A Theorem on n-Extendable Graphs

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ABSTRACT. A graph G having a 1-factor is called n-extendable if every matching of size n extends to a 1-factor. We show that if G is a connected graph of order $2p$ ($p \geq 3$), and q and n are integers, $1 \leq n < q < p$, such that every induced connected subgraph of order $2q$ is n-extendable, then G is n-extendable.

We consider only finite, simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A \subset V(G)$, $G[A]$ denotes the subgraph of G induced by A and $G \backslash A$ is the subgraph of G induced by $V(G) \backslash A$. If A and B are disjoint subsets of $V(G)$, then $E(A, B)$ denotes the set of edges with one end in A and the other in B. Further, for $e \in E(G)$, $V(e)$ is the set of endvertices of e. Let n and p be positive integers with $n \leq p-1$, G a graph with $2p$ vertices having a 1-factor (a perfect matching). Then G is said to be n-extendable if every matching of size n in G extends to a 1-factor. Other terminology and notation not defined here can be found in [1].

In [4], Sumner proved that the following theorem.

Theorem 1. *Let G be a connected graph of order $2p$, and let q be a integer such that $1 \leq q < p$. Suppose for some integer q, every induced connected subgraph of order $2q$ has a 1-factor. Then G has a 1-factor.*

In this note, we prove a similar theorem for n-extendable graphs.

Theorem 2. *Let G be a connected graph of order $2p$ ($p \geq 3$), and let q and n be integers such that $1 \leq n < q < p$. Suppose for some integer q, every induced connected subgraph of order $2q$ is n-extendable. Then G is n-extendable.*

We use the following lemmas in the proof of Theorem 2.

Lemma 3 (Plummer[3]).

(i) If G is n-extendable, then G is $(n-1)$-extendable.

(ii) If G is a connected n-extendable graph, then G is $(n+1)$-connected.

Lemma 4 (Egawa, Enomoto, and Saito[2]). *If G $(\neq K_2)$ is a connected graph having a 1-factor $F: = \{f_k \in E(G) | 1 \leq k \leq p\}$, then there exist two distinct elements of F, say f_i and f_j, such that both $G \backslash V(f_i)$ and $G \backslash V(f_j)$ are connected.*

Proof of Theorem 2: We use induction on $p-q$. First we prove the case $p-q=1$. Suppose that $G \backslash V(e)$ is n-extendable for every $e \in E(G)$ such that $G \backslash V(e)$ is connected. Obviously, by Theorem 1, G has a 1-factor. Further, by Lemma 4, there exists an edge $e = x_0 y_0$ such that $G \backslash V(e)$ is connected. Let $H: = G \backslash V(e)$. Then H is a connected n-extendable graph. Let $f_1, f_2, \ldots, f_n$ be any n independent edges in G. Since H is connected n-extendable graph, if $\{f_1, f_2, \ldots, f_n\} \subset E(H) \cup \{x_0 y_0\}$, then we can easily see that G has a 1-factor which contains $\{f_1, f_2, \ldots, f_n\}$ by Lemma 3 (i). Thus we may assume $\{f_1, f_2, \ldots, f_n\} \cap E(V(e), V(H)) \neq \emptyset$.

We distinguish two cases.

Case 1. $n = 1$.

Let $f_1 \in E(V(e), V(H))$. Without loss of generality, we may assume $f_1 = x_0 v$, where $v \in V(H)$. Since H is a connected graph having a 1-factor, H has an edge f which does not have a vertex v as its endvertex such that $H \backslash V(f)$ is connected by Lemma 4 (note that $|H| \geq 4$ from the hypothesis). Now $G[(V(H) \backslash V(f)) \cup V(e)] (= G \backslash V(f))$ is connected. Therefore $G \backslash V(f)$ is 1-extendable by the assumption. Then $G \backslash V(f)$ has a 1-factor F_1 containing f_1. $F_1 \cup \{f\}$ is a 1-factor of G, which contains f_1.

Case 2. $n \geq 2$.

Note that by Lemma 3 (ii), we may assume that H is at least 3-connected in this case. Let $f_1 \in E(V(e), V(H))$ and $\{f_2, \ldots, f_n\} \subset E(H)$. Clearly, $H \backslash V(f_n)$ is connected and also $G[(V(H) \backslash V(f_n)) \cup V(e)]$ $(= G \backslash V(fn))$ is connected. Therefore $G \backslash V(f_n)$ is n-extendable. By Lemma 3 (i), $G \backslash V(f_n)$ has a 1-factor F_2 containing f_i $(1 \leq i \leq n-1)$. Then $F_2 \cup \{f_n\}$ is a 1-factor of G which contains $\{f_1, \ldots, f_n\}$.

Let $f_1 \in E(x_0, V(H))$, $f_2 \in E(y_0, V(H))$ and $\{f_3, \ldots, f_n\} \subset E(H)$, where $x_0 y_0 = e$. If $n \geq 3$, then we can show that G has a 1-factor containing $\{f_1, \ldots, f_n\}$ by using the same argument as in the previous paragraph. If $n = 2$, then since $|H| \geq 6$ from the hypothesis, there exists an edge $f \in E(G)$ such that $H \backslash V(f)$ is connected and $V(f) \cap (V(f_1) \cup V(f_2)) = \emptyset$. Then we can make f do the work of an edge f_n in the case $n \geq 3$. This completes the proof of the case where $p-q=1$.

Next we show the case where $p - q \geq 2$. Let G be a connected graph satisfying the condition of Theorem 2. We suppose that the theorem holds for smaller values of $p - q$. Further, suppose that $G \backslash V(e)$ is connected for some edge e. Let G_0 be an arbitrary induced connected subgraph of $G \setminus V(e)$ with order $2q$. Then G_0 is also an induced connected subgraph of G. Since G_0 is n-extendable by the assumption, $G \backslash V(e)$ is n-extendable by the induction hypotheoses. For every edge e, this situation holds whenever $G \setminus V(e)$ is connected. Thus, from the proof of the case where $p - q = 1$, G is n-extendable. □

References

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