On Edge-Graceful and Super-Edge-Graceful Graphs[1]

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The notion of an edge-graceful graph is that of distinct integer labels on the edges of a graph inducing distinct integer labels on the vertices of the graph, where each vertex label is a modular sum of the labels on all incident edges. Several classes of graphs have been shown to be edge-graceful and of particular interest is Lee's conjecture [5] that all trees of odd order are edge-graceful. While by no means proving this conjecture, we introduce a variation of edge-gracefulness which for trees of odd order implies edge-gracefulness. Such a tool gives an alternate proof of the edge-gracefulness of regular spider graphs and extends the known domain for classes of edge-graceful trees, as well as being an interesting notion in its own right. Finally, we give several algorithms which when applied to edge-graceful graphs generate new edge-graceful graphs.

For the graph G with vertex set $V(G)$ and edge set $E(G)$ with $p = |V(G)|$ and $q = |E(G)|$, let (ℓ, ℓ^*) be a *function pair* which assigns integer labels to the edges and vertices; that is, $\ell : E(G) \to Z$ and $\ell^* : V(G) \to Z$. Following Lo [6], define G as *edge-graceful* if there is a function pair (ℓ, ℓ^*) such that ℓ is *onto* $\{1, \ldots, q\}$ and ℓ^* is *onto* $\{0, \ldots, p-1\}$, and

$$\ell^*(v) = \Big(\sum_{uv \in E(G)} \ell(uv)\Big) \bmod p.$$

Let

$$Q = \begin{cases} \{\pm 1, \ldots, \pm \frac{q}{2}\}, & \text{if } q \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{q-1}{2}\}, & \text{if } q \text{ is odd,} \end{cases}$$

$$P = \begin{cases} \{\pm 1, \ldots, \pm \frac{p}{2}\}, & \text{if } p \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{p-1}{2}\}, & \text{if } p \text{ is odd.} \end{cases}$$

[1]This work was completed while both authors were Fulbright Scholars in the Mathematics Department at the University of Botswana in Gaborone, Botswana, during the 1990-1991 academic year.

Dropping the modularity operator and pivotting on symmetry about zero, define a graph G as a *super-edge-graceful graph* if there is a function pair (ℓ, ℓ^*) such that ℓ is onto Q and ℓ^* is onto P, and

$$\ell^*(v) = \sum_{uv \in E(G)} \ell(uv).$$

Furthermore, if (ℓ, ℓ^*) is a super-edge-graceful function pair for G, and H is a subgraph of G, then the *restriction* of ℓ^* to H, denoted ℓ^*_H, is defined as

$$\ell^*_H(v) = \sum_{uv \in H} \ell(uv).$$

As an example of a super-edge-graceful graph, consider the hexagon with a *Star of David* inscribed therein, denoted C_6^2, following the notation of [3]. With the edges labeled ± 1 to ± 6, as indicated below, the vertices are labeled ± 1 to ± 3.

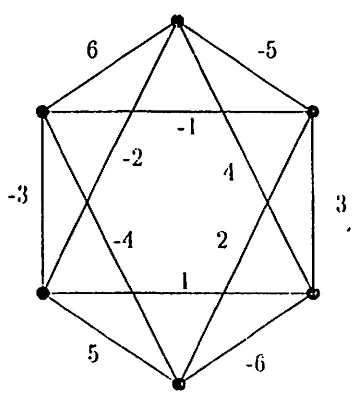

Figure 0.1: A Super-Edge-Graceful Labeling of C_6^2.

In certain instances, super-edge-gracefulness implies edge-gracefulness, as the following theorem makes precise.

Theorem 1. *If G is a super-edge-graceful graph and*

$$q \equiv \begin{cases} -1 \bmod p, & \text{if } q \text{ is even,} \\ 0 \bmod p, & \text{if } q \text{ is odd,} \end{cases}$$

then G is also edge-graceful.

Proof: Let (ℓ, ℓ^*) be a super-edge-graceful function pair for G. Define an edge-graceful function pair (μ, μ^*) for G as follows. Let

$$r = \begin{cases} q+1, & \text{if } q \text{ is even,} \\ q, & \text{if } q \text{ is odd.} \end{cases}$$

In either case, note that $r \equiv 0 \bmod p$. Define

$$\mu(uv) = \begin{cases} \ell(uv), & \text{if } \ell(uv) > 0, \\ \ell(uv) + r, & \text{if } \ell(uv) \leq 0. \end{cases}$$

For both q being odd or even, μ is clearly onto $\{1, \ldots, q\}$. Now

$$\begin{aligned} \mu^*(v) &= (\textstyle\sum_{uv \in E(G)} \mu(uv)) \bmod p \\ &= (\textstyle\sum_{\ell(uv)>0} \ell(uv) + \sum_{\ell(uv) \leq 0}(\ell(uv) + r)) \bmod p \\ &= (\textstyle\sum_{\ell(uv)>0} \ell(uv) + \sum_{\ell(uv) \leq 0} \ell(uv)) \bmod p = \ell^*(v) \bmod p, \end{aligned}$$

which means that μ^* is onto $\{0, 1, \ldots, p-1\}$. □

Corollary 2. *Super-edge-graceful trees of odd order are edge graceful.*

Proof: The corollary follows by noting that $q + 1 = p$ for all trees. □

However super-edge-gracefulness does not in general imply edge-gracefulness. Lo [6] showed that a necessary condition for a graph to be edge-graceful is that p divides

$$(q^2 + q + \frac{p(p-1)}{2}).$$

Thus no cycle of even order is edge-graceful. But the cycle of order eight, C_8, is super-edge-graceful, as shown in the figure below. Furthermore C_6^2 is not edge-graceful by Lo's condition, but is super-edge-graceful as already illustrated. Whether edge-gracefulness of a graph implies super-edge-gracefulness is an open question. Lest one think that every graph is super-edge-graceful, note for example that C_4 and C_6 are not super-edge-graceful.

The definition of super-edge-gracefulness yields simple proofs showing that certain graphs are super-edge-graceful and thus, by Theorem 1, are also edge-graceful. For example we have

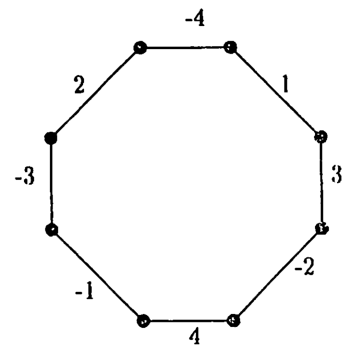

Figure 0.2: A Super-Edge-Graceful Cycle.

Proposition 3: A Growing Tree Algorithm. *Let T be a super-edge-graceful tree with $2n$ edges. If any 2 edges are appended to the same vertex of T, then the new tree is also super-edge-graceful.*

Proof: This follows by noting that the new edges can be labeled $\pm(n+1)$ leaving the original vertex labels unchanged, with two new vertex labels of $\pm(n+1)$. □

The following result with respect to edge-gracefulness was shown in [3]. The easiness of its super-edge-gracefulness proof, however, shows the power of the concept.

Corollary 4. *If G is a non-trivial tree with only one vertex of even degree, then G is super-edge-graceful.*

Proof: Every tree as described in the hypothesis can be constructed by starting with P_3, which is super-edge-graceful, and successively appending two edges to a vertex of the tree under construction, which by the above proposition gives a super-edge-graceful tree at each stage of the iteration. □

We now begin our proof of the main result that spiders are super-edge-graceful, and show how this result easily increases the set of proven edge-graceful trees. A *spider graph* is a tree with a *core* vertex c of degree at least 2 and all other vertices of degree at most 2. (Unless it is a path, a spider has a unique core vertex.) A spider is *regular* if the distance from the core vertex to each end vertex is the same. The path from the core to any end or exterior vertex is called a *leg* of the spider. Small [7], whose proof was simplified by Cabaniss, *et. al.*, [3], showed that regular spiders of odd degree are edge-graceful, using a fairly complicated (non super-edge-graceful) pair of algorithms. Using two fairly simple algorithms we give an alternate proof of this theorem. Before doing so, we illustrate the algorithms.

The Shuttle Algorithm. Consider the regular spider with 6 legs of length 7. Arrange the necessary edge labels as the sequence

$$S = \{21, -1, 20, -2, 19, -3, \ldots, 2, -20, 1, -21\}.$$

Index the legs as L_1 to L_6. Represent the edges of each leg, with exterior vertices on the left and the core on the right, as a succession of blanks to be labeled.

$$\begin{array}{l} L_1 = \{\_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\} \\ L_2 = \{\_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\} \\ L_3 = \{\_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_\} \end{array},$$

with L_4, L_5, L_6 similarly represented. As a shuttlecock being shunted back and forth on a loom, enter the terms of S in the blanks, proceeding first to the right on L_1, then to the left on L_2, then to the right on L_3, and so on, resulting in

$$\begin{array}{llllllll} L_1 = \{21 & -1 & 20 & -2 & 19 & -3 & 18\} \\ L_2 = \{-7 & 15 & -6 & 16 & -5 & 17 & -4\} \\ L_3 = \{14 & -8 & 13 & -9 & 12 & -10 & 11\}, \end{array}$$

with L_4, L_5, L_6 being the *inverses* of L_3, L_2, L_1, respectively, (that is, $L_4 = -L_3$, and so on), and so the spider is successfully vertex labeled as can be easily verified.

The Shoelace Algorithm. Consider the regular spider with 7 legs of length 6. From the edge labels ± 1 to ± 21, remove the multiples of 7, and arrange the remaining integers into two alternating sequences:

$$S = \{20, -19, 18, -17, 16, -15, 13, -12, 11, -10, 9, -8, 6, -5, 4, -3, 2, -1\},$$

$$T = \{1, -2, 3, -4, 5, -6, 8, -9, 10, -11, 12, -13, 15, -16, 17, -18, 19, -20\}.$$

Index the legs as L_0 to L_6. L_0 is labeled with the non-zero multiples of 7.

$$L_0 = \{21 \quad -7 \quad 14 \quad -14 \quad 7 \quad -21\}.$$

Insert the terms from S and T into the blanks of L_1, L_2, L_3 following a *shoelace* path connecting the extreme unlabeled blanks of the legs as in the figure below, alternating between S and T until all blanks have labels.

That is, inserting the first 6 terms of S leaves the following.

$$\begin{array}{llllll} L_1 = \{20 & \_ & \_ & \_ & \_ & -15\} \\ L_2 = \{16 & \_ & \_ & \_ & \_ & -19\} \\ L_3 = \{18 & \_ & \_ & \_ & \_ & -17\} \end{array}$$

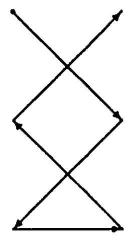

Figure 0.3: A Shoelace Pattern.

Inserting the first 6 terms of T leaves the following.

$$\begin{array}{ll} L_1 = \{20 & -1 \quad \_ \quad \_ \quad 6 \quad -15\} \\ L_2 = \{16 & -5 \quad \_ \quad \_ \quad 2 \quad -19\} \\ L_3 = \{18 & -3 \quad \_ \quad \_ \quad 4 \quad -17\} \end{array}$$

Inserting the next 6 terms of S leaves the following.

$$\begin{array}{llllll} L_1 = \{20 & -1 & 13 & -8 & 6 & -15\} \\ L_2 = \{16 & -5 & 9 & -12 & 2 & -19\} \\ L_3 = \{18 & -3 & 11 & -10 & 4 & -17\} \end{array}$$

Then let L_4, L_5, L_6 be the inverses of L_1, L_2, L_3. The spider is thus successfully vertex labeled as can be easily verified.

Theorem 5. *Every regular spider with an even number of edges is super-edge-graceful.*

Proof: Consider first spiders with an even number of legs. Let $2k$ be the number of legs; let n be the length of each leg. Let

$$S = \{kn, -1, kn-1, -2, kn-2, \ldots, -kn\}$$

be the sequence of $2kn$ terms.

Then

$$V = \{kn, kn-1, kn-2, kn-3, \ldots, -kn\}$$

is the sequence obtained by including the first and last terms of S and by finding sums of adjacent terms of S. Let $S_j, 1 \leq j \leq 2k$, be a partition of S, where S_1 is the first n terms of S, S_2 is the next n terms of S, but in *reverse* order, S_3 is the next n terms, S_4 is the next n terms in reverse order, and so on. Interpret S_j as the edge labeling of a leg of the spider, with the edge labels proceeding from the exterior to core as we proceed from left to right along the sequence. Therefore all $2k$ legs of the spider are edge labeled.

The core vertex is thus labeled 0. Let V_j be the n vertex labels as induced by S_j; (we exclude the last term of S_j from V_j). To see that no duplicate vertex label occurs, note that the first term of each S_j is a different non-zero multiple of n. Let $\overline{V}_1$ be the first n terms of V, $\overline{V}_2$ be the next n terms of V, and so on. Note that the term $-kn$ is not used in this partition of V. Then the interior vertex labels of V_j correspond with the last $n-1$ terms of $\overline{V}_j$ (although the correspondence may be in reverse order). Since these interior vertex labels are distinct integers, none of which are multiples of n, then no duplication occurs within the set of interior vertex labels. Therefore this spider is super-edge-graceful.

Now consider spiders with an odd number of legs of even length. Let $m = 2k+1$, be the number of legs. Let $2n$ be the length of each leg. For $1 \le i \le n$, $1 \le j \le k$, let

$$\begin{cases} a_{i,j} = m(i-1)+j, \\ b_{i,j} = mi - j, \end{cases}$$

be an enumeration of the positive integers from 1 to mn, excluding the integer multiples of m.

Define edge labelings along k legs L_j, $1 \le j \le k$, as follows, where the integer sequence from left to right are the labels of the edges from an exterior edge to the core. If j is odd let

$$\begin{aligned} L_j &= \{b_{n,j}, -a_{1,j}, b_{n-1,j}, -a_{2,j}, \ldots, b_{1,j}, -a_{n,j}\} \\ &= \{mn-j, -j, m(n-1)-j, -m-j, \ldots, m-j, -m(n-1)-j\}, \end{aligned}$$

giving rise to the vertex labels, from exterior vertex to the core, but not including the core,

$$\begin{aligned} V_j &= \{mn-j, mn-2j, m(n-1)-2j, \ldots, m(2-n)-2j\} \\ &= \{v_{1,j}, v_{2,j}, \ldots, v_{2n,j}\}. \end{aligned}$$

If j is even let

$$\begin{aligned} L_j &= \{a_{n,j}, -b_{1,j}, a_{n-1,j}, -b_{2,j}, \ldots, a_{1,j}, -b_{n,j}\} \\ &= \{m(n-1)+j, -m+j, m(n-2)+j, \ldots, j, -mn+j\}, \end{aligned}$$

giving rise to the vertex labelings,

$$\begin{aligned} V_j &= \{m(n-1)+j, m(n-2)+2j, m(n-3)+2j, \ldots, m(-n)+2j\} \\ &= \{v_{1,j}, v_{2,j}, \ldots, v_{2n,j}\}. \end{aligned}$$

For each j, $1 \le j \le k$, let $L_{-j} = -L_j$, the *inverse* leg of the spider, giving rise to the inverse vertex labels, $V_{-j} = -V_j$.

Finally, define the edge labels along the last leg, L_o, using multiples of m.

$$L_0 = \{mn, -m, m(n-1), -2m, \ldots, m, -mn\},$$

giving rise to the vertex labels of

$$V_0 = \{mn, m(n-1), m(n-2), \ldots, -mn\}.$$

Note that mn is an exterior vertex label and that $-mn$ is the core label.

It is clear that all edge labels from ± 1 to $\pm km$ have been used. It remains to show that the vertex labelling uses all values from $0, \pm 1$, to $\pm km$. First of all, note that for each V_j, $-k \leq j \leq k$, except for the exterior vertex label, all vertex labels belong to that equivalence class modulo m containing $-2j$ (if j is either a positive odd integer or a negative even integer) or $2j$ (if j is either a negative odd integer or a nonnegative even integer). Since $0, \pm 2j$, where $1 \leq j \leq k$ are distinct integers modulo m, then among all interior vertices in all m arms there are no duplicate vertex labels.

Hence it remains to show that no exterior vertex label duplicates another vertex label. The k positive vertex labels (omitting L_0, whose two end labels duplicate no other vertex label) are $m(n-1)+j$, when j is even and $mn - j$ when j is odd.

Clearly, no exterior vertex labels are the same for different even j's or for different odd j's. Setting $m(n-1)+j_0 = mn - j_1$ for some j_0 and j_1 yields $j_0 + j_1 = m$, but $j_0 + j_1 \leq 2k < m$. So there are no duplicates among the exterior vertices with positive label. The same argument applies for the exterior vertices with negative labels.

To see that no exterior vertex label duplicates an interior label observe that for $1 \leq j_0, j_1 \leq k$ and $n \geq 2$

$$(*) \quad v_{1,j_1} > v_{3,j_0} \text{ and } v_{1,j_1} > -v_{2n-1,j_0}.$$

Since interior vertex labels along V_{j_0} decrease from the positive value v_{2,j_0} to the negative value v_{2n,j_0}, then in order for an exterior vertex label to duplicate an interior vertex label, by (*) that duplicate label occurs either at a vertex adjacent to an exterior vertex or adjacent to the core for any $n \geq 1$.

Consider the exterior vertex label $m(n-1)+j_0$ where j_0 is even. The two possible duplicate values are the phrases $m(n-2)+2j_1$ or $mn - 2j_1$ for some j_1. In either case, equating $m(n-1)+j_0$ with each phrase results in a parity contradiction on m.

Similarly, for the exterior vertex label $mn - j_0$ where j_0 is odd, equating $mn - j_0$ with each phrase results in a parity contradiction, this time on j_0.

Thus the vertex labels include all values from $0, \pm 1$, to $\pm mn$, and hence these spiders are super-edge-graceful. □

Corollary 6: Small's Theorem. *Every regular spider with an even number of edges is edge-graceful.*

Proof: Apply Corollary 2 and Theorem 5. □

Corollary 7. *Let T be a tree obtained from a regular spider of odd order by appending $2n$ vertices, 2 at a time where each time the 2 have a common parent. Then T is super-edge-graceful, as well as edge-graceful.*

Proof: This follows immediately from Theorem 5 and repeated application of Proposition 3. □

Many of the trees in Corollary 7 were not previously known to be edge-graceful. Hence one advantage of a super-edge-graceful labeling over an arbitrary edge-graceful labeling is that when edges are appropriately appended to trees, then the new trees are also super-edge-graceful, and the edges remaining from the old trees need not be relabeled!

The super-edge-graceful concept is interesting from the standpoint that while no tree of even order is edge-graceful as follows from Lo's condition, some of these graphs are super-edge-graceful, as shown in the next proposition.

Proposition 8. *For $n \geq 2$, let $T_{2,n}$ be the tree which has exactly 2 adjacent vertices of degree $n+1$ and all other vertices of degree 1. $T_{2,n}$ is super-edge-graceful for all $n \geq 2$.*

Proof: Since $T_{2,n}$ has $2n+2$ vertices, the edges must be labeled from $0, \pm 1$, to $\pm n$. Call the vertices of degree $n+1$, u and v. Label the edge between u and v with 0. If n is even, label two edges incident to u with n and 1, and label two edges incident to v with $-n$ and -1. If n is odd, label three edges incident to u with $-1, 2, n$, and label three edges incident to v with $1, -2, -n$. Label the other edges incident to u with inverse pairs, $\pm(n-1), \pm(n-2), \ldots$, and label the other edges incident to v with the remaining labels. In either case u and v have labels $\pm(n+1)$, and the remaining vertices have the labels $\{\pm 1, \ldots, \pm n\}$, which means that these graphs are super-edge-graceful. □

The remainder of this paper is a collection of algorithms, which when applied to super-edge-graceful graphs yield new super-edge-graceful graphs.

Proposition 9: Insertion of an Edge Algorithm. *If G is super-edge-graceful and q is even and u,v are nonadjacent vertices of G, then inserting the edge uv into G results in a super-edge-graceful graph.*

Proof: Label edge uv with 0, and note that no vertex labels have changed, making the new graph super-edge-graceful. □

With this proposition it is easy to prove the following about paths and cycles, two graph classes shown to be edge-graceful in [6].

Corollary 10. *P_{2n+1} and C_{2n+1} are super-edge-graceful for all $n \geq 1$.*

Proof: P_{2n+1} is a regular spider with 2 legs of length n. Thus it is super-edge-graceful. Now join the 2 end vertices with an edge labeled 0, and thus C_{2n+1} is super-edge-graceful. □

Proposition 11: A Vertex Fusion Algorithm. *Let (ℓ, ℓ^*) be a super-*

edge-graceful function pair for G. Let $S = \{u,v,w\}$ or $S = \{u,v\}$ be a set of vertices in G such that

$$\ell^*(u) = \frac{p-1}{2}, \quad \ell^*(v) = -\frac{p-1}{2}, \quad \ell^*(w) = 0,$$

or

$$\ell^*(u) = \frac{p}{2}, \quad \ell^*(v) = -\frac{p}{2}.$$

Let F be the graph (or multi-graph) obtained from G by identifying the vertices of S with one of its members. Then F is super-edge-graceful.

Proof: To define a super-edge-graceful function pair (μ, μ^*) on F let $\mu \equiv \ell$ on $G \cap F$. For other edges in F, choose u in S; define $\mu(zu) = \ell(zx)$ if zx is in $E(G)$ and x is in S. Provided no two elements of S are adjacent or are adjacent to the same vertex, μ is well-defined. Otherwise μ is a multi-valued edge function, labeling loops and multiple edges. In either case, it is clear that μ^* is onto P, since either $S = \{u,v,w\}$, in which case

$$\mu^*(u) = \ell^*(u) + \ell^*(v) + \ell^*(w) = 0,$$

or $S = \{u,v\}$, in which case

$$\mu^*(u) = \ell^*(u) + \ell^*(v) = 0.\square$$

For example, successively identifying the labeled inverse pairs of vertices for the spider with 7 legs of length 6 (see the example given for the shoelace algorithm) results in the following succession of super-edge-graceful graphs. That is, identify the vertices with labels $\{0, \pm 21\}$, denoted by larger nodes in part *i* of the figure below, resulting in *ii*. Identify the vertices with labels $\{0, \pm 20\}$, resulting in *iii*. Allowing multiple loops, we can continue identifying vertices in the manner indicated. After a total of 6 iterations the super-edge-graceful butterfly in *iv* results from *folding up* the spider! (The four loops in *iv* have been moved out of the clutter at the core via Proposition 15.)

Proposition 12: A Vertex Fission Algorithm. *Let G_1, G_2 be subgraphs of G with super-edge-graceful function pair (ℓ, ℓ^*) and w be a vertex of G such that $G = G_1 \cup G_2$, $G_1 \cap G_2$ contains no edge incident to w, both G_1 and G_2 contain at least one edge incident to w, and*

$$\ell^*_{G_1}(w) = \frac{p+1}{2} = -\ell^*_{G_2}(w).$$

Rename w in G_1 with a new vertex u, renaming the graph as G_3. Let $F = G_2 \cup G_3$. Then F is super-edge-graceful.

Proof: Immediate. $\square$

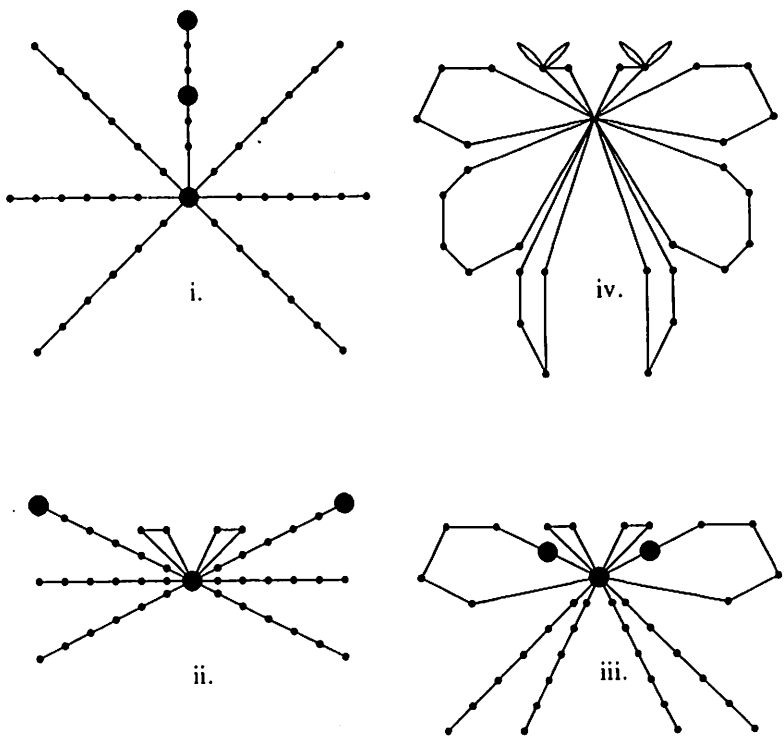

Figure 0.4: The Metamorphosis of a Super-Edge-Graceful Butterfly.

Corollary 13. *Let S_1, S_2 be disjoint spider graphs each with an even number of legs all of the same odd length. Then $S_1 \cup S_2$ is a super-edge-graceful graph.*

Proof: Let S be the regular spider having $2m$ legs all of length $2n+1$, where $m \geq 2, n \geq 0$. Let S have the edge labeling of Theorem 5. On the first leg of S, the edge incident to the core is labeled $(2n+1)m-n$. On the *inverse* of the second leg of S, the edge incident to the core is labeled $n+1$. Let S_1 be the spider formed by these two legs along with any number of pairs of inverse legs, so that its core is labeled $(2n+1)m+1$. Let S_2 be the spider formed by the remaining legs so that its core is labeled $-((2n+1)m+1)$. By Proposition 12, $S_1 \cup S_2$ is super-edge-graceful. □

Corollary 14. *P_{8n} and P_{8n-2} are super-edge-graceful for all $n \geq 1$.*

Proof: By Corollary 13, $2P_{4n-1}$ is super-edge-graceful, since P_{4n-1} is a spider with 2 legs, each of length n. By Proposition 9, insert an edge labeled 0 between the two paths to form P_{8n-2}, which is thus super-edge-graceful.

To see that P_{8n} is super-edge-graceful, label the $4n-1$ edges of P_{4n} according to the sequence

$$\{4n-1, -1, 4n-2, -2, \ldots, -n+1, 4n-n, n, -4n+n+1, n+1, -4n+n+2, \ldots, -2n\}.$$

As before, label another copy of P_{4n} with the inverse sequence, insert an edge labeled 0 between these two paths. The resulting path P_{8n} is super-edge-graceful.□

Determination of the super-edge-gracefulness of the remaining paths of even order greater than 6 is an open question. We thank the referee for pointing out the algorithm showing that P_{8n} is super-edge-graceful.

To consider two additional super-edge-graceful graph generating algorithms, let v be a *cut-vertex* of G with $V_1, V_2, \ldots, V_t$ being the vertex sets of the *components* of $G \backslash \{v\}$. Then a *cut-vertex decomposition* (G_1, G_2, v) is the pair of subgraphs G_1, G_2 where G_1 is *induced* by the union of some collection of the V_i with $\{v\}$ and G_2 is *induced* by the union of the remaining V_i with $\{v\}$.

The next proposition which follows immediately from the definitions shows how to cut a graph into two pieces and paste it together so that the resulting graph is as super-edge-graceful as the original. (The next two propositions also have edge-graceful analogs.)

Proposition 15: A Cut & Paste Algorithm. *Let (G_1, G_2, v) be a cut-vertex decomposition of G which has a super-edge-graceful function pair (ℓ, ℓ^*) with $\ell^*_{G_1}(v) = 0$. In G_1 rename v as the new vertex w, and let x be any vertex in G_2. Regard G_1, G_2 as disjoint graphs using the labels already assigned. The graph which results from identifying w and x is super-edge-graceful.*

Corollary 16. *Let T_1 be a non-trivial tree with only the root of even degree. Form tree T_2 by replacing each edge of T_1 with a path of $n \geq 1$ edges. Then T_2 is super-edge-graceful.*

Proof: T_1 must have an even number, say $2k$, of edges. As in the proof of Corollary 4 there is a super-edge-graceful labeling of T_1 using, successively, $\pm 1, \pm 2, \ldots, \pm k$ on pairs of edges with a common parent. In the super-edge-graceful labeling of the spider with $2k$ legs of length n the legs are partitioned into inverse pairs. Use these inverse pairs of legs to form T_2 with its super-edge-graceful labeling in the same way that T_1 was labeled. Thus by Proposition 15, T_2 is super-edge-graceful. □

Corollary 17. *Let T_1 be a non-trivial tree with no vertex of even degree. Replace each edge of T_1 with a path of $2n$ edges, $n \geq 1$, forming tree T_2. Then T_2 is super-edge-graceful.*

Proof: T_1 has an odd number, $2k+1$, of edges. It can be formed from the spider with $2k+1$ legs of length 1 by moving pairs of edges successively as in Corollary 16. Consider the spider with $2k+1$ legs of length $2n$ which is labeled as in Theorem 5. By successively moving inverse legs we obtain T_2 and a super-edge-graceful labeling of it. □

Proposition 18: A Pruning & Grafting Algorithm. *Let F and G be graphs with the same parameters p and q. Let (G_1, G_2, v) be a cut-vertex decomposition of G, which has a super-edge-graceful function pair (ℓ, ℓ^*) with $\ell^*_{G_2}(v) = 0$. Let (G_2, G_3, v) be a cut-vertex decomposition of F. Let (μ, μ^*) be a function pair for G_3 such that $\mu^*(u) = \sum_{uw \in E(G_3)} \mu(uw)$ and*

$$\mu(E(G_3)) = \ell(E(G_1)) \text{ and } \mu^*(V(G_3)) = \ell^*(V(G_1)).$$

Then F is super-edge-graceful.

In the words of pruning and grafting, G_1 is *pruned* from G, and G_3 is *grafted* in, resulting in F.

Proof: To define a super-edge-graceful function pair (ϕ, ϕ^*) for F, let

$$\phi(e) = \begin{cases} \ell(e), & \text{if } e \in E(G_2), \\ \mu(e), & \text{if } e \in E(G_3), \end{cases}$$

$$\phi^*(v) = \begin{cases} \ell^*(v), & \text{if } v \in V(G_2), \\ \mu^*(v), & \text{if } v \in V(G_3). \end{cases}$$

It is clear that (ϕ, ϕ^*) is a super-edge-graceful function pair for F.□

Corollary 19: A Customized Spider Leg. *Let G be a regular spider with $2k+1$ legs of length $2n$. Let H be any super-edge-graceful tree of order $2n+1$. Let F be the graph formed by pruning a leg of G and grafting in H. Then F is super-edge-graceful.*

Proof: Let L_0 be the leg of G labeled with multiples of $2k+1$ as given in Theorem 5. Note that L_0 is the path P_{2n+1}, and one super-edge-graceful labeling of P_{2n+1} results from dividing each edge label of L_0 by $2k+1$. That is, if $(\ell, \ell^*), (\lambda, \lambda^*)$ are the super-edge-graceful function pairs for G, P_{2n+1}, respectively, then

$$\ell_{L_0} \equiv (2k+1)\lambda \text{ and } \ell^*_{L_0} \equiv (2k+1)\lambda^*.$$

Let (ϕ, ϕ^*) be a super-edge-graceful function pair for H. In order to use Proposition 18 to show that F is super-edge-graceful we need to define an appropriate function pair (μ, μ^*) for H. For each edge e in H define $\mu(e) = (2k+1)\phi(e)$. It is clear that μ^* gives the appropriate vertex labels and hence by Proposition 18, F is super-edge-graceful. □

For example let G be the spider of figure 4(i). Let G_1 be a leg of the spider, G_2 be the spider with 6 legs, and G_3 be the tree with 7 vertices, super-edge-graceful labeled on its edges as indicated. The vertex v is the enlarged node on all the graphs G_1, G_2, G_3. Apply Corollary 19. Then $G_2 \cup G_3$ is a super-edge-graceful graph.

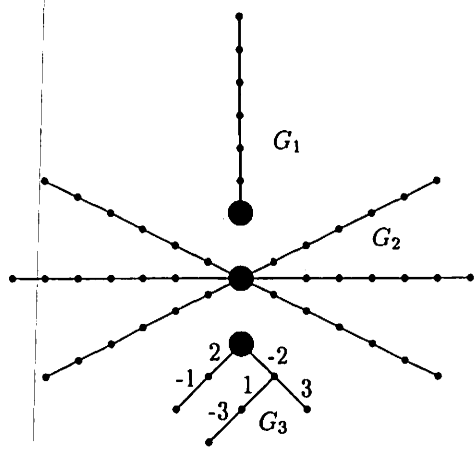

Figure 0.5: A Super Leg for a Super-Edge-Graceful Spider.

Readers interested in the general problem of labeling the edges of a graph with the integers 1 through q so as to induce a labeling of the vertices of the graph are referred to [1], [2], and [5], where the notions of magic graphs and gracefully weighted graphs are developed.

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