On the Iteration of Graph Labelings

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Abstract. A labeling (function) of a graph G is an assignment f of nonnegative integers to the vertices of G. Such a labeling of G induces a labeling of $L(G)$, the line graph of G, by assigning to each edge uv of G the label $|f(u) - f(v)|$. In this paper we investigate the iteration of such graph labelings.

1. Introduction

Given any finite simple graph G, $V(G)$ and $E(G)$ will denote the sets of vertices and edges in G, respectively. The line graph of a graph G containing at least one edge, denoted by $L(G)$, is defined as follows: $V(L(G)) = E(G)$, and two vertices are adjacent in $L(G)$ if the corresponding edges of G are both incident with a common vertex of G. For any positive integer n, $L^n(G)$ will denote the line graph of $L^{n-1}(G)$, where $L^0(G) = G$. The reader is referred to [4] for properties and characterizations of a line graph.

A *labeling (function)* of a graph G is an assignment f of nonnegative integers to the vertices of G. We use $F(G)$ to denote the set of all labelings of a graph G. We then define the function $I_G : F(G) \to F(L(G))$ as follows: given any $f \in F(G)$, $I_G(f)(uv) = |f(u) - f(v)|$ for each vertex uv in $L(G)$. I_G will be abbreviated as I if the context is clear. In general, we define $I^n : F(G) \to F(L^n(G))$ so that $I^n(f) = I(I^{n-1}(f))$ for any $f \in F(G)$. Such a graph labeling has its origin from the graceful labeling of graphs. A graph G is called *graceful* if there exists an injection $f : V(G) \to \{0, 1, \ldots, |E(G)|\}$ such that the induced mapping $f^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$, defined by $f^*(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$ is a bijection. Interest in graceful graphs began in the mid 1960's with a conjecture of G. Ringel [10] and a paper by A. Rosa [11]. In the intervening two decades, the so-called Ringel-Kotzig Conjecture that all trees are graceful has been the focus of a large number of papers. Numerous variations of graceful labelings have been investigated. For further details on graceful graphs and their applications, we refer the readers to [3] and [7].

In this paper we will consider the iteration of graph labelings. A labeling f of a graph G is called *convergent* if $I^n(f) = 0$ or $L^n(G)$ is undefined for some nonnegative integer n (the smallest such n will be called the *convergence rate of* f, denoted by $r_G(f)$, or simply $r(f)$), and f is called *divergent* otherwise (in which case we write $r(f) = \infty$). We also define $M(f) = \max\{f(u) \mid u \in V(G)\}$ and $m(f) = \min\{f(u) \mid u \in V(G)\}$. Notice that $r(f) = r(af + b)$ for any nonnegative integers a and b, where $a \neq 0$. Also, any constant labeling

f has convergence rate either zero or one, depending on whether $f = 0$, and any nonzero labeling of a graph containing no edges has convergence rate 1.

For any graph G and positive integer N, define $r(G,N) = \max\{r(f) \mid f$ is a convergent labeling of G such that $M(f) = N\}$. Let P_n denote the path containing n vertices, then we have $L^n(P_n)$ is undefined. Hence any labeling of P_n has convergence rate at most n. On the other hand, for any positive integer N, if we assign one of the two vertices in P_n of degree 1 by N and the rest of the vertices of P_n by 0, then the resulting labeling of P_n has convergence rate n. Therefore we have $r(P_n, N) = n$. Also notice that $r(G,N) = \max\{r(G_i,N) \mid 1 \leq i \leq k\}$, where $\{G_i \mid 1 \leq i \leq k\}$ is the set of components of G. Hence we will assume G is a connected graph other than P_n for the remainder of this paper.

2. A General Result

Let C_n denote the cycle on n vertices. Then the following lemma can be easily checked:

Lemma 2.1. *If $G \neq C_{2n}$ for any n, then $L(G)$ is connected and nonbipartite.*

Lemma 2.1 implies that given any graph G, $L^n(G)$ is undefined for some positive integer n if and only if G is a path. Hence for the rest of this paper we will assume $L^n(G)$ is defined for every positive integer n. We now have the following:

Lemma 2.2. *Let f be a labeling of a nonbipartite graph G. If f is not a constant function, then f is divergent.*

Proof: Let $n = r(f)$ be the convergence rate of f, and suppose $2 \leq n < \infty$. This implies that $I^n(f) = 0$, and hence $I^{n-1}(f)$ is a constant labeling in $F(L^{n-1}(G))$. By Lemma 2.1 we have $L^{n-2}(G)$ is a nonbipartite graph, hence we may select vertices $v_1, \ldots, v_t$ such that v_i is adjacent to v_{i+1} in $L^{n-2}(G)$ for $1 \leq i \leq t$, where $v_1 = v_{t+1}$ and $t \geq 3$ is odd. Assume $I^{n-2}(f)(v_i) = x_i$ for $1 \leq i \leq t$. Then we have $|x_1 - x_2| = |x_2 - x_3| = \cdots = |x_t - x_1| = I^{n-1}(f)$. This implies that $x_i - x_{i+1} = \delta_i(x_{t-1} - x_t)$ for $1 \leq i \leq t$, where $\delta_i = \pm 1$. Adding all of the t inequalities together, we obtain $(\delta_1 + \delta_2 + \cdots + \delta_t)(x_{t-1} - x_t) = 0$. Notice that $\delta_1 + \delta_2 + \cdots + \delta_t \neq 0$ since t is odd. This implies that $I^{n-1}(f) = |x_{t-1} - x_t| = 0$, which contradicts the fact that $r(t) = n$. This proves Lemma 2.2. ∎

Corallary 2.3. *Given any labeling f of a bipartite graph G with maximum valency at least 3, we have either $r(f) \leq 2$ or $r(f) = \infty$.*

Proof: By Lemma 2.1 we have $L(G)$ is nonbipartite. Hence $I(f)$ is either constant or divergent, which implies that either $r(f) \leq 2$ or $r(f) = \infty$. ∎

Given any bipartite graph G with bipartition X and Y, if we label the vertices in X with a positive N and the vertices in Y with 0, then the resulting labeling of G has convergence rate 2. Hence by Lemma 2.2 and Corollary 2.3. we have shown the following:

Theorem 2.4. *$r(G,N) = 1$ if G is nonbipartite and $r(G,N) = 2$ if G is bipartite with maximum valency at least 3.*

By Theorem 2.4 it suffices to consider the labelings of an even cycle C_n on n vertices $v_1, v_2, \dots, v_n$, where v_i is adjacent to v_{i+1} for $1 \leq i \leq n$ (the computation is reduced to modulus n). We will use $r(n, N)$ to denote $r(C_n, N)$ for the remainder of this paper.

3. Labeling Even Cycles

We write $n = 2^t q$, where t and q are both positive integers and q is odd. Clearly, any labeling f of C_n can be abbreviated as an n-tuple $(a_1, a_2, \dots, a_n)$, where $a_i = f(v_i)$ for $1 \leq i \leq n$. Hence we may assume $F(C_n)$ is the set of all n-tuples over nonnegative integers. Notice that $(a_1, \dots, a_n)$ and $(a_i, a_{i+1}, \dots, a_n, \dots, a_{i-1})$ have the same convergence rate for any $2 \leq i \leq n$. Hence we may assume $I(f) = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_n - a_1|)$. We also define $S(f) = (a_1 + a_2, a_2 + a_3, \dots, a_n + a_1)$ for any labeling $f = (a_1, \dots, a_n)$ of C_n, and $S^k(f) = S(S^{k-1}(f))$ in general. A labeling $f = (a_1, \dots, a_n)$ of C_n is *even (odd)* if a_i is even (odd) for all $1 \leq i \leq n$. Clearly we have $I^i(f) + S^i(f)$ is even for any positive integer i. The following Lemma can be easily checked (a proof of part ii) can be seen in [6]):

Lemma 3.1. *i) For any odd integer $q \geq 1$, there exists an integer $k \geq 1$ such that $q \mid (2^k - 1)$; ii) For any positive integers k and i such that $0 \leq i \leq 2^k$, the number $\binom{2^k}{i}$ is even.*

Lemma 3.2. *If a labeling $f = (a_1, a_2, \dots, a_n)$ is convergent, where $n = 2^t q$, then $a_i + a_{2^t+i}$ is even for all $1 \leq i \leq n$.*

Proof: First of all, it can be easily checked that for any positive integer k we have $S^k(f) = (x_1, x_2, \dots, x_n)$, where $x_i = \sum_{j=0}^{k} \binom{k}{j} a_{i+j}$ for $1 \leq i \leq n$. Let $k \geq 1$ be an integer such that $q \mid (2^k - 1)$ by Lemma 3.1. Then we have $n \mid (2^{(t+k)} - 2^t)$. Assume $S^{2^t}(f) = (x_1, x_2, \dots, x_n)$ and $S^{2^{(t+k)}}(f) = (y_1, y_2, \dots, y_n)$. Then for $1 \leq i \leq n$, $y_i = \sum_{j=0}^{2^{(t+k)}} \binom{2^{(t+k)}}{j} a_{i+j} \equiv a_i + a_{i+2^{(t+k)}} = a_i + a_{i+2^t} \equiv \sum_{j=0}^{2^t} \binom{2^t}{j} a_{i+j} = x_i \pmod 2$. So $S^{2^t}(f) + S^{2^{(t+k)}}(f)$ is even. Similarly, we have $S^{2^{(t+k)}}(f) + S^{2^{(t+2k)}}(f)$ is even, and so on. Since $I^i(f) + S^i(f)$ is even, $I^{2^t}(f) + I^{2^{(t+ik)}}(f)$ is even for any positive integer i. This indicates that $I^{2^t}(f)$ is even, otherwise f would be divergent. Hence $S^{2^t}(f)$ is also even, or equivalently, $a_i + a_{2^t+i}$ is even for all $1 \leq i \leq n$. This completes the proof of Lemma 3.2. ∎

The proof of Lemma 3.2 immediately implies the following:

Corallary 3.3. *Given any labeling $f = (a_1, \dots, a_n)$ of C_n, where $n = 2^t q$, we have*

i) *If f is convergent, then $I^{2^t}(f)$ is even;*

ii) *$I^n(f)$ is even whenever n is a power of 2;*
iii) *If $M(f) = 1$, then f is convergent if and only if $I^{2^t}(f) = 0$ if and only if $a_i = a_{i+2^t}$ for all $1 \leq i \leq n$.*

Applying Lemma 3.2, Corollary 3.3, and mathematical induction, we immediately have the following:

Corollary 3.4. *All labelings of C_n converge if and only if n is a power of 2.*

Corollary 3.4 was first proved in [5] (the case $n = 4$ was also indicated in [8]). A different proof using some basic properties of polynomial rings was given in [12].

Theorem 3.5. $r(2^t q, N) \leq (2^t - 1)\log_2(N+1) + 1$ *with equality if* $N = 1$.

Proof: Let $f = (a_1, \ldots, a_n)$ be a convergent labeling of $C_{2^t q}$ such that $M(f) = N$. If f is not a constant, then write $f = xf_0 + y$ so that x and y are both nonnegative integers and f_0 is a labeling which is neither even nor odd. By Corollary 3.3 i), we have $I^{2^t}(f_0)$ is even. Let k_1 denote the smallest nonnegative integer so that $I^{(k_1+1)}(f_0)$ is even, which implies that $I^{k_1}(f_0)$ must be odd by the choice of k_1. But then $k_1 \geq 1$ by the choice of f_0. If $I^{k_1}(f_0)$ is not a constant, then w rite $I^{k_1}(f_0) = x_1 f_1 + y_1$ such that $x_1 \geq 2$ and $y_1 \geq 1$ are both integers and f_1 is neither even nor odd. Similarly, let $k_2 \geq 1$ be the smallest integer such that $I^{k_2}(f_1)$ is odd, and write $I^{k_2}(f_1) = x_2 f_2 + y_2$ if $I^{k_2}(f_1)$ is not a constant, where $x_2 \geq 2$ and $y_2 \geq 1$ are both integers and f_2 is neither even nor odd, and so on. Finally, there exists a positive integer s such that $I^{k_s}(f_{s-1})$ is an odd constant labeling, and we write $I^{k_s}(f_{s-1}) = f_s$. Hence we have obtained a finite sequence of labelings $f_0, f_1, \ldots, f_s$ such that i) $r(f) = r(f_0)$; ii) $M(f) \geq M(f_0)$, $M(f_{s-1}) \geq M(f_s)$ and $M(f_{i-1}) \geq 2M(f_i) + 1$ for $1 \leq i \leq s-1$; iii) $r(f_{i-1}) = r(f_i) + k_i$ for $1 \leq i \leq s$, where $1 \leq k_i \leq 2^t - 1$; iv) f_s is an odd constant, and hence $r(f_s) = 1$. But then we have $N = M(f) \geq M(f_0) \geq 2M(f_1) + 1 \geq \cdots \geq 2^{s-1}M(f_{s-1}) + 2^{s-2} + \cdots + 1 \geq 2^s - 1$, which implies that $s \leq \log_2(N+1)$. Hence we have $r(f) = r(f_0) = r(f_s) + k_1 + k_2 + \cdots + k_s \leq s(2^t - 1) + 1 \leq (2^t - 1)\log_2(N+1) + 1$. So $r(n, N) \leq (2^t - 1)\log_2(N+1) + 1$. On the other hand, consider the labeling $f = (a_1, \ldots, a_n)$ of C_n such that $M(f) = 1$ and $a_i = 1$ if and only if $i \equiv 1 \pmod{2^t}$ for $1 \leq i \leq n$. Then we have $r(f) = 2^t$. This completes the proof of Theorem 3.5. ∎

To give a global lower bound for $r(n, N)$, we need a preliminary result. Given any labeling $f = (a_1, a_2, \ldots, a_{2s})$ of C_{2s}, define $\bar{f} = (\bar{a}_1, \ldots, \bar{a}_{2s})$, where $\bar{a}_i = M(f) - a_i$ for $1 \leq i \leq 2s$. Clearly, we have $m(\bar{f}) = 0$ and $r(f) = r(\bar{f})$. Moreover, if $m(f) = 0$, then $M(f) = M(\bar{f})$. Then we have the following:

Lemma 3.6. *Given any labeling f of C_{2s} such that $m(f) = 0$, there exists a labeling g of C_{2s} with the following properties. i) $m(g) = 0$; ii) $r(g) = r(f) + 1$ if f is convergent; and iii) $2M(f) \leq M(g) \leq 3(s-1)M(f)$.*

Proof: Given a labeling $f = (a_1 \ldots, a_{2s})$ of C_{2s} such that $m(f) = 0$, let $(i_1, i_2, \ldots, i_{2s})$ be a permutation of $1, 2, \ldots, 2s$ so that $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{2s}} = 0$. Define

$$\alpha = (a_{i_1} + a_{i_2} + \cdots + a_{i_{2s-3}}) - (a_{i_2} + a_{i_4} + \cdots + a_{i_{2s}}) - a_{i_{2s-1}} \tag{1}$$

and

$$\beta = (\bar{a}_{i_4} + \bar{a}_{i_6} + \cdots + \bar{a}_{i_{2s}}) - (\bar{a}_{i_1} + \bar{a}_{i_3} + \cdots + \bar{a}_{i_{2s-1}}) - \bar{a}_{i_2} \tag{2}$$

Then we see that $\alpha = a_{i_1} + (a_{i_3} - a_{i_2}) + \cdots + (a_{i_{2s-3}} - a_{i_{2s-4}}) - a_{i_{2s-2}} - a_{i_{2s-1}} - a_{i_{2s}} \leq a_{i_1} = M(f)$, and similarly, $\beta \leq \bar{a}_{i_{2s}} = M(f)$. Adding (1) and (2) together and noticing the fact that $a_{i_{2s}} = 0$, we can easily check that $\alpha + \beta = 2(a_{i_3} - a_{i_4} + a_{i_5} - a_{i_6} + \cdots + a_{i_{2s-3}} - a_{i_{2s-2}}) \geq 0$. This implies that either $\alpha \geq 0$ or $\beta \geq 0$. There are two cases.

i) $\alpha \geq 0$.

Let $g' = (x_1, x_2, \ldots, x_{2s})$, where $x_i = 2a_i + \alpha \leq 3M(f)$ for $1 \leq i \leq 2s$. Define

$$\delta_i = \begin{cases} 1 & \text{if } i = i_k \text{ for some odd } k \neq 2s-1 \\ -1 & \text{otherwise.} \end{cases}$$

Then we have $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{2s} x_{2s} = 0$. Now let $x = \min\{\delta_1 x_1 + \cdots + \delta_k x_k \mid 1 \leq k \leq 2s\}$, and $g = (-x, \delta_1 x_1 - x, \delta_1 x_1 + \delta_2 x_2 - x, \ldots, \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{2s-1} x_{2s-1} - x)$. Then $m(g) = 0$ and g is a labeling of C_{2s}. Also $I(g) = g' = 2f + \alpha$, which implies that $r(g) = r(f) + 1$ if f is convergent. Moreover, for some $1 \leq j \leq 2s$, we have $M(g) = \delta_1 x_1 + \cdots + \delta_j x_j - x \leq x_{i_1} + x_{i_3} + x_{i_5} + \cdots + x_{i_{2s-3}} \leq 3(s-1)M(f)$.

ii) $\beta \geq 0$

Let $g' = 2\bar{f} + \beta$. Then similar to the case i), we can find a labeling g of C_{2s} such that $m(g) = 0$, $I(g) = g'$, and $M(g) \leq 3(s-1)M(f)$. Hence in either case we have the required result. ∎

Now start with the labeling $f_0 = (1, 0, 1, 0, \ldots, 1, 0)$ of C_{2s}. Notice that $r(f_0) = 2$ and $m(f_0) = 0$. Hence we can obtain a sequence of labelings $f_0, f_1, f_2, \ldots$ of C_{2s} by Lemma 3.6 such that $r(f_k) = r(f_{k-1}) + 1$ and $2M(f_{k-1}) \leq M(f_k) \leq 3(s-1)M(f_{k-1})$ for any positive integer k.

For any natural number N, assume $M(f_k) \leq N < M(f_{k+1})$ for some k. Let $f = f_k + N - M(f_k)$. Then $M(f) = N$ and $r(f) = r(f_k) = k + 2$. On the other hand, $N + 1 \leq M(f_{k+1}) \leq 3(s-1)M(f_k) \leq \cdots \leq [3(s-1)]^{k+1}$. So $k + 1 \geq \log_2(N+1)/\log_2(3(s-1))$, which implies that $r(f) = k + 2 \geq \log_2(N+1)/\log_2(3(s-1)) + 1$. Hence we have shown the following:

Theorem 3.7. $r(2^t q, N) \geq \log_2(N+1)/\log_2(3(2^{(t-1)}q - 1)) + 1$.

4. The Case $n = 4$

When $n \equiv 0 \pmod 4$, we will show a better bound than that given in Theorem 3.7. Notice that $r(n, N) \geq r(m, N)$ if $m \mid n$ (in fact we guess that $r(2^t q, N) = r(2^t, N)$ if $t \geq 2$). Hence any lower bound on $r(4, N)$ would be also a lower bound for $r(n, N)$ when $n \equiv 0 \pmod 4$.

For $k \geq 3$, note that the equation

$$x^{k-1} + \cdots + x^2 + x = 1 \tag{3}$$

has a unique positive real solution, which is denoted by p_k. In fact it can be checked that $\frac{1}{2} < p_{k+1} < p_k < 1$ for any $k \geq 3$. We will use p to denote $p_4 \doteq .543689012$ for convenience. For any real number x, we use $[x]$ to denote the integer n nearest x (if $x + \frac{1}{2}$ is an integer, then define $[x] = x + \frac{1}{2}$). Given any n-tuple $f = (a_1, \ldots, a_n)$ over the set of real numbers (hence f is not necessarily a labeling of C_n), we also define $I(f) = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_n - a_1|)$. In particular, if we let $g_n = (p_n^{n-1}, \ldots, p_n^2, p_n, 1)$, then Professor Rick Luttmann observed that $I(g_n) = \frac{p_n}{1-p_n} g_n$ for $n \geq 3$. This implies that if we label the vertices of C_n by real numbers, g_n would be divergent. In fact, it was proved in [9] that g_4 is essentially the only divergent 4-tuple over the set of real numbers. Hence by Corollary 3.4, it is natural to believe that for a given natural number N and $n = 2^t$, $g'_n = ([p_n^{(n-1)} N], \ldots, [p_n^2 N], [p_n N], N)$ has a large convergence rate, which was also noticed in [2] for the case $n = 4$. For the rest of this section, we will study the number $r(4, N)$.

Let $f_0 = (0,0,1,1)$, $f_1 = (1,1,1,3)$, $f_2 = (0,1,2,3)$, and for $k \geq 3$, define

$$f_k = \begin{cases} f_{k-3} + f_{k-2} & \text{if } k \equiv 0 \pmod 3 \\ f_{k-3} + 2 f_{k-2} & \text{otherwise.} \end{cases}$$

Assume $f_k = (a_k, b_k, c_k, d_k)$ for each $k \geq 0$. Then we can see that $d_k = a_k + b_k + c_k$ for each k. Assume $p^3 d_k = a_k + x_k$, $p^2 d_k = b_k + y_k$, $p d_k = c_k + z_k$, and let $\alpha_k = \max\{|x_k|, |y_k|, |z_k|\}$ for all $k \geq 0$. Then by Equation (3) we have

$$x_k + y_k + z_k = 0 \quad \text{for all } k \geq 0 \tag{4}$$

The following lemma summarizes some of the properties of the sequence $\{f_k\}$.

Lemma 4.1.

i) *Let* $f'_k = (0, a_k, a_k + b_k, d_k)$, *then* $I(f'_k) = f_k$ *for each* $k \geq 0$;

ii) *For each* $k \geq 1$, *we have*

$$f_k = \begin{cases} (c_{k-1} - a_{k-1}, c_{k-1} + a_{k-1}, a_{k-1} + 2b_{k-1} + c_{k-1}, a_{k-1} \\ +2b_{k-1} + 3c_{k-1}) & \text{if } k \equiv 1 \pmod 3 \\ \frac{1}{2}(c_{k-1} - a_{k-1}, c_{k-1} + a_{k-1}, a_{k-1} + 2b_{k-1} + c_{k-1}, a_{k-1} \\ +2b_{k-1} + 3c_{k-1}) & \text{otherwise} \end{cases}$$

iii) $r(f_k) = r(f_{k-1}) + 1$, *and hence* $r(f_k) = k+3$ *for each* $k \geq 0$;
iv) $f_{k+3} = (c_k, a_k+b_k+c_k, a_k+2b_k+2c_k, 2a_k+3b_k+4c_k)$ *for each* $k \geq 0$;
v) *For any* $k \geq 0$, *we have* $\alpha_{3k+1} \geq \alpha_{3k+i}$ *for* $2 \leq i \leq 4$. *In particular* $\alpha_6 \leq 0.17$, *and* $\alpha_m \leq \alpha_7 \leq 0.17$ *for all* $m \geq 7$;
vi) $p^{-2k} \geq \frac{(1-p^2)M(f_{3k+i})}{(1-p^2)M(f_i)+0.17}$ *for each* $k \geq 0$ *and* $i \geq 3$.

Proof: The truth of i) is obvious, and ii) can be proved by induction. From ii) we can easily see that $I(f_k) = af_{k-1}$, where $a = 2$ or 1 depending on whether $k \equiv 1 \pmod 3$. Hence we have $r(f_k) = r(f_{k-1}) + 1$, which implies that $r(f_k) = k+3$ for all $k \geq 0$. By definition and ii) we can also see that iv) holds.

We now show that v) holds. For any $k \geq 0$, we will prove $\alpha_{3k+4} \leq \alpha_{3k+1}$ only, the proof of the other cases are similar. By iv) and the equations (3) and (4), we have

$$\begin{aligned} z_{3k+4} &= pd_{3k+4} - c_{3k+4} \\ &= p(2a_{3k+1} + 3b_{3k+1} + 4c_{3k+1}) - (a_{3k+1} + 2b_{3k+1} + 2c_{3k+1}) \\ &= (2p-1)(p^3 d_{3k+1} - x_{3k+1}) + (3p-2)(p^2 d_{3k+1} - y_{3k+1}) \\ &\qquad + (4p-2)(pd_{3k+1} - z_{3k+1}) \\ &= (1-2p)x_{3k+1} + (2-3p)y_{3k+1} + (2-4p)z_{3k+1} \\ &= (p-1)x_{3k+1} - pz_{3k+1} \end{aligned}$$

Hence we have $|z_{3k+4}| \leq (1-p)\alpha_{3k+1} + p\alpha_{3k+1} = \alpha_{3k+1}$. Similarly, we can prove that $|x_{3k+4}| \leq \alpha_{3k+1}$ and $|y_{3k+4}| \leq \alpha_{3k+1}$, which implies that $\alpha_{3k+4} \leq \alpha_{3k+1}$. Moreover, we can easily see that $\alpha_i \leq 0.17$ for $6 \leq i \leq 7$. Hence $\alpha_m \leq \alpha_7 \leq 0.17$ for all $m \geq 7$.

Clearly we have $d_i = b_{i+3}$ for any $i \geq 0$ by v). Hence we have $M(f_i) = d_i = p^2 M(f_{3+i}) - y_{3+i} = p^{2k} M(f_{3k+i}) - p^{2k-2} y_{3k+i} - \cdots - p^2 y_{6+i} - y_{3+i}$. Therefore, when $i \geq 3$, we obtain $p^{2k} M(f_{3k+i}) = M(f_i) + p^{2k-2} y_{3k+i} + \cdots + p^2 y_{6+i} + y_{3+i} \leq M(f_i) + 0.17(p^{2k-2} + \cdots + p^2 + 1)$, which is simplified to

$$p^{-2k} \geq \frac{(1-p^2)M(f_{3k+i}) + 0.17}{(1-p^2)M(f_i) + 0.17} > \frac{(1-p^2)M(f_{3k+i})}{(1-p^2)M(f_i) + 0.17}.$$

This completes the proof of Lemma 4.1. ∎

Applying Lemma 4.1 we have the following:

Theorem 4.2. $r(4, N) \geq -\frac{1.5}{\log_2 p} \log_2(N+1) + 1.5$.

Proof: Clearly we may assume $N \geq 3$ and $M(f_{3k-1}) \leq N < M(f_{3k+2})$, where $k \geq 1$. First suppose $M(f_{3k-1}) \leq N < M(f_{3k})$. Now let $f = f'_{3k-1} + N - M(f_{3k-1})$, then $r(f) = r(f'_{3k-1}) = r(f_{3k-1}) + 1 = 3k+3$ by Lemma 4.1. Applying Lemma 4.1 vi) for $i = 3$ we have

$$p^{-2(k-1)} \geq \frac{(1-p^2)M(f_{3k})}{(1-p^2)M(f_3) + 0.17} \geq \frac{(1-p^2)(N+1)}{4(1-p^2) + 0.17}$$

which implies that

$$r(4,N) \geq r(f) = 3k+3 \geq \frac{1.5}{-\log_2 p}\log_2(N+1) + 2.443 .$$

Similar to the above proof, we have $r(4,N) \geq \frac{1.5}{-\log_2 p}\log_2(N+1) + 1.526$ when $M(f_{3k}) \leq N \leq M(f_{3k+1})$, and $r(4,N) \geq \frac{1.5}{-\log_2 p}\log_2(N+1) + 2.044$ when $M(f_{3k+1}) \leq N < M(f_{3k+2})$. This completes the proof of Theorem 4.2.

Theorem 4.2 implies that if $n \equiv 0 \pmod 4$, then we have $r(n,N) \geq r(4,N) \geq 1.706\log_2(N+1)+1.5$. For any natural number N, if $M(f_k) \leq N < M(f_{k+1})$ for some $k \geq 0$, we guess that $r(4,N) = k+4$, which would imply that the bound given in Theorem 4.2 is essentially optimal.

5. The Case $n = 6$

We will need the following result of Euler (e.g., see [1], p. 19):

Lemma 5.1. *If $|t| < 1$ and $|x| < 1$, then*

$$\prod_{k=0}^{\infty}(1+tx^k) = 1 + \prod_{k=1}^{\infty}\frac{t^k x^{k(k-1)/2}}{(1-x)(1-x^2)\ldots(1-x^k)}.$$

Applying Lemma 5.1 we have the following:

Corollary 5.2. *If $0 < t < 1, 0 < x < 1$, and $tx/(1-x^2) < 1$, then*

$$\prod_{k=0}^{\infty}(1+tx^k) < \frac{1+t-x^2}{1-tx-x^2}$$

Proof: Notice that $\frac{tx^{k-1}}{1-x^k} > \frac{tx^k}{1-x^{k+1}}$ for any natural number k. Hence Lemma 5.1 implies that

$$\prod_{k=0}^{\infty}(1+tx^k) < 1 + \frac{t}{1-x}\prod_{k=0}^{\infty}\left(\frac{tx}{1-x^2}\right)^k = \frac{1+t-x^2}{1-tx-x^2}.$$

∎

Let $g_0 = (1,3,1,3,1,3)$, and in general we define $g_{k+1} = Tg_k$, where g_k is a labeling of C_6 for every $k \geq 0$ and

$$T = \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & -2 & 1 & 0 \\ 1 & 2 & 3 & -2 & 3 & 0 \end{pmatrix}$$

Then we can easily check that $\omega = (\sqrt{5}-2, (3-\sqrt{5})/2, (\sqrt{5}-1)/2, 1, (3-\sqrt{5})/2, (\sqrt{5}-1)/2)$ is an eigenvector corresponding to the eigenvalue $\lambda = \sqrt{5}+1$ of the matrix T.

Write $g_k = M_k(\omega + x_k) = (a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}, a_{k_5}, a_{k_6})$, where $x_k = (x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}, x_{k_5}, x_{k_6})$ and $M_k = a_{k_4}$ (and hence $x_{k_4} = 0$) for each $k \geq 0$. Then it can be seen that $a_{k_6} = a_{k_1} + a_{k_2} + a_{k_3} - a_{k_4} + a_{k_5}$. Define $\epsilon_k = \max\{|x_{k_i}|, 1 \leq i \leq 5\}$. Then we have the following:

Lemma 5.3.

i) $r(g_k) = r(g_{k-1}) + 1$, *and hence* $r(g_k) = k+2$ *for all* $k \geq 0$*;*
ii) $\epsilon_{k+2} < c\epsilon_k$ *for all* $k \geq 2$, *where* $c \doteq .713457843$*;*
iii) $M(g_k) = M_k$ *for each* $k \geq 0$*;*
iv) *Let* $g_k' = (0, a_{k_1}, a_{k_1} + a_{k_2}, a_{k_1} + a_{k_2} + a_{k_3}, a_{k_1} + a_{k_2} + a_{k_3} - a_{k_4}, a_{k_1} + a_{k_2} + a_{k_3} - a_{k_4} + a_{k_5})$, *then* $M(g_k') \leq M(g_k)$ *and* $I(g_k') = g_k$ *for each* $k \geq 0$.

Proof: It can be easily checked that i) is true, $\epsilon_2 = (7 - 3\sqrt{5})/2$, and that $M_{k+2} = (\lambda^2 + 5x_{k_1} + 8x_{k_2} + 9x_{k_3} + 7x_{k_5})M_k$ for every $k \geq 0$. Now suppose $\epsilon_k \leq \epsilon_2$ for some $k \geq 2$. Then we have $g_{k+2} = M_{k+2}(e + x_{k+2}) = T^2 g_k = T^2(\omega + x_k)M_k = (\lambda^2\omega + T^2 x_k)M_k$ which implies that

$$x_{k+2} = \frac{T^2 x_k - (5x_{k_1} + 8x_{k_2} + 9x_{k_3} + 7x_{k_5})\omega}{\lambda^2 + 5x_{k_1} + 8x_{k_2} + 9x_{k_3} + 7x_{k_5}}$$

Some tedious calculation then shows that $\epsilon_{k+2} \leq c\epsilon_k$ where $c = (39 - 15\sqrt{5})/(\lambda^2 - 29\epsilon_2) \doteq .713457843$, and hence the truth of ii). The truth of iii) and iv) follow from ii). This completes the proof of Lemma 5.3. ∎

We are now ready to prove the main result of this section.

Theorem 5.4. $r(6, N) \geq \frac{\log(N+1)}{\log(\sqrt{5}+1)} - 1$.

Proof: The result is clear when $N < 39$. Hence we will assume $N \geq 39 > M_2$. Now suppose $M_{2k} \leq N < M_{2k+2}$ for some $k \geq 2$. Let $g = g_{2k}' + N - M_{2k}$, then $r(g) = r(g_{2k}') = r(g_{2k}) + 1 = 2k+3$, and $M(g) = N$. This implies that $r(6, N) \geq 2k + 3$.

On the other hand, Lemma 5.3 implies that $N + 1 \leq M_{2k+2} = (\lambda^2 + 5x_{(2k)1} + 8x_{(2k)2} + 9x_{(2k)3} + 7x_{(2k)5})M_{2k} \leq (\lambda^2 + 29\epsilon_{2k})M_{2k} \leq (\lambda^2 + 29c^{k-1}\epsilon_2)M_{2k} = \lambda^2(1 + tx^{k-1})M_{2k}$ where $x = c$ and $t = 29\epsilon_2/\lambda^2$. Now Corollary 5.2 implies that

$$N + 1 \leq \lambda^{2k} M_2 \prod_{k=0}^{\infty}(1 + tx^k) \leq 2.6\lambda^{2k}.$$

Therefore, we have

$$r(6,N) \geq 2k+3 \geq \frac{\log(N+1) - \log 2.6}{\log(\sqrt{5}+1)} + 3 \geq \frac{\log(N+1)}{\log(\sqrt{5}+1)} - 1.$$

This completes the proof of Theorem 5.4. ∎

The idea of constructing a labeling with large convergence rate can be generalized. Let $h_0 = (1,3,1,3,\ldots,1,3)$, and in general we define $h_{k+1} = T_{2q}h_k$, where $q \geq 3$ is odd, h_k is a labeling of C_{2q} for each $k \geq 0$, and $T_{2q} = (t_{ij})$ is a matrix of order $2q$ defined as follows: $t_{11} = -1$, and if $(i,j) \neq (1,1)$, then

$$t_{ij} = \begin{cases} 0 & \text{if } i \leq j \text{ and } j \text{ is even} \\ 1 & \text{if } i \leq j \text{ and } j \text{ is odd} \\ -2 & \text{if } i > j \geq 3 \text{ and } j \text{ is even} \\ 3 & \text{if } i > j > 3 \text{ and } j \text{ is odd} \\ 2 & \text{if } i > j = 2 \\ 1 & \text{if } i > j = 1 \end{cases}.$$

Our intuition is that T_{2q} contains a unique eigenvalue λ_{2q} such that $q < \lambda_{2q} < q+1$, and the norm of every other eigenvalue of T_{2q} is at most $q+1$. If this is the case, we would have the following (by some arguments in terms of matrices):

$$r(2q,N) \geq \frac{\log(N+1)}{\log \lambda_{2q}} + C(q),$$

where $C(q)$ is independent of N. This would be an improvement of Theorem 3.7.

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