Almost Vertex Bipancyclic Graphs

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Abstract. A connected balanced bipartite graph G on $2n$ vertices is almost vertex bipancyclic (i.e. G has cycles of length $6, 8, \ldots, 2n$ through each vertex of G) if it satisfies the following property $P(n)$: *if* $x, y \in V(G)$ *and* $d(x, y) = 3$ *then* $d(x) + d(y) \geq n + 1$. Furthermore, all graphs except C_6 on $2n$ ($n \geq 3$) vertices satisfying $P(n)$ are bipancyclic (i.e. there are cycles of length $4, 6, \ldots, 2n$ in the graph).

1. Introduction

Throughout this paper, we consider only simple undirected graphs. Notation follow Y.P. Liu [1] unless otherwise specified.

With $G = (A, B; E)$ we denote the bipartite graph G with edge-set E and vertex-set $V = A \cup B$, where A and B are the two sides of G. If $|A| = |B|$ then we say that G is a *balanced* bipartite graph. If G is a bipartite graph on $2n$ vertices with cycles of all even lengths $4, 6, \ldots, 2n$ then we say that G is *bipancyclic*. If G has cycles of length $6, 8, \ldots, 2n$ through any given vertex then we call G *almost vertex bipancyclic*.

We use $\varepsilon(G)$ and $\delta(G)$ to denote the number of edges and the minimum degree of vertices in the graph respectively. The valence of a vertex $v \in V$ is written as $d(v)$ and the distance between two vertices x and y is written as $d(x, y)$. For $U \subseteq V$ we use $N(U)$ to denote the set of vertices $v \in V \setminus U$ such that v is adjacent to some vertex in U.

A balanced bipartite graph G is said to have the property $P(n)$ if G is connected and,

$$\text{for } u, v \in V, d(u, v) = 3 \quad \Rightarrow \quad d(u) + d(v) \geq n + 1$$

It is known that Fan's condition on general 2-connected graphs:

$$d(u, v) = 2 \quad \Rightarrow \quad \max\{d(u), d(v)\} \geq n/2$$

ensures the pancyclicity of the graph (with three exceptions) ([2], [3]).

As one can check, for bipartite graphs, only $K_{n,n}$ and $K_{n,n} - e$ (delete edge e from $K_{n,n}$) satisfy Fan's condition. So Fan and Tian-Shi's theorem do not apply to bipartite graphs.

There are some sufficient conditions which guarantee the bipancyclicity of bipartite graphs, for example (see [6]), Moon-Moser's condition [4]: $\varepsilon(G) \geq n(n-1)+1$, and Mitchem-Schmeichel's condition [6]: $\delta(G) \geq (n+1)/2$. In addition, We have following

Theorem A. *(Entringer and Schmeichel, [5]): Let G be a hamiltonian bipartite graph on $2n$ vertices. If $\varepsilon(G) > n^2/2$, then G is bipancyclic.*

In this note we give a new sufficient condition which is parallel to Fan's condition, that is: if $G(\neq C_6)$ is a balanced bipartite graph on $2n$ $(n \geq 3)$ vertices satisfying $P(n)$ then G is bipancyclic. As we shall see in section 3 that Moon-Moser's theorem and Mitchem-Schmeichel's theorem can be treated as corollaries of our result.

Furthermore we show that every bipartite graph satisfying $P(n)$ $(n \geq 2)$ is almost vertex bipancyclic. The method we use in proving this theorem is similar to Cai Xiao-tao's method in [7].

2. Hamiltonian Bipartite Graphs

If $u \in A, v \in B, (u,v) \notin E$ and $d(u) + d(v) \geq n+1$ then we add a new edge (u,v) to G and continue doing this until we at last come to the graph $\tilde{G}$, to which no new edge can be added as above. Call $\tilde{G}$ the *biclosure* of G. Clearly G is Hamiltonian if and only if $\tilde{G}$ is Hamiltonian. We show that if G satisfies $P(n)$ $(n \geq 3)$, then $\tilde{G}$ is a balanced, complete bipartite graph $K_{n,n}$, hence G is Hamiltonian.

Theorem 2.1. *For a balanced bipartite graph $G = (A, B; E)$ of order $2n(n \geq 3)$ if G satisfies $P(n)$, then $\tilde{G} = K_{n,n}$.*

Proof: We prove the following five statements first.

(2.1) If $U \subseteq A$ (or B) then $|N(U)| \geq |U|$, where the identity holds if and only if $U = A$ (or B).

In fact, if $U = A$, from the connectedness of G, we see that $N(A) = B$, i.e. $|N(U)| = |U| = n$; If $U \subset A$, we show that $|N(U)| > |U|$. Suppose $|U| = r$, $|N(U)| = s$. If $N(U) = B$, then clearly $|N(U)| > |U|$. If $N(U) \neq B$, since G is connected, there exists $b_1 \in B - N(U), a \in A - U, b \in N(U), a_1 \in U$ such that $a_1 b a b_1$ is a path joining a_1 and b_1 and $(a_1, b_1) \notin E$, thus $d(a_1, b_1) = 3$. By $P(n)$, we have $d(b_1) + d(a_1) \geq n+1$. But $d(b_1) \leq n-r, d(a_1) \leq s$, thus $n-r+s \geq n+1$. So $s \geq r+1$, i.e. $|N(U)| > |U|$.

(2.2) Let $S = \{v \in G | d(v) \geq (n+1)/2\}$, then $N(S)$ induces a complete bipartite subgraph of $\tilde{G}$.

For $u \in N(S)$, without loss of generality, suppose $u \in A$, then there exists a vertex, say $v \in S \cap B$ such that $(u,v) \in E$. By (2.1), $|N(N(v))| \geq |N(v)| \geq (n+1)/2$. For each $w \in N(N(v))$, either $(u,w) \in E$ or $d(u,w) = 3$. If

$d(u,w) = 3$, then $d(u)+d(w) \geq n+1$ by $P(n)$. In both cases, $(u,w) \in E(\widetilde{G})$. Thus $\widetilde{d}(u) \geq |N(N(v))| \geq (n+1)/2$, where $\widetilde{d}(u)$ is the valence of u in $\widetilde{G}$. From above, we see that each vertex $u \in S \cup N(S)$ has valence no less than $(n+1)/2$ in $\widetilde{G}$. By definition of $\widetilde{G}$, we arrive at (2.2).

From (2.2), if $|S \cup N(S)| = 2n$, then $\widetilde{G} \cong K_{n,n}$, Theorem 1 holds. Therefore, we just need to consider the case when $|S \cup N(S)| < 2n$.

(2.3) Let $S \cup N(S) = A_1 \cup B_1$, $A_1 \subset A$, $B_1 \subset B$. Then both $A_1 \cap S$ and $B_1 \cap S$ are not empty.

If $B \subseteq S$, since $\sum_{v \in A} d(v) = \sum_{u \in B} d(u)$ and $|A| = |B|$, then we have some vertex $v \in A$ such that $d(v) \geq (n+1)/2$. So $v \in A_1 \cap S \neq \phi$.

If $B \not\subseteq S$, suppose $u \in B$ and $d(u) < (n+1)/2$. Since G is connected, there exists some vertex, say $v \in A$, which satisfies $d(u,v) = 3$. By $P(n)$, $d(u) + d(v) \geq n+1$. Then $d(v) > (n+1)/2$, $v \in S \cap A_1$. Hence $A_1 \cap S \neq \phi$. Similarly, $B_1 \cap S \neq \phi$.

(2.4) $|A_1| \geq (n+1)/2$, $|B_1| \geq (n+1)/2$.

From (2.3), $A_1 \cap S \neq \phi$, if $v \in A_1 \cap S$, then by the definition of S, we know $d(v) \geq (n+1)/2$, then, of course, $|B_1| \geq (n+1)/2$. Similarly $|A_1| \geq (n+1)/2$.

(2.5) If $v \in V(G) \setminus (S \cup N(S))$, then $\widetilde{d}(v) \geq (n+1)/2$.

In fact, suppose $v \in A$, then from $v \notin S$, $d(v) < (n+1)/2$. Also by connectedness of G, we have some vertex $u \in B$ such that $d(v,u) = 3$. By $P(n)$, $d(u) + d(v) \geq n+1$. Thus $d(u) > (n+1)/2$. For arbitrary $x \in B_1$, we have $\widetilde{d}(x) \geq |A_1|$ by (2.2), where clearly $|A_1| \geq d(u)$. So $\widetilde{d}(x) + \widetilde{d}(v) \geq d(u) + d(v) \geq n+1$ and $(x,v) \in E(\widetilde{G})$. Therefore v is adjacent to each vertex of B_1 in G. As a result, from (2.3), $\widetilde{d}(v) \geq |B_1| \geq (n+1)/2$.

Finally from (2.1) to (2.5) above we see that $\widetilde{d}(v) \geq (n+1)/2$ for all $v \in V(G) = A \cup B$. This implies $\widetilde{G} \cong K_{n,n}$. ∎

We present below an obvious corollary without proof.

Corrollary 2.1. *For a balanced bipartite graph $G = (A,B;E)$, if G satisfies $P(n)$, then G is Hamiltonian.*

We close this section by the following remark: we have examples showing that the inequality in $P(n)$ can not be replaced by $d(u)+d(v) \geq n$ or $\max\{d(u), d(v)\} \geq (n+1)/2$. So, in this sense we can say that Theorem 2.1 is best possible.

3. Vertex Bipancyclicity

Theorem 3.1 *Suppose $G = (A, B; E)$ is a bipartite graph on $2n$ vertices with the property $P(n)(n \geq 4)$. For arbitrary $x \in A, y \in B$, if G has a path $P_{2k}(x,y)$ with $2k(k \geq 4)$ vertices linking x and y, then G also has a path $P_{2k-2}(x,y)$.*

Proof: Suppose $P_{2k}(x,y) = (1,2,\ldots,2k)$ (where $x = 1, y = 2k, k \geq 4$). Let $d_1(v) = |N(v) \setminus P_{2k}(x,y)|$, $d_2(v) = |\{u \in N(v) \mid 1 \leq u \leq 8\}|$, and $d_3(v) = |\{u \in N(v) \mid 9 \leq u \leq 2k\}|$. Thus $d(v) = d_1(v) + d_2(v) + d_3(v)$. By contradiction, suppose G has no $P_{2k-2}(x,y)$. We prove the following twelve assertions:

(1) $d(1) + d(4) + d(5) + d(8) \geq 2n+2$.

Since G has no $P_{2k-2}(x,y)$, we see that $(1,4),(5,8) \notin E$. That is, $d(1,4) = d(5,8) = 3$. By $P(n)$, $d(1) + d(4) \geq n+1$, $d(5) + d(8) \geq n+1$. So (1) holds.

(2) $d_2(4) = d_2(5) = 2$.

(3) $d_1(1) + d_1(5) \leq n-k$, $d_1(4) + d_1(8) \leq n-k$.

In fact, if $i \in V(G) \setminus P_{2k}(x,y)$ and $(1,i) \in E$, then $(5,i) \notin E$, since otherwise G has $P_{2k-2}(x,y) = (1,i,5,6,\ldots,2k)$, a contradiction. So $d_1(1) + d_1(5) \leq n-k$. Similarly $d_1(4) + d_1(8) \leq n-k$.

From (1), (2) and (3), we get

(4) $d_2(1) + d_2(8) + d_3(1) + d_3(4) + d_3(5) + d_3(8) \geq 2k-2$ for $k \geq 4$.

If $k = 4$, then (4) becomes $d_2(1) + d_2(8) \geq 6$. Since $(1,4),(5,8) \notin E$, we deduce that $(1,6),(1,8),(3,8) \in E$. But then we have $P_6(x,y) = (1,6,5,4,3,8)$, a contradiction. So Theorem 3.1 holds while $k = 4$. We assume $k > 4$ afterwards.

(5) $d_3(1) + d_3(4) \leq k-3$, $d_3(5) + d_3(8) \leq k-3$. Furthermore, if $(4,9) \notin E$, then $d_3(1) + d_3(4) \leq k-4$; if $(5,10) \notin E$, then $d_3(5) + d_3(8) \leq k-4$.

In fact, for $9 < i < 2k$, if $(1,i) \in E$, then $(4,i+1) \notin E$. Since otherwise we get $P_{2k-2}(x,y) = (1,i,i-1,\ldots,4,i+1,\ldots,2k)$, a contradiction. So $d_3(1) + d_3(4) \leq k-3$. If $(4,9) \notin E$, then among $k-4$ vertices $9,11,\ldots,2k-1$, there are at least $d_3(1)$ vertices that are not adjacent to vertex 4. So $d_3(1) + d_3(4) \leq k-4$.

If $(5,i) \in E$ for some even i $(10 \leq i \leq 2k)$, then $(8,i+1) \notin E$. Since otherwise $(1,2,3,4,5,i,i-1,\ldots,9,8,i+1,i+2,\ldots,2k)$ would be a P_{2k-2} in G, a contradiction. Hence, among $k-6$ vertices $13,15,\ldots,2k-1$ there are at least $d_3(5) - 1$ vertices that are not adjacent to vertex 8. This gives $d_3(5) + d_3(8) \leq k-3$.

If $(5,10) \notin E$, then together with $(8,11) \notin E$ and $(8,9) \in E$ we have: $d_3(8) \leq 1 + (k-6) - (d_3(5) - 1)$, i.e., $d_3(5) + d_3(8) \leq k-4$.

(6) $(1,8) \in E$, $(4,9) \notin E$.

From (4) and (5) we have $d_2(1)+d_2(8) \geq 4$. If $(1,8) \notin E$, then together with $(1,4),(5,8) \notin E$ and $d_2(1) + d_2(8) \geq 4$, we get $(1,6),(3,8) \in E$. But in this case G has $P_{2k-2}(x,y) = (1,6,5,4,3,8,9,10,\ldots,2k)$, a contradiction. So $(1,8) \in E$.

If $(4,9) \in E$, then G has $P_{2k-2}(x,y) = (1,8,7,6,5,4,9,10,\ldots,2k)$, a contradiction. This conflict implies that $(4,9) \notin E$.

(7) $(5,10) \in E$, $(3,8) \notin E$.

If $(5,10) \notin E$, together with $(4,9) \notin E$, we have $d_3(1) + d_3(4) + d_3(5) + d_3(8) \leq 2k-8$ by (5). From this and (4) we obtain $d_2(1) + d_2(8) \geq 6$. This would again force $(1,6),(1,8),(3,8) \in E$, which then gives $P_{2k-2}(x,y)$ in G, a contradiction. So $(5,10) \in E$.

If $(3,8) \in E$, noticing that $(5,10) \in E$, we get $P_{2k-2}(x,y) = (1,2,3,8,7,6,5,10,\ldots,2k)$, a contradiction. So $(3,8) \notin E$.

(8) $(1,6) \in E$.

By (6) and (5) we have $d_3(1) + d_3(4) \leq k-4, d_3(5) + d_3(8) \leq k-3$. Furthermore by (4), we get $d_2(1) + d_2(8) \geq 5$. But $(1,4),(5,8) \notin E$ and $(3,8) \notin E$ by (7), then $d_2(1) + d_2(8) \geq 5$ holds only if $(1,6) \in E$.

(9) $(2,7) \notin E$.

If otherwise $(2,7) \in E$, then $(1,8,7,2,3,4,5,10,\ldots,2k)$ would be a path $P_{2k-2}(x,y)$, a contradiction.

From above we get

(10) $d_2(i) = 2$ for all $i = 3,4,7$ and 8.

(11) $d_3(3) + d_3(4) + d_3(7) + d_3(8) \geq 2k-6$.

From $(1,8) \in E$ and $P_{2k}(x,y)$ we have $P_4(3,8) = (3,2,1,8)$. Since $(3,8) \notin E$ then $d(3,8)=3$. Of course $d(4,7)=3$. By $P(n)$, we have $d(3) + d(8) \geq n+1$, $d(4)+d(7) \geq n+1$. As in (3), we also have $d_1(3)+d_1(7) \leq n-k$ and $d_1(4) + d_1(8) \leq n-k$. From these four inequalities and (10) we get (11).

(12) For any $10 \leq i < 2n$, if $(3,i) \in E$ then $(4,i+1) \notin E$; if $(7,i) \in E$ then $(8,i+1) \notin E$.

In fact, if both $(3,i)$ and $(4,i+1) \in E$, then G has $P_{2k-2}(x,y) = (1,6,7,\ldots,i,3,4,i+1,\ldots,2k)$, a contradiction. If both $(7,i)$ and $(8,i+1) \in E$, then G has $P_{2k-2}(x,y) = (1,2,3,4,5,10,11,\ldots,i,7,8,\ldots,i+1,i+2,\ldots,2k)$, a contradiction. So (12) holds.

Now we are ready to prove the theorem. By (6), $(4,9) \notin E$. Then from (12), we deduce that $d_3(3)+d_3(4) \leq k-4$. By the assumption that G has no $P_{2k-2}(x,y)$, we see that $(7,10),(8,11) \notin E$. Thus also by (12), $d_3(7)+d_3(8) \leq k-4$. Sum up the above two inequalities we see that $d_3(3)+d_3(4)+d_3(7)+d_3(8) \leq 2k-8$, a contradiction to (11).

This final conflict implies that Theorem 3.1 holds. ∎

Theorem 3.2. *Let $G = (A, B; E)$ be a graph of order $2n$ with the property $P(n), n \geq 4$. If e is an edge of G such that G has a Hamilton cycle through it, then G also has cycles of lengths $6, 8, \ldots, 2(n-1)$ through it. Hence, G has cycles of lengths $6, 8, \ldots, 2n$ through any given vertex of G.*

Theorem 3.3. *If $G = (A, B; E)$ is a graph of order $2n$ with property $P(n)$ $(n \geq 3)$, then G is bipancyclic unless $G = C_6$.*

Proof: Suppose G is a graph of order $2n$ with property $P(n)$ $(n \geq 3)$. If $n = 3$, and $G \neq C_6$, then it is easy to check that G is bipancyclic. Now assume $n \geq 4$. By Theorem 3.2, G has cycles of lengths $6, 8, \ldots, 2n$, so it suffices to show that G has at least a cycle of length 4. By Theorem 2.1, G has a Hamilton cycle, say, $(1, 2, \ldots, 2n, 1)$. If for some $i (1 \leq i \leq 2n), (i, i+3) \in E$, where the addition is taken module $2n$, then apparently G has a cycle of length 4. If $(i, i+3) \notin E$ for all $1 \leq i \leq 2n$, then $d(i, i+3) = 3$. By $P(n)$, we have $d(i) + d(i+3) \geq n+1$, for all i from 1 to $2n$. Then $d(1) + d(2) + \cdots + d(2n) \geq n(n+1)$. Thus $\varepsilon(G) \geq n(n+1)/2$. Hence by Theorem A, G is bipancyclic. ∎

Now we list two theorems that can be treated as corollaries of the Theorem above.

Corrollary 3.1. ([4], [5] and [6]) *Let G be a balanced bipartite graph on $2n (\geq 4)$ vertices. If $\varepsilon(G) > n(n-1) + 1$, then G is bipancyclic.*

Proof: We show that in this case, G has no vertices $x \in A$ and $y \in B$ such that $(x, y) \notin E(G)$ and $d(x) + d(y) < n+1$. Otherwise at the extremal case, $\varepsilon(G) \leq d(x)+d(y)+(n-1)(n-1) < n+1+(n-1)(n-1) = n(n-1)+2$, i.e., $\varepsilon(G) \leq n(n-1) + 1$, a contradiction. Thus G satisfies $P(n)$. By Theorem 3.3, G is bipancyclic. ∎

Corrollary 3.2. ([4], [5] and [6]) *Let G be a balanced bipartite graph of $2n (\geq 4)$ vertices. If $\delta(G) \geq (n+1)/2$, then G is bipancyclic.*

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