|v_f(i) - v_f(j)| \leq 1 \text{ for all } i, j \in \{0, 1, \ldots, k-1\}.$$

The labelling is called edge-k-equitable if

$$|e_f(i) - e_f(j)| \leq 1 \text{ for all } i,j \in \{0,1,\dots,k-1\}.$$

A labelling is called k-equitable if it is both edge- and vertex-k-equitable. A graph G is said to be k-equitable if it admits a k-equitable labelling.

Remark 2.2: The definition of k-equitability was introduced by I. Cahit [1]. In the case $k = 2$ the labelling was called cordial by Cahit [2].

Cahit proved that every tree is 2-equitable [2], and that the path on n vertices, P_n, is 3-equitable for any $n \in N$ [1].

Theorem 2.3. *The path on n vertices, P_n, is k-equitable for any $k, n \in N$.*

Proof: Consider the following labelling: for any $h = 0,1,\dots$

$$f(v_{2hk+2i+1}) = f(v_{2hk-2i}) = i \text{ where } i = 0,1,\dots,[k/2]$$
$$f(v_{2hk+2i}) = f(v_{2hk-(2i-1)}) = k-1-(i-1) \text{ where } i=0,1,\dots,[k/2]-1$$
if k is odd and $i=0,1,\dots,k/2$ if k is even.

Thus, as the first $2k$ vertex-labels we have:

$$f(v_1) = 0 = f(v_{2k})$$
$$f(v_2) = k-1 = f(v_{2k-1})$$
$$f(v_3) = 1 = f(v_{2k-2})$$
$$f(v_4) = k-2 = f(v_{2k-3})$$
$$\dots$$
$$f(v_k) = [k/2] = f(v_{2k-(k-1)}).$$

The actual labelling of the path consists of copies of this sequence of vertex-labels. It is clear that the labelling is vertex-k-equitable, and it is easy to check that it is edge-k-equitable as well. ∎

Remark 2.4: From now on, the labelling f defined in the proof of Theorem 2.3. will be called the "basic" labelling.

Example 2.5: 5-equitable labelling of P_{18}:

0 4 1 3 2 2 3 1 4 0 0 4 1 3 2 2 3 1

3. The k-equitability of cycles.

Let C_n denote the cycle on n vertices.

Theorem 3.1. *C_n is 2-equitable if and only if $n \not\equiv 2 \pmod 4$. See [2].*

Theorem 3.2. *C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$. See [3]. Now suppose $k > 3$ and let $n \equiv r \pmod{2k}, 0 \leq r < 2k$.*

Lemma 3.3. *The sum of the edge-labels over a closed path must be even. See [10].*

Theorem 3.4. *There is no k-equitable labelling of C_n if any of the following conditions holds:*

(i) $n = k$;
(ii) $n = k - 1$ *and* $k \equiv 2, 3 \pmod 4$;
(iii) $r = k$ *and* $k \equiv 2, 3 \pmod 4$.

Proof: (i) In a vertex-k-equitable labelling we have to use k different vertex-labels, so there will be no edge-label 0 among the k edge-labels, therefore, at least one edge-label will occur twice, and the labelling cannot be edge-k-equitable; therefore, it cannot be k-equitable.

(ii) In a vertex-k-equitable labelling we have to use $k-1$ different vertex-labels, so there will be no edge labelled 0 among the $k-1$ edges. If the labelling is edge-k-equitable as well, then all the edge-labels are different and, thus, the sum of the edge-labels is $k(k-1)/2$. This sum must be even by Lemma 3.3.

Therefore, if $k \equiv 2, 3 \pmod 4$, there is no k-equitable labelling of C_{k-1}.

(iii) If $n = (2A+1)k$ and $A \neq 0$, then each edge- and vertex-label must occur exactly $(2A+1)$ times in a k-equitable labelling. The sum of the edge-labels is $(2A+1)(k-1+0)k/2 = (2A+1)(k-1)k/2$. This sum must be even by Lemma 3.3. Therefore, there is no k-equitable labelling of $C_{(2A+1)k}$ if $k \equiv 2, 3 \pmod 4$.

Remark 3.5: In part (ii) k-equitable labelling would mean the same as graceful labelling since it is obvious from the definitions that a graph G with e edges is $(e+1)$-equitable if and only if G is graceful.

Theorem 3.6. *C_n is k-equitable for any n and k if and only if none of the conditions (i), (ii), (iii), in Theorem 3.5 hold.*

Proof: The necessity follows from Theorem 3.5. We prove the sufficiency by constructing k-equitable labellings for every other case.

Recall that $n \equiv r \pmod{2k}, 0 \leq r < 2k$.

The "basic" labelling (*cf.* Theorem 2.3 and Remark 2.4) gives k-equitable labelling in the following cases:

Case 1. $r = 0$.
Case 2. $r \leq k/2$ and r is odd.
Case 3. $r > k$ and r is odd.

The truth of this statement can be easily seen from the definition of the "basic" labelling.

Case 4. $r = 2$.

Use the "basic" labelling for the first $n-4$ vertices. In order to get a k-equitable labelling, for the remaining 4 vertices and the 5 edges incident with them we need the

vertex-labels : $k-1, 0, a, b$
the edge-labels: $k-1, k-2, 0, c, d$

where $a \neq b$ and $c \neq d$.

Define the labelling function f' in the following way:

$$f'(v_i) = f(v_i) \text{ if } i = 1, 2, \ldots, n-4$$

where $f(v_i)$ is the corresponding vertex-label in the "basic" labelling

$$\begin{aligned} f'(v_{n-3}) &= 1 \\ f'(v_{n-2}) &= k-2 \\ f'(v_{n-1}) &= 0 \\ f'(v_n) &= k-1. \end{aligned}$$

Now f' is a k-equitable labelling since $f'(v_1) = f(v_1) = 0$ and $f'(v_{n-4}) = f(v_{n-4}) = f(v_{2k-2}) = 1$ and the labels of the edges incident with the last four vertices are $k-1, k-1, k-2, k-3$, and 0. ∎

Example: 4-equitable labelling of C_{10}:

0 3 1 2 2 1 1 2 0 3

. — . — . — . — . — . — . — . — . — .

v_1 v_{10}

Case 5. $r \leq k/2, r = 2h, h \neq 1$.

Define f' such that if f is the "basic" labelling, then $f'(v_i) = f(v_i)$ if $i = 1, 2, \ldots, n-1$ and $f'(v_n) = [k/2]$. Then $f'(v_{n-1}) = f(v_{n-1}) = f(v_{2k+2h-1}) = h-1$ where $h = 2, \ldots, [k/4]$ and f' is obviously a vertex-k-equitable labelling. In $C_n \backslash \{v_n\}$ the labels of the edges incident with the last $r-1$ vertices are decreasing by one starting with $k-1$ if we do not consider the edge labelled 0. There are $r-2$ edges of this type, thus, each edge-label

$$\geq k-1-(r-2)+1 = k-r+2 \geq k-k/2+2 \geq k/2+2.$$

The edges incident with v_n have labels

$$\begin{aligned} f'(e_n) &= |f'(v_n) - f'(v_1)| = [k/2] - 0 = [k/2] \quad \text{and} \\ f'(e_{n-1}) &= |f'(v_n) - f'(v_{n-1})| = [k/2] - (h-1) < [k/2] \end{aligned}$$

Example: 10-equitable labelling of C_{24}. The sequence of the vertex-labels is:

$$f'(v_1) = 0-0-1-8-2-7-3-6-4-5-5-4$$
$$-6-3-7-2-8-1-9-0-0-1-9-5 = f'(v_{24})$$

Case 6. $r > k$ and r is even.

If f is the "basic" labelling, then we define f' in the following way:

$$f'(v_i) = f(v_i) \text{ if } i = 1,2,\ldots,n-1 \text{ and } f'(v_n) = 0.$$

f' is obviously a vertex-k-equitable labelling. In the last $r-k$ steps the edge-labels are increasing in the "basic" labelling, so the largest is:

$$f'(e_{n-2}) = |f'(v_{n-1}) - f'(v_{n-2})| = k - 2(i+1).$$

On the other hand,

$$f'(e_{n-1}) = |f'(v_n) - f'(v_{n-1})| = k - (i+1)$$

since

$$f'(v_{n-1}) = f(v_{n-1}) = f(v_{2k-2i-1}) = k - 1 - i$$

and

$$f'(v_{n-2}) = f(v_{n-2}) = f(v_{2k-2i-2}) = i + 1.$$

Since $f'(e_n) = 0$ we can conclude that f' is a k-equitable labelling.

Example: 10-equitable labelling of C_{18}. The sequence of the vertex-labels is:

$$f'(v_1) = 0-9-1-8-2-7-3-6-4-5-5-4-6-3-7-2-8-0 = f'(v_{18})$$

Case 7. $k/2 < r < k$ and $k-r$ is even.

Let $n = 2Ak + r$ and f be the "basic" labelling. Define the labelling f' in the following way:

$$f'(v_i) = f(v_i) \text{ if } i = 1,2,3,\ldots,2Ak$$
$$f'(v_{2Ak+1}) = f(v_k)$$
$$f'(v_{2Ak+2}) = f(v_{k-1})$$
$$\ldots$$
$$f'(v_{2Ak+r}) = f(v_{k-(r+1)})$$

Though $v_{2Ak+r+1}$ and $v_{2Ak+r+2}$ do not exist, we may define the following two numbers:

$$f'(v_{2Ak+r+1}) = f(v_{k-r})$$
$$f'(v_{2Ak+r+2}) = f(v_{k-(r+1)}).$$

Now define the labelling f'' such that

$$f''(v_i) = f'(v_i) \text{ if } i = 1, 2, \ldots, 2Ak$$
$$f''(v_{2Ak+1}) = f'(v_{2Ak+1})$$
$$f''(v_{2Ak+i}) = f'(v_{2Ak+i}) \text{ if } f'(e_{i-2}) \leq [k/2] - 1$$
$$f''(v_{2Ak+j}) = f'(v_{2Ak+j+2}) \text{ if } f'(e_{j-2}) > [k-2] - 1 \text{ and } j < r$$
$$f''(v_n) = 0.$$

That is, f'' differs from f' in the following way: $f''(v_n) = 0$, and instead of the label of the vertex which is adjacent to the edges labelled $[k/2]$ and $[k/2]+1$ in the f' labelling, we use the second next vertex-label (from f') as the f'' label and continue the labelling from there.

In this way f'' is a k-equitable labelling.

Indeed, the vertex-k-equitability is obvious. For the edge-equitability note that

$$f''(v_{2Ak}) = f(v_{2Ak}) = 0$$
$$f''(v_{2Ak+1}) = f(v_k) = [k/2]$$
$$f''(v_{2Ak+r-1}) = f''(v_{n-1}) = f(v_{k-r})$$
$$= (k-1) - ((k-r)/2 - 1) = k - k/2 + r/2 = k/2 + r/2 \geq r + 1.$$

Thus,

$$f''(e_{2Ak}) = [k/2]$$
$$f''(e_{n-1}) = f''(e_{2Ak+r-1}) = f''(v_{2Ak+r-1})$$
$$f''(e_n) = 0$$

by definition, and the edges between e_{2Ak+1} and e_{n-2} have the labels $1, \ldots, r$. (It is an increasing sequence, and we cut out the edge-labels $[k/2]$ and $[k/2]+1$.) Therefore, f'' is a k-equitable labelling.

Example: $k = 10$ $[k/2] = 5$.

The sequence of the vertex-labels in the "basic" labelling is: 0, 9, 1, 8, 2, 7, 3, 6, 4, 5, ...

$f''(C_{2Ak+r})$ if $r = 8$:

0 9 ... 0 5 4 6 3 7 1 9 0

v_1 v_2 ... v_{2Ak} ... v_n

Case 8. $k/2 < r < k-1$ and $k-r$ is odd.

This case is very similar to Case 7. The only difference is that we do not define the label of the last vertex separately, that is, $f''(v_n) = f'(v_{2Ak+r+2})$.

Thus, f'' is a k-equitable labelling since

$$f''(v_n) = f'(v_{2Ak+r+2}) = f(v_{k-(r+1)}) = (k-1)-(k-(r+1))/2$$
$$= k/2 + r/2 + 1/2$$

$$f''(v_{n-1}) = \begin{cases} f'(v_{2Ak+r+1} & \text{if } f'(e_{r-1}) > [k/2]-1 \\ f'(v_{2Ak+r-1}) & \text{if } f'(e_{r-1}) \le [k/2]-1 \end{cases}$$

$$f'(v_{2Ak+r+1}) = f(v_{k-r}) = (k-r-1)/2$$

$$f'(v_{2Ak+r-1}) = f(v_{k-r+2}) = (k-r+2-1)/2 = (k-r+1)/2.$$

Therefore,

$$f''(e_n) = f''(v_n) = (k+r+1)/2 > r+1$$

$$f''(e_{n-1}) = |f''(v_n) - f''(v_{n-1})|$$
$$= \begin{cases} (k+r+1)/2-(k-r-1)/2 = r+1 \text{ if } f'(e_{r-1}) > [k/2]-1 \\ (k+r+1)/2-(k-r+1)/2 = r \text{ if } f'(e_{r-1}) \le [k/2]-1 \end{cases}$$

$$f''(e_{2Ak}) = f''(v_{2Ak+1}) = [k/2].$$

The edges between e_{2Ak+1} and e_{n-2} have the labels $1,\dots,r-1$ if $f'(e_{r-1}) \le [k/2]-1$ or $1,\dots,r+1$ if $f^P(e_{r-1}) > [k/2]-1$.

Example: $k = 10$, $r = 7$.

0 9 0 5 4 6 3 7 1 9

v_1 v_2 v_{2Ak} v_n

Case 9. $r = k-1$.

If $n = 2Ak + k - 1$ for some $A \in N$, then if $A = 0$, we have the same problem as that of the graceful labelling of C_{n-1}, which was solved in [10] (*cf.* Remark 3.6).

If $A \neq 0$ and $k \equiv 0, 1 \pmod 4$, we can use the "basic" labelling for the first $2Ak$ vertices and a graceful labelling for the last $k-1$ vertices. If $A \neq 0$ and $k \equiv 2, 3 \pmod 4$, then in a k-equitable labelling one edge- and one vertex-label appears one time less than all the others. The sum of the edge-labels must be even, no matter what the "missing" vertex-label is.

In this case $k(k-1)/2$ is odd and $[k/2]$ is also odd, so we can leave out the edge-label $[k/2]$, for example.

Let f be the "basic" labelling, and define f' in the following way:

$$f'(v_i) = f(v_i) \text{ if } i \le (2A-1)k$$

for the last $3k-1$ vertices let

$$f'(v_{2(A-1)k+1}) = f'(v_{2(A-1)k+3}) = 0$$
$$f'(v_{2(A-1)k+2j+1}) = h \text{ if } j = 3h \text{ or } j = 3h+1 \text{ or } j = 3h-1 \text{ and } 2 \le j \le 3k/2 - 1$$
$$f'(v_{2(A-1)k+2j}) = k - p \text{ if } j = 3p \text{ or } j = 3p-1 \text{ or } j = 3p-2 \text{ and } 0 < j < (3k-1)/2$$
$$f'(v_{2(A-1)k+3k-1}) = [k/2].$$

In this way, $f'(v_{2(A-1)k+3k-2}) = [k/2]$ since

a)

If k is odd then $3k - 2 = 2j + 1$, that is, $2j + 1 \equiv (-2) \pmod 3$.
If $j = 3p$ then $2j + 1 = 6p + 1 \equiv (-2) \pmod 3$,
if $j = 3p - 1$ then $2j + 1 = 6p - 1 \equiv (-1) \pmod 3$,
if $j = 3p - 2$ then $2j + 1 = 6p - 3 \equiv 0 \pmod 3$,
thus, $j = 3p$ and $3k-2 = 6p + 1$ $p = (3k-3)/6 = k/2 - 1/2 = [k/2]$.

b)

If k is even then $3k - 2 = 2j$, that is, $2j \equiv (-2) \pmod 3$.
If $j = 3h$ then $2j = 6h \equiv 0 \pmod 3$,
if $j = 3h + 1$ then $2j = 6h + 2 \equiv 2 \pmod 3$,
if $j = 3h - 1$ then $2j = 6h - 2 \equiv (-2) \pmod 3$,
thus, $j = 3h - 1$ and $3k - 2 = 6h - 2$ $h = k/2 = [k/2]$.

Now define the labelling f'' such that

$$f''(v_i) = f'(v_{i+1}) \text{ if } f'(e_{i-1}) = f'(e_i) = [k/2],$$
$$f''(v_i) = f'(v_i) \text{ otherwise.}$$

It is easy to see that f'' is a k-equitable labelling. The vertex-k-equitability is clear. For the edge-k-equitability note that among the last $3k - 1$ edge-labels in the labelling f' we have 3 of each edge-label except 0 and $[k/2]$. 0 occurs once as the label of e_{n-1} and $[k/2]$ occurs four times, since $f'(e_n) = [k/2]$. The difference between f' and f'' is that we interchange the labels of the two vertices that are incident with exactly two edges labelled $[k/2]$. In this way we lose three $[k/2]$ edge-labels, but gain two 0 and one $[k/2]$ edge-label.

Hence, f'' is a k-equitable labelling.

Example: $k = 10$ $r = 9$

$f'(C_n)$:

$0 - \ldots - 0 - 9 - 0 - 9 - 1 - 9 - 1 - 8 - 1 - 8 - 2 - 8 - 2 - 7 - 2$
$- 7 - 3 - 7 - 3 - 6 - 3 - 6 - 4 - 6 - 4 - 5 - 4 - 5 - 5$

$f''(C_n)$:

$0 - \ldots - 0 - 9 - 0 - 9 - 1 - 9 - 1 - 8 - 1 - 8 - 2 - 8 - 2 - 2 - 7$
$- 7 - 3 - 7 - 3 - 6 - 3 - 6 - 4 - 6 - 4 - 5 - 4 - 5 - 5$

Case 10. $r = k$ that is, $n = (2A+1)k$ for some $A \in N$.

The non-existence of a k-equitable labelling for $A = 0$ and for $k \equiv 2,3 \pmod 4$ when $A \neq 0$ is given by Theorem 3.5. We construct k-equitable labellings for the remaining cases.

a) $k \equiv 0 \pmod 4$ that is, $k = 4B$ for some $B \in N$.

Let f be the "basic" labelling and use the notation $v'_i = v_{2(A-1)k+i}$. Now define the labelling f' in the following way:

$$
\begin{aligned}
f'(v_i) &= f(v_i) \text{ if } 1 \le i \le 2(A-1)k \\
f'(v'_1) &= f'(v'_2) = f'(v'_4) = 0 \\
f'(v'_{2i+1}) &= k - p \text{ if } i = 3p \\
&\quad \text{or } i = 3p-1 \text{ or } i = 3p-2 \text{ and } 1 \le i \le 3B \\
f'(v'_{2i}) &= p \text{ if } i = 3p \\
&\quad \text{or } i = 3p+1 \text{ or } i = 3p+2 \text{ and } 3 \le i \le 3B \\
f'(v'_{6B+2}) &= f'(v'_{6B+4}) = f'(v'_{6B+8}) = 3B-1 \\
f'(v'_{6B+3}) &= f'(v'_{6B+5}) = B \\
f'(v'_{6B+4i+2}) &= 3B - 2p \text{ if } i = 3p-2 \\
&\quad \text{or } i = 3p-1 \text{ or } i = 3p \text{ and } 1 \le i < (6B-5)/4 \\
f'(v'_{6B+4i}) &= 3B - (2p+1) \text{ if } i = 3p \\
&\quad \text{or } i = 3p+1 \text{ or } i = 3p+2 \text{ and } 3 \le i < (6B-7)/4 \\
f'(v'_{6B+2i+1}) &= B + p \text{ if } i = 3p \\
&\quad \text{or } i = 3p+1 \text{ or } i = 3p+2 \text{ and } 3 \le i \le 3B-4 \\
f'(v'_{12B-6}) &= f'(v'_{12B-1}) = f'(v'_{12B}) = 2B \\
f'(v'_{12B-5}) &= f'(v'_{12B-3}) = f'(v'_{12B-2}) = 2B-1 \text{ and} \\
f'(v'_{12B-4}) &= 2B+1.
\end{aligned}
$$

That is, the sequence of the last $3k$ vertex-labels is:

$$
\begin{aligned}
&0 - 0 - (k-1) - 0 - (k-1) - 1 - (k-1) - 1 - (k-2) - 1 - (k-2) \\
&- 2 - (k-2) - 2 - \ldots - (B-1) - 3B - (B-1) - 3B - B - 3B \\
&- - - - -(3B-1) - B - (3B-1) - B - - - - -(3B-2) - (B+1) \\
&- (3B-1) - (B+1) - (3B-2) - (B+1) - (3B-3) - (B+2) \\
&- (3B-2) - (B+2) - \ldots - (2B+1) - (2B-2) - (2B+2) \\
&- (2B-2) - (2B+1) - (2B-2) - - - - -2B - (2B-1) \\
&- (2B+1) - (2B-1) - (2B-1) - 2B - 2B.
\end{aligned}
$$

The labelling f' is obviously k-vertex-equitable and it is easy to check that it is k-edge-equitable as well.

Example: $k = 8$ $B = 2$.

$$f'(v_{2(A-1)k}) = 0-0-0-7-0-7-1-7-1-6-1-6-2$$
$$-6--5-2-5-2--4-3-5-3-3-4-4 = f'(v_n)$$

b) $k \equiv 1 \pmod 4$ that is, $k = 4B+1$ for some $B \in N$. As in part a) let f be the "basic" labelling and use the notation $v'_i = v_{2(A-1)k+i}$. If $k = 5$, then define the labelling f' in the following way:

$$f'(v_i) = f(v_i) \text{ if } 1 \le i \le 2(A-1)k$$
$$f'(v'_1) = f'(v'_2) = f'(v'_4) = 0$$
$$f'(v'_3) = f'(v'_5) = f'(v'_7) = 4$$
$$f'(v'_6) = f'(v'_8) = f'(v'_{10}) = 1$$
$$f'(v'_9) = f'(v'_{12}) = f'(v'_{13}) = 3$$
$$f'(v'_{11}) = f'(v'_{14}) = f'(v'_{15}) = 2.$$

That is:

$$f'(v_{(2A-1)k}) = 0-0-0-4-0-4-1-4-1$$
$$-3-1-2-3-3-2-2 = f'(v_n).$$

Suppose $k > 5$. Now we can define a k-equitable labelling in a very similar way to part a). Let

$$f'(v_i) = f(v_i) \text{if } 1 \le i \le 2(A-1)k$$
$$f'(v'_1) = f'(v'_2) = f'(v'_4) = 0$$
$$f'(v'_{2i+1}) = k-p \text{ if } i = 3p$$
$$\text{or } i = 3p-1 \text{ or } i = 3p-2 \text{ and } 1 \le i \le 3B+1$$
$$f'(v'_{2i}) = p \text{ if } i = 3p$$
$$\text{or } i = 3p+1 \text{ or } i = 3p+2 \text{ and } 3 \le i \le 3B+2$$
$$f'(v'_{6B+5}) = f'(v'_{6B+7}) = f'(v'_{6B+11}) = B+1$$
$$f'(v'_{6B+6}) = f'(v'_{6B+8}) = 3B$$
$$f'(v'_{6B+2i}) = 3B-(p-1) \text{ if } i = 3p-2$$
$$\text{or } i = 3p-1 \text{ or } i = 3p \text{ and } 5 \le i < (6B-7)/2$$
$$f'(v'_{6B+4i+1}) = B+2p \text{ if } i = 3p-1$$
$$\text{or } i = 3p \text{ or } i = 3p+1 \text{ and } 2 \le i \le (6B-6)/4$$
$$f'(v'_{6B+4i-1}) = B+2p-1 \text{ if } i = 3p-2$$
$$\text{or } i = 3p-1 \text{ or } i = 3p \text{ and } 4 \le i \le (6B-8)/4$$
$$f'(v'_{12B-6}) = f'(v'_{12B-1}) = f'(v'_{12B}) = 2B$$
$$f'(v'_{12B-5}) = f'(v'_{12B-3}) = f'(v'_{12B-2}) = 2B+1 \text{ and}$$
$$f'(v'_{12B-4}) = 2B-1.$$

The sequence of the last $3k$ vertex-labels is:

$$0 - 0 - (k-1) - 0 - (k-1) - 1 - (k-1) - 1 - \ldots$$
$$- (3B+1) - B - (3B+1) - B - 3B - B$$
$$- -(B+1) - 3B - (B+1) - 3B - -(B+2) - (3B-1)$$
$$- (B+1) - (3B-1) - (B+2) - (3B-1) - (B+3)$$
$$- \ldots - (2B-2) - (2B+2) - (2B-3) - (2B+2)$$
$$- (2B-2) - -2B - (2B+1) - (2B-1) - (2B+1)$$
$$- (2B+1) - 2B - 2B.$$

Example: $k = 9$.

$$f'(v_{2(A-1)k}) = 0 - 0 - 0 - 8 - 0 - 8 - 1 - 8 - 1 - 7 - 1 - 7 - 2 - 7 - 2$$
$$- 6 - 2 - -3 - 6 - 3 - 6 - -4 - 5 - 3 - 5 - 5 - 4 - 4 = f'(v_n)$$

4. k-equitable labellings of K_n.

Denote by K_n the complete graph on n vertices. Thus, K_n has n vertices and $n(n-1)/2$ edges. First note that in a k-equitable labelling, if a number occurs at least twice as a label, then all the possible corresponding labels (that is, vertex-labels or edge-labels) must have been used.

Case 4.1. $k = 2$.

Suppose we have h vertices labelled 0 and m labelled 1 in an equitable labelling. Then the number of the edges labelled 0 is $h(h-1) + m(m-1))/2$ and the number of the edges labelled 1 is $(hm + mh)/2$. The labelling is edge-equitàble exactly if $|(h(h-1) + m(m-1)/2 - 2hm/2| \leq 1$. The labelling is vertex-equitable exactly if $|h - m| \leq 1$. If we have a vertex-equitable labelling, then we have the following possibilities:

a) $h = m$ b) $h = m + 1$ c) $h = m - 1$.

A vertex-equitable labelling is equitable if and only if it is edge-equitable as well. This means we must have the following conditions:

a) (when $h = m$).

$|2(h^2 - h) - 2h^2| \leq 2$ that is, $|2h| \leq 2$ that is, $h \leq 1$, thus, $h = m = 1$ or $h = m = 0$.

b) (when $h = m + 1$).

$|(m+1)m + m(m-1) - 2m(m+1)| \leq 2$ that is, $|-2m| \leq 2$ that is, $m \leq 1$, thus, $m = 1$ and $h = 2$, or $m = 0$ and $h = 1$.

c) (when $h = m - 1$).

Because of the symmetry this gives the same result as part b). Therefore, the only 2-equitable complete graphs are K_1, K_2 and K_3.

Case 4.2, $k > 2$.

Let $n > k$. Assume we have a vertex-equitable labelling. Denote by A the maximum number of occurences of the vertex-labels. Naturally $A \geq 1$. If $A = 1$, then there cannot be an edge labelled 0 in a vertex-equitable labelling since each vertex-label can occur at most once. Hence, $k \leq n$. Thus, if $n > k$, then $A > 1$.

In a vertex-equitable labelling we have the following possibilities:

	number of occurences			
vertex-label 0	A	A	A-1	A-1
vertex-label $(k-1)$	A	A-1	A	A-1
edge-label $(k-1)$	A^2	A(A-1)		$(A-1)^2$
case	a	b		c

Each vertex-label occurs at least $(A-1)$ times in a vertex-equitable labelling, hence, the label 1 occurs at least

a) $2A(A-1) + (k-3)(A-1)^2$ b) $A(A-1) + (k-2)(A-1)^2$ c) $(k-1)(A-1)^2$ times.

In an edge-equitable labelling the number of the edges labelled 1 and $k-1$ can differ at most by one. This gives the following conditions:

a) $|2A(A-1) + (k-3)(A-1)^2 - A^2| \leq 1$ that is, $|A^2 - 2A + (k-3)(A-1)^2| \leq 1$.

If $A > 2$ then $A^2 - 2A \geq 3$ and $(k-3)(A-1)^2 \geq 0$, hence, there is no solution to the inequality if $A > 2$.

If $A = 2$ then $|4 - 4 + (k-3)1^2| = |k-3| \leq 1$, hence, $k \leq 4$.

b) $|A(A-1) + (k-2)(A-1)^2 - A(A-1)| \leq 1$ that is, $|(k-2)(A-1)^2| \leq 1$.

Since $A > 1$, this can happen if and only if $|k-2| \leq 1$, that is, $k \leq 3$. Therefore, $k = 3$, since $k > 2$, and, hence, $A = 2$.

c) $|(k-1)(A-1)^2 - (A-1)^2| \leq 1$ that is, $|(k-2)(A-1)^2| \leq 1$.

Similarly to part b) we get $k = 3$ and $A = 2$.

Thus, if $A > 1$ then $A = 2$, and k can be 3 or 4.

In a k-equitable labelling of K_n, therefore, each of the vertex-labels can occur at most twice, at least one of them does occur twice, and, hence, each of the vertex-labels occurs at least once. Thus, $k + 1 \leq n \leq 2k$.

Hence, if $n = k + i$ then obviously the number of the edges labelled 0 is exactly i, since the labelling is k-equitable.

Now let us consider the number of the edges labelled 1. When we use each vertex-label exactly once, we have $k - 1$ edges labelled 1. After this each new non-zero vertex-label gives at least two edges labelled 1, while a vertex labelled 0 gives at least one edge labelled 1. Thus, the number of the edges labelled 1 is at least $k - 1 + 1 + 2(i-1) = k + 2i - 2$. Since the number of the edges labelled 0 is i, we must have $|(k + 2i - 2) - i| \leq 1$, that is, $|k + i| \leq 3$. Since $i \geq 1$ and $k \geq 3$, there is, therefore, no k-equitable labelling of K_n if $k < n$.

Therefore, let $n \leq k$.

Then in a k-equitable labelling of K_n the edge-label 0 cannot occur, since all the vertex-labels must be different, thus, K_n is k-equitable if and only if all the $n(n-1)/2$ edges have different labels. Therefore, $k > n(n-1)/2$ is a natural necessary condition for K_n to be k-equitable. To find a sufficient condition one should solve the classical combinatorial problem involving the notching of a metal bar of length k at integer points in such a way that all the distances between two notches, or between a notch and an end point, are distinct. If there are $(n-2)$ notches and 2 end points, then there are $n(n-1)/2$ lengths which must be distinct. Hence, this problem is equivalent to finding the smallest k for which K_n is k-equitable.

According to [6] the best known k values for K_n if $2 \leq n \leq 10$ are:

$n = 2$	$k = 2$	$n = 7$	$k = 26$
$n = 3$	$k = 4$	$n = 8$	$k = 37$
$n = 4$	$k = 7$	$n = 9$	$k = 49$
$n = 5$	$k = 12$	$n = 10$	$k = 65$
$n = 6$	$k = 18$		

5. k-equitable labellings of complete bipartite graphs.

Denote by $K_{m,n}$ the complete bipartite graph on $m + n$ vertices; that is, the graph $K_{m,n}$ has the vertex set $V(K_{m,n}) = V_1 \cup V_2$ where $|V_1| = m$ and $|V_2| = n$ and $K_{m,n}$ has the edge set $E(K_{m,n}) = \{e_{i,j} = (u_i, v_j) \colon u_i \in V_1, v_j \in V_2\}$. $K_{1,n}$ is frequently called a star on n vertices and denoted by S_n.

Lemma 5.1. *S_n is k-equitable for any $k, n \in N$.*

Proof: Label the vertex in the centre by 0 and label the end-vertices with copies of the decreasing sequence $(k-1), (k-2), \ldots, 0$. This labelling is obviously k-equitable. ∎

Lemma 5.2. *In a k-equitable labelling of $K_{2,n}$, V_1 consists of vertices labelled 0 and $k-1$, whenever $n > k+2$.*

Proof: Two identical vertex-labels cannot occur in V_1, because if they do, in a vertex-equitable labelling there would be at least two fewer edge-labels 0 than any other edge-labels.

In order to have edges labelled $k-1$, either the vertex-label 0 or the vertex-label $k-1$ must occur in V_1. Suppose only one of them occurs. There are $n+2$ vertices in $K_{2,n}$, which means that in a vertex-equitable labelling at most $[(n+2)/k] + 1$ vertex-labels of the same kind can occur. Hence, this is the maximum number of the edge-labels $k-1$ as well.

There are $2n$ edges in $K_{2,n}$, thus, in a k-equitable labelling there are at least $[2n/k]$ of each different edge-label. Therefore, in an edge-equitable labelling the following relation must hold: $[(n+2)/k] + 1 \geq [2n/k]$. Thus, we must have $(n+2)/k + 1 \geq 2nk$, that is, $n + 2 + k \geq 2n$, that is, $2 + k \geq n$ which contradicts our assumption that $n > k+2$. ∎

Theorem 5.3. *Let $n \equiv h \pmod{k}, 0 \leq h < k$. Then $K_{2,n}$ is k-equitable if and only if any of the following conditions hold:*

(i) $h = k - 1$
(ii) $h \leq k/2 - 1$
(iii) $n = [k/2]$ *and* k *is odd.*

Proof: Necessity: Let $n > k + 2$. By Lemma 5.2 the vertex-labels in V_1 are 0 and $k - 1$. Hence, the edge-label 0 can occur only if 0 or $k - 1$ is used as a vertex-label in V_2. In order to get a vertex-equitable labelling, the vertex-labels 0 and $k - 1$ can be used a certain number of times in $V - 2$ only if all the other vertex-labels have already been used as many times. This can happen when $h = 0$ or $h = k - 1$.

If $0 < h < k - 1$, then in an edge-equitable labelling each edge-label occurs at least $[2n/k]$ times and cannot occur more than $[2n/k] + 1$ times. To have $[2n/k]$ edges labelled 0, we have to use at least $[n/k]$ of each of the vertex-labels 0 and $k - 1$. That means that all the other vertex-labels must have been used at least $[n/k]$ times as well. For vertex-equitability the remaining vertices must have different labels, and for edge-equitability, the arising $2h$ edge-labels must be all different as well. Among these $2h$ edge-labels there is neither 0 nor $k - 1$, hence, $2h \leq k - 2$, that is, $h \leq k/2 - 1$. Let $n \leq k - 2$. Then there is no edge labelled 0 among the $2n$ edges in a vertex-equitable labelling. Therefore, if the labelling is edge-equitable as well, all the edge-labels must be different. Thus, $2n \leq k - 1$ and $n \leq k/2 - 1/2$.

If k is even then $n \leq k/2 - 1/2$ if and only if $n \leq k/2 - 1$.
If k is odd then $n \leq k/2 - 1/2$ if and only if $n \leq [k/2]$.

Sufficiency: (i) In V_1 use the vertex-labels 0 and $k - 1$. In V_2 use the vertex-label 0 exactly one time less than any other vertex-labels. This labelling is k-equitable. Indeed, the vertex-equitability is obvious. For the edge-equitability note that there are $2Ak + 2k - 2$ edges in $K_{2,n}$, where $n = Ak + k - 1$. The "missing" edge-labels, that is, the edge-labels that occur one time less than any other, are 0 and $k - 1$.

(ii) In V_1 use the vertex-labels 0 and $k - 1$ again, in V_2 use the vertex-labels $1, 2, \ldots, h$ one more time than any other vertex-labels.

This labelling is obviously vertex-k-equitable. For the edge-equitability note that the labels of the edges adjacent to the vertex labelled i are $|0 - i| = i$ and $|k - 1 - i| = k - 1 - i$ when $1 \leq i \leq h$. Moreover, $i \leq k/2 - 1$, hence, $|0 - i| \leq k/2 - i$ and $|k - 1 - i| \geq k - 1 - (k/2 - 1) = k/2$.

(iii) In V_1 use the vertex-labels 0 and 1, and in V_2 use the vertex-labels $2i$ where $0 \leq i \leq [k/2]$. Thus, the vertex-labels are obviously different and the edge-labels are different as well, since the labels of the two edges adjacent to any vertex in V_2 have different parity. Hence, the labelling is k-equitable.

Conclusion.

Many open problems concerning k-equitability remain. Of these, probably the most "hopeful" problem is to determine necessary and sufficient conditions for k-equitability of complete bipartite graphs $K_{m,n}$ with $n \geq 3$. As for trees, even to prove k-equitability ($k \geq 4$) of all caterpillars appears to be quite difficult: it is probably more realistic to try to prove this for some special classes of caterpillars. On the other hand, it is possible that the proving of the k-equitability of the caterpillars or even the k-equitability of all trees for some specific values of k (for example, $k = 3$) is an easier problem.

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