Counting Young Tableaux When Rows Are Cosets

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Abstract

Let $T(m,n)$ denote the number of $m \times n$ rectangular standard Young tableaux with the property that the difference of any two rows has all entries equal. Let $\hat{T}(n) := \sum_{d\,|\,n} T(d, n/d)$. We find recurrence relations satisfied by the numbers $T(m,n)$ and $\hat{T}(n)$, compute their generating functions, and express them explicitly in some special cases.

1 Introduction

For $m, n \geq 1$, let $T(m,n)$ denote the number of rectangular $m \times n$ integer arrays A with the following properties:

1. A is a standard Young tableau (that is, its rows and columns are nondecreasing and it contains all integers between 0 and $mn - 1$),
2. the rows of A are additive cosets (that is, the componentwise difference of any two rows is a vector with all components equal).

For short, we call such an array a *coset tableau*. Table 1 shows three coset tableaux with $m = n = 4$.

0	1	2	3	0	1	4	5	0	1	8	9
4	5	6	7	2	3	6	7	2	3	10	11
8	9	10	11	8	9	12	13	4	5	12	13
12	13	14	15	10	11	14	15	6	7	14	15

Table 1: Some 4×4 coset tableaux.

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The purpose of this paper is to find recurrence relations satisfied by the numbers $T(m,n)$, compute their generating functions, and, if possible, express them explicitly as functions of m and n. Our original motivation was to show that $T(p,q) = 2$ when p and q are primes.

In Section 2, we prove a theorem describing the structure of coset tableaux. This theorem leads to recurrence (5) in Section 3, from which we compute $T(m,n)$ when at least one of the parameters is a prime power. We also derive recurrence relations for $\hat{T}(n) := \sum_{d\mid n} T(d,n/d)$. In Section 4, we use these relations to find the ordinary and Dirichlet generating functions for $T(m,n)$ and $\hat{T}(n)$, from which we derive some explicit formulae.

2 A structure theorem

We will index the rows and columns of coset tableaux with $0,1,\ldots,m-1$ and $0,1,\ldots,n-1$, respectively.

If A is a coset tableau, it is clear that $a_{00} = 0$. Also, either $a_{01} = 1$ or $a_{10} = 1$; we call A *upper* in the former case, and *lower* in the latter. The three tableaux shown in Table 1 are all upper. Furthermore,

$$a_{ij} = a_{i0} + a_{0j} \quad (\text{for } 0 \le i < m,\ 0 \le j < n).$$

This implies that if A is an $m \times n$ coset tableau then its transpose is an $n \times m$ coset tableau. Hence $T(n,m) = T(m,n)$ for all $m,n \ge 1$.

Theorem 1 *Let $m,n > 1$. Let A be an upper $m \times n$ coset tableau, $d := a_{10}$, and $0 \le j < n$. Then*

(*i*) *if $j \not\equiv -1 \pmod d$, then $j+1 < n$ and $a_{0,j+1} = a_{0,j} + 1$,*

(*ii*) $n \equiv 0 \pmod d$,

(*iii*) $a_{ij} \equiv j \pmod d$ $\quad$ (*for* $0 \le i < m$).

Proof: (*i*) We proceed by induction on j. For $j = 0$, $n > 1$ and $a_{01} = a_{00} + 1$.

Now assume that $0 < j < n$, $j \not\equiv -1 \pmod d$, and that the assertion holds for all smaller j. Let $x := a_{0j}$. As $m > 1$, there exists an element of A, say a_{st}, such that $a_{st} = x + 1$.

If $t < j$ we distinguish two cases. If $t \not\equiv 0 \pmod d$, then, by induction hypothesis, $a_{0t} = a_{0,t-1} + 1$. Hence $a_{s,t-1} = a_{s0} + a_{0t} - 1 = a_{st} - 1 = x = a_{0j}$, implying that $s = 0$ and $t - 1 = j$, in contradiction with $t < j$.

If $t \equiv 0 \pmod d$, then $t + k \not\equiv -1 \pmod d$ for $0 \le k \le d-2$. If $t+d-2 < j$ then the induction hypothesis applies to all these indices and we have

$$a_{0,t+1} - a_{0t} = a_{0,t+2} - a_{0,t+1} = \ldots = a_{0,t+d-1} - a_{0,t+d-2} = 1$$

and therefore $a_{0,t+d-1} = a_{0t} + d - 1$. It follows that

$$a_{s,t+d-1} = a_{s0} + a_{0,t+d-1} = a_{s0} + a_{0t} + d - 1 = a_{st} + d - 1 = x + d = a_{1j}\,,$$

and so $j = t + d - 1 \equiv -1 \pmod d$, a contradiction.

If $t + d - 2 \geq j$ then we can only infer that $a_{0j} - a_{0t} = j - t$. But now $j - t \leq d - 2$ and therefore

$$a_{sj} = a_{s0} + a_{0j} = a_{s0} + a_{0t} + j - t = a_{st} + j - t \leq (x+1) + d - 2 = a_{1j} - 1\,.$$

Hence $a_{sj} < a_{1j}$, implying that $s = 0$. But then $a_{0j} = x < x + 1 = a_{0t}$, in contradiction with $t < j$.

So we must have $t \geq j$. If $t = j$ then $a_{s0} = a_{st} - a_{0t} = a_{st} - a_{0j} = 1$, but this is impossible as $1 = a_{01}$. Therefore $t > j$. Now

$$1 = a_{st} - a_{0j} = a_{s0} + (a_{0t} - a_{0j}) \geq s + 1$$

so that $s = 0$ and $a_{0t} - a_{0j} = 1$. It follows that $t = j+1$ and that $a_{0,j+1} = a_{0j} + 1$ as desired. Also, as $t < n$, this proves that $j + 1 < n$.

(*ii*) If n is not divisible by d then we can let $j = n - 1$ in part (*i*) to get $n < n$, a contradiction.

(*iii*) Part (*i*) implies that each of the sets $\{a_{i,jd+k} \mid 0 \leq k < d\}$, for $0 \leq i < m$, $0 \leq j < n/d$, contains d consecutive integers. As these sets represent a partition of the set $\{0, 1, \ldots, mn - 1\}$, we must have $a_{i,jd} \equiv 0 \pmod d$ for all i and j, and therefore $a_{ij} \equiv j \pmod d$ for all i and j. □

3 Recurrence relations

For $m, n \geq 1$ and $mn > 1$, let $S(m,n)$ denote the number of upper $m \times n$ coset tableaux. In particular, $S(1,n) = 1$ for $n > 1$, while $S(m,1) = 0$ for $m > 1$. Clearly, the number of lower $m \times n$ coset tableaux is $S(n,m)$. Therefore

$$T(m,n) = \begin{cases} 1, & \text{if } m = n = 1 \\ S(m,n) + S(n,m), & \text{otherwise} \end{cases} \tag{1}$$

Let A be an upper $m \times n$ coset tableau with $m, n > 1$, and $d := a_{10}$. By Theorem 1(*ii*), d is a divisor of n. By Theorem 1(*iii*), the multiples of d in A are gathered in columns j with $j \equiv 0 \pmod d$. By deleting all other columns and dividing the remaining ones by d, we obtain a lower $m \times (n/d)$ coset tableau. Since according to Theorem 1(*i*), A is uniquely determined by the columns retained, there is a 1–1 correspondence between these two types of tableaux. Therefore

$$S(m,n) = \sum_{\substack{d \mid n \\ d \neq 1}} S(n/d, m) = \sum_{\substack{d \mid n \\ d \neq n}} S(d, m)\,. \tag{2}$$

If we additionally define

$$S(1,1) := 1\,, \tag{3}$$

then (2) is valid for all $m, n \geq 1$ with $mn > 1$. From (1), (2) and (3) it follows that

$$T(m,n) = \sum_{d\,|\,n} S(d,m) \tag{4}$$

for all $m, n \geq 1$. Considering m as a parameter and applying Möbius inversion to (4), we obtain

$$S(n,m) = \sum_{d\,|\,n} \mu(d) T(m, n/d)$$

Together with (1), this implies

$$T(m,n) = \begin{cases} 1, & \text{if } m = n = 1 \\ \sum_{d\,|\,n} \mu(d) T(m,n/d) + \sum_{d\,|\,m} \mu(d) T(m/d,n)\,, & \text{otherwise} \end{cases} \tag{5}$$

From (5) we can easily prove by induction on $k + l$ that for p and q prime,

$$T(p^k, q^l) = \binom{k+l}{k}\,. \tag{6}$$

Proposition 1 *Let p be a prime and $m, k \geq 1$. Then*

$$T(m, p^k) = \sum_{d\,|\,m} T(d, p^{k-1})\,.$$

Proof: By (5),

$$T(m, p^k) = -\sum_{\substack{d\,|\,m \\ d \neq 1}} \mu(d) T(m/d, p^k) + T(m, p^{k-1})\,,$$

so that

$$T(m, p^{k-1}) = \sum_{d\,|\,m} \mu(d) T(m/d, p^k)\,.$$

Applying Möbius inversion in reverse, we get the desired conclusion. □

In particular, $T(m,p) = \tau(m)$, the number of divisors of m.

Proposition 2 *Let m and n be relatively prime, and let p be a prime. Then*

$$T(mn, p^k) = T(m, p^k) T(n, p^k)\,.$$

Proof: By induction on k. As $T(m,1) = 1$ for any m, the assertion holds for $k = 0$.

Let $k > 0$. Then, by Proposition 1 and by induction hypothesis,

$$\begin{aligned} T(mn, p^k) &= \sum_{d \mid mn} T(d, p^{k-1}) \\ &= \sum_{d_1 \mid m} \sum_{d_2 \mid n} T(d_1 d_2, p^{k-1}) \\ &= \sum_{d_1 \mid m} \sum_{d_2 \mid n} T(d_1, p^{k-1}) T(d_2, p^{k-1}) \\ &= \sum_{d_1 \mid m} T(d_1, p^{k-1}) \sum_{d_2 \mid n} T(d_2, p^{k-1}) \\ &= T(m, p^k) T(n, p^k) \quad \square \end{aligned}$$

In general, $T(m,n)$ is not multiplicative. For example, $T(6,6) = 14$ while $T(2,6) \times T(3,6) = 4 \times 4 = 16$.

Corollary 1 *Let $p_1, p_2, \ldots, p_r$ be distinct primes, and p a prime. Then*

$$T(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, p^k) = \prod_{j=1}^{r} \binom{k_j + k}{k}.$$

Proof: This follows from (6) and Proposition 2. $\square$

Now define

$$\hat{T}(n) := \sum_{d \mid n} T(d, n/d).$$

$\hat{T}(n)$ is the number of all coset tableaux with n elements; it is obtained from $T(m,n)$ by convolution. Convolving both sides of (5) one gets

$$\hat{T}(n) = 2 \sum_{d \mid n} \mu(n/d) \hat{T}(d) \quad (n \geq 2), \tag{7}$$

and from this, by means of reverse Möbius inversion,

$$\hat{T}(n) = \sum_{\substack{d \mid n \\ d \neq n}} \hat{T}(d) + 1 \quad (n \geq 1). \tag{8}$$

We omit the details of the proofs of (7) and (8).

Tables 2 and 3 give some values of $T(m,n)$ and $\hat{T}(n)$ computed from (5) and (7), respectively.

m \ n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	3	2	4	2	4	3	4
3	1	2	2	3	2	4	2	4	3	4
4	1	3	3	6	3	9	3	10	6	9
5	1	2	2	3	2	4	2	4	3	4
6	1	4	4	9	4	14	4	16	9	14
7	1	2	2	3	2	4	2	4	3	4
8	1	4	4	10	4	16	4	20	10	16
9	1	3	3	6	3	9	3	10	6	9
10	1	4	4	9	4	14	4	16	9	14

Table 2: Some values of $T(m,n)$.

4 Generating functions and explicit formulae

Define G_T as a power series in infinitely many variables:

$$G_T(x_1,x_2,\ldots;y_1,y_2,\ldots) := \sum_{k,l} T(p_1^{k_1}p_2^{k_2}\cdots,\ p_1^{l_1}p_2^{l_2}\cdots)\, x_1^{k_1}x_2^{k_2}\cdots y_1^{l_1}y_2^{l_2}\cdots,$$

where $\{p_1,p_2,\ldots\}$ is the set of all primes and the summation is over all pairs of infinite vectors, $k=(k_1,k_2,\ldots)$ and $l=(l_1,l_2,\ldots)$, with non-negative integer entries, at most finitely many of which are non-zero. Similarly, define

$$G_{\hat{T}}(x_1,x_2,\ldots) := \sum_{k} \hat{T}(p_1^{k_1}p_2^{k_2}\cdots)\, x_1^{k_1}x_2^{k_2}\cdots.$$

Theorem 2 *Let G_T and $G_{\hat{T}}$ be as above. Then*

$$G_T(x_1,x_2,\ldots;y_1,y_2,\ldots) = \frac{1}{\prod_{i=1}^{\infty}(1-x_i)+\prod_{i=1}^{\infty}(1-y_i)-1}, \quad (9)$$

$$G_{\hat{T}}(x_1,x_2,\ldots) = \frac{1}{2\prod_{i=1}^{\infty}(1-x_i)-1}. \quad (10)$$

Proof: The recursion given in Equation (5) implies that

$$\begin{aligned} G_T &= \sum_{k,l} T(p^k,p^l)\, x^k y^l \\ &= \sum_{k,l}\left(\sum_{d\,|\,p^l}\mu(d)T(p^k,\ p^l/d)\, x^k y^l + \sum_{d\,|\,p^k}\mu(d)T(p^k/d,\ p^l)\, x^k y^l\right) - 1 \\ &= \sum_{k,l}\left(\sum_{h} T(p^k,\ p^l)(-1)^{|h|} x^k y^{l+h} + \sum_{h} T(p^k,\ p^l)(-1)^{|h|} x^{k+h} y^l\right) - 1, \end{aligned}$$

n	$\hat{T}(n)$	n	$\hat{T}(n)$	n	$\hat{T}(n)$	n	$\hat{T}(n)$	n	$\hat{T}(n)$
1	1	21	6	41	2	61	2	81	16
2	2	22	6	42	26	62	6	82	6
3	2	23	2	43	2	63	16	83	2
4	4	24	40	44	16	64	64	84	88
5	2	25	4	45	16	65	6	85	6
6	6	26	6	46	6	66	26	86	6
7	2	27	8	47	2	67	2	87	6
8	8	28	16	48	96	68	16	88	40
9	4	29	2	49	4	69	6	89	2
10	6	30	26	50	16	70	26	90	88
11	2	31	2	51	6	71	2	91	6
12	16	32	32	52	16	72	152	92	16
13	2	33	6	53	2	73	2	93	6
14	6	34	6	54	40	74	6	94	6
15	6	35	6	55	6	75	16	95	6
16	16	36	52	56	40	76	16	96	224
17	2	37	2	57	6	77	6	97	2
18	16	38	6	58	6	78	26	98	16
19	2	39	6	59	2	79	2	99	16
20	16	40	40	60	88	80	96	100	52

Table 3: Some values of $\hat{T}(n)$.

where $\sum_h$ is the sum over all infinite vectors $h = (h_1, h_2, \ldots)$ of zeros and ones with only finitely many ones, and $|h|$ is the sum of the components of h. Continuing the derivation given above:

$$\begin{aligned} G_T &= G_T \sum_h (-1)^{|h|} y^h + G_T \sum_h (-1)^{|h|} x^h - 1 \\ &= G_T \left(\prod_{i=1}^{\infty} (1 - y_i) + \prod_{i=1}^{\infty} (1 - x_i) \right) - 1, \end{aligned}$$

proving (9). – Equation (10) can be proved similarly using (7). □

Let $p_1, p_2, \ldots, p_r$ be distinct primes. Since $G_{\hat{T}}(x_1, x_2, \ldots)$ is a symmetric function of the x_i, the value of $\hat{T}(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})$ depends only on r and the k_i. Let

$$G_{\hat{T},r}(x_1, x_2, \ldots, x_r) := \sum_{k_1, k_2, \ldots, k_r = 0}^{\infty} \hat{T}(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})\, x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}\,.$$

Now $G_{\hat{T},r}(x_1, x_2, \ldots, x_r) = G_{\hat{T}}(x_1, x_2, \ldots, x_r, 0, 0, \ldots)$, hence from (10) it follows that

$$G_{\hat{T},r}(x_1, x_2, \ldots, x_r) = \frac{1}{2\prod_{j=1}^{r}(1-x_j) - 1}.$$

So, for example, the Bell series of $\hat{T}$ (cf. [2], Sections 2.16 and 2.17) is

$$G_{\hat{T},1}(x) = \frac{1}{1-2x},$$

and

$$G_{\hat{T},2}(x,y) = \frac{1}{1-2x-2y+2xy}.$$

If we expand these functions into power series, we find that for p prime,

$$\hat{T}(p^n) = 2^n, \tag{11}$$

and, for $m \le n$ and p, q distinct primes,

$$\hat{T}(p^m q^n) = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{m-k} \binom{n+k}{n} 2^{n+k}. \tag{12}$$

Finally, we note that from (5) it is easy to compute the Dirichlet generating functions

$$D_T(s,t) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{T(m,n)}{m^s n^t}$$

$$D_{\hat{T}}(s) := \sum_{n=1}^{\infty} \frac{\hat{T}(n)}{n^s}$$

of $T(m,n)$ and $\hat{T}(n)$, respectively. The result is

$$D_T(s,t) = \frac{1}{1/\zeta(s) + 1/\zeta(t) - 1} \tag{13}$$

and

$$D_{\hat{T}}(s) = \frac{2}{2-\zeta(s)} - 1 \tag{14}$$

where ζ is the Riemann zeta function. We refer to [2] or [3] for the relevant theory.

For $n > 1$, let f_n denote the number of ordered factorizations of n into factors larger than 1, and let $f_1 := 1$. Then

$$\sum_{n=1}^{\infty} \frac{f_n}{n^s} = \frac{1}{2-\zeta(s)}$$

(see [1], p. 202). Comparing this with (14) and using uniqueness of expansion into Dirichlet series, we see that $\hat{T}(n) = 2f_n$ when $n > 1$.

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