Domination Numbers of Complete Grid Graphs, I

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ABSTRACT. This paper concerns the domination numbers $\gamma_{k,n}$ for the complete $k \times n$ grid graphs for $1 \leq k \leq 10$ and $n \geq 1$. These numbers were previously established for $1 \leq k \leq 4$. Here we present dominating sets for $5 \leq k \leq 10$ and $n \geq 1$. This gives upper bounds for $\gamma_{k,n}$ for k in this range. We discuss evidence that indicates that these upper bounds are also lower bounds.

A *dominating set* in a graph is a set of vertices having the property that every vertex not in the set is adjacent to a vertex in the set. The *domination number* $\gamma(G)$ of a graph G is the cardinality of a smallest dominating set in G. For an extensive survey of domination problems and a comprehensive biography we refer the reader to the recent survey volume edited by Hedetniemi and Laskar [?]. In this paper we establish upper bounds for the domination numbers $\gamma_{k,n} = \gamma(P_k \times P_n)$ of the (complete) grid graphs $P_k \times P_n$ for $5 \leq k \leq 10$ and $n \geq 1$.

Jacobson and Kinch [?] established $\gamma_{k,n}$ for $k = 1, 2, 3, 4$ for all $n \geq 1$. Beyond $k = 4$ the problem becomes more difficult. E.O. Hare [?] developed an algorithm to compute $\gamma_{k,n}$ and using the output of an implementation of her algorithm she found simple formulas for $\gamma_{k,n}$ when $1 \leq k \leq 10$ agreeing with her data. Chang and Clark [?] proved Hare's formulas for $k = 5$ and 6 and $n \geq 1$. Note that Hare's algorithm does not produce a dominating set for $P_k \times P_n$ as it computes $\gamma_{k,n}$. Here we present dominating sets which agree in cardinality with Hare's calculated domination numbers and extend these sets periodically to all n, thus obtaining upper bounds.

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Hare's calculations and the above mentioned results of Chang and Clark lend credence to the conjecture that the upper bounds obtained here are in fact also the lower bounds.

As mentioned by Hedetniemi and Laskar [?] the complexity of finding $\gamma(P_k \times P_n)$ is still not known. However the problem of finding the domination number of an arbitrary *grid graph* (= subgraph of $P_k \times P_n$) is NP-complete [?]. In this paper we consider only so-called *complete grid graphs*, i.e., the graphs $P_k \times P_n$.

Below we give the known values of $\gamma_{k,n}$ from the sources mentioned above. For ease of comparison we have converted all of these formulas to the standard form:

$$\lfloor \frac{an+b}{p} \rfloor, \tag{1}$$

where a, b and p are positive integers depending on k and in some cases b depends on the value of n modulo p.

Known domination numbers for $P_k \times P_n$ for $1 \leq k \leq 10$

$$\gamma_{1,n} = \lfloor \frac{n+2}{3} \rfloor, \ n \geq 1$$

$$\gamma_{2,n} = \lfloor \frac{n+2}{2} \rfloor, \ n \geq 1$$

$$\gamma_{3,n} = \lfloor \frac{3n+4}{4} \rfloor, \ n \geq 1$$

$$\gamma_{4,n} = \begin{cases} n+1, & n = 1,2,3,5,6,9 \\ n, & \text{otherwise for } n \geq 1 \end{cases}$$

$$\gamma_{5,n} = \begin{cases} \lfloor \frac{6n+6}{5} \rfloor, & n = 2,3,7 \\ \lfloor \frac{6n+8}{5} \rfloor, & \text{otherwise for } n \geq 1 \end{cases}$$

$$\gamma_{6,n} = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor, & n \geq 6 \text{ and } n \equiv 1 \bmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor, & \text{otherwise if } n \geq 4 \end{cases}$$

$$\gamma_{7,n} = \lfloor \frac{5n+3}{3} \rfloor, \ 2 \leq n \leq 500$$

$$\gamma_{8,n} = \lfloor \frac{15n+14}{8} \rfloor, \ 7 \leq n \leq 500$$

$$\gamma_{9,n} = \lfloor \frac{23n+20}{11} \rfloor, \; 4 \le n \le 233$$

$$\gamma_{10,n} = \begin{cases} \lfloor \frac{30n+37}{13} \rfloor, & \text{for } n \equiv 0 \text{ or } 3 \bmod 13 \text{ and } n \ne 13, 16 \\ \lfloor \frac{30n+24}{13} \rfloor, & \text{otherwise for } 10 \le n \le 125 \end{cases}$$

1 Representation of Dominating Sets

The presentation of our dominating sets will be highly pictorial. For example, a dominating set of cardinality 84 for the 19 × 19 grid graph is represented as in Figure 1 below.

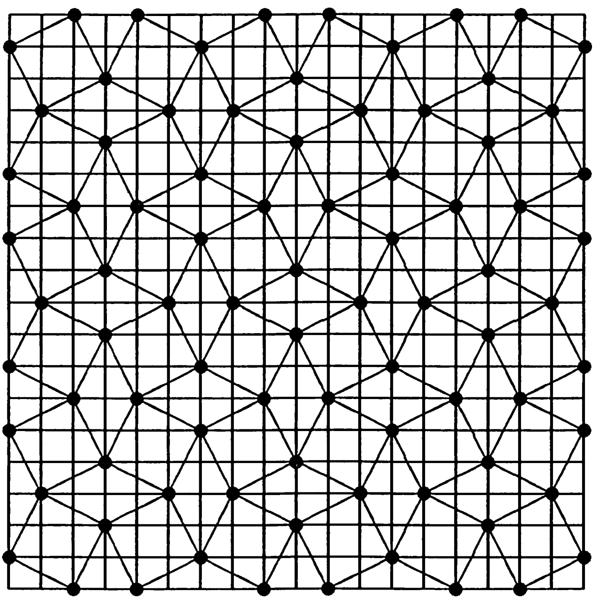


Figure 1: An 84 stone dominating set of $P_{19} \times P_{19}$.

The vertices of the graph are the intersections of the vertical and horizontal lines which represent the edges of the graph. *The diagonal lines with slopes* ± 2 *and* $\pm(1/2)$ *are not part of the graph. They are drawn in just to bring out the pattern formed by the vertices of the dominating set.* We place black dots at the vertices that belong to our dominating (or would be dominating) set. Using the terminology of the oriental game of Go we call the elements of a dominating set *stones.* The Go board is a 19 × 19 grid graph. At this time we do not know the domination number $\gamma_{19,19}$ of the Go board. From [?] we have $76 \le \gamma_{19,19} \le 84$. A dominating set of 84 stones for the 19 × 19 grid is shown in Figure 1. A different 84 stone dominating set was given in [?] for the 19×19 grid graph.

Using the above described representation we give in Figure 2 some sample dominating sets for the $k \times n$ grid graphs for k from 2 to 10 whose cardinalities are given by the formulas in Table 1. That these are the dominating sets of minimal cardinality follows from the above mentioned paper of Jacobson and Kinch [?], the computer calculations of E.O. Hare [?] and the results of Chang and Clark [?] in the case of $k = 5$ and $k = 6$.

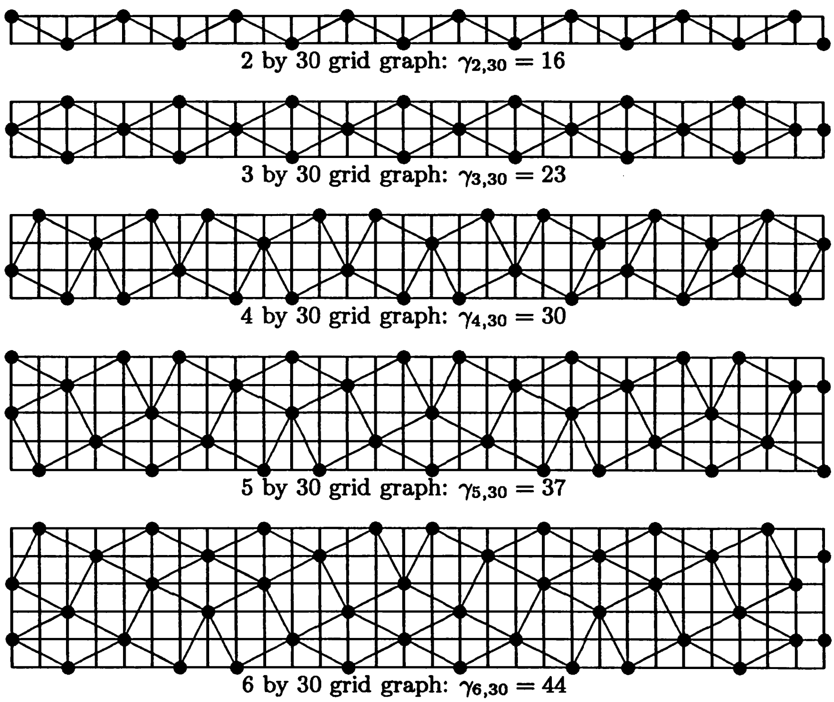
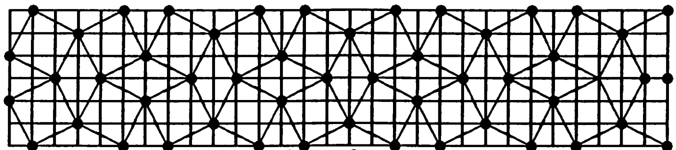
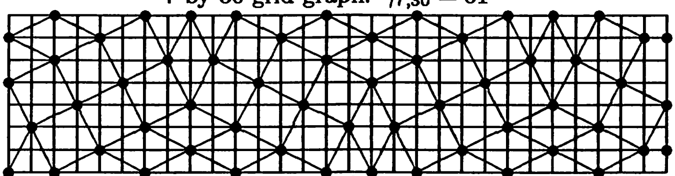


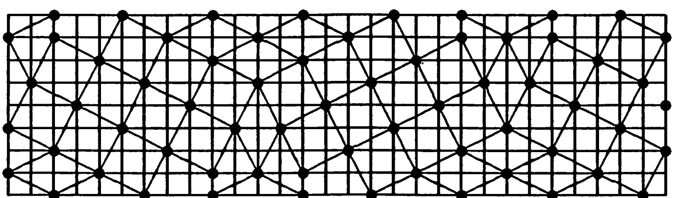
Figure 2: (a) Some dominating sets of minimal cardinality



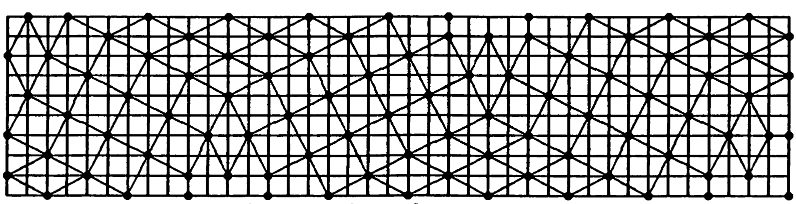
7 by 30 grid graph: $\gamma_{7,30} = 51$



8 by 30 grid graph: $\gamma_{8,30} = 58$



9 by 30 grid graph: $\gamma_{9,30} = 64$



10 by 40 grid graph: $\gamma_{10,40} = 94$

Figure 2: (b) More dominating set of minimal cardinality

Note that aside from irregularities along the boundaries and especially at the ends we obtain tessellations by diamonds and squares of rectangles enclosing the grids. Each diamond leads to a double covering of the vertex in its center, so diamonds are less desirable than squares in regard to minimizing dominating sets.

We note for further convenience that if f is a function satisfying

$$f(n) = \lfloor \frac{an+b}{p} \rfloor, \; n \geq 0,$$

then whenever $n = pq + r$, $r \geq 0$, we have

$$f(n) = aq + \lfloor \frac{ar+b}{p} \rfloor. \qquad (2)$$

2 Dominating Sets for $5 \times n$ grid graphs, $n \geq 5$

We construct here dominating sets S for $P_5 \times P_n$, when $n \geq 5$, of cardinality

$$\phi_5(n) = \lfloor \frac{6n+8}{5} \rfloor.$$

(Note that the construct works for $n = 7$, but in this case does not give the correct domination number. (see Table 1))

Since $n \geq 5$, n may be written in the form

$$n = 5q + r \text{ where } 1 \leq r \leq 5 \text{ and } q \geq 0.$$

Then,

$$\phi_5(n) = 6q + b_r,$$

where from (2)

$$b_1 = 2,\ b_2 = 4,\ b_3 = 5,\ b_4 = 6,\ b_5 = 7. \tag{3}$$

To construct our dominating set S for $P_5 \times P_n$ we use the blocks A, B, B_1, B_2, B_3, B_4 and B_5 of Figure 3. Note that the blocks A and B each contain 6 stones and the blocks B_r each contain b_r stones. (It is a curiosity that blocks A and B are reflections of each other. Indeed this holds for the blocks A and B used below for $k \times n$ grids, $k = 6, 7, 8, 9,$ *and* 10.)

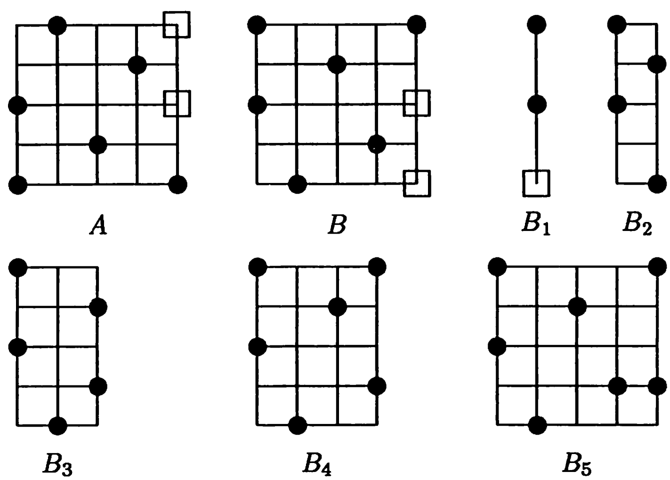
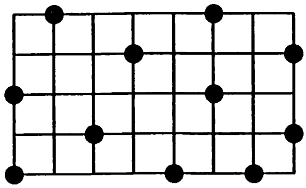


Figure 3: Blocks for constructing dominating sets for $P_5 \times P_n$.

We construct S by concatenating the blocks in Figure 4. We explain what we mean by *concatenation* by an example: If we concatenate the 5×5 block A and the 5×3 block B_3 we obtain the 5×8 block AB_3 pictured in Figure 4.



AB_3

Figure 4: Concatenation of A and B_3.

The black dots in each of the blocks in Figure 3 represent the stones of the dominating set S we wish to construct. The squared vertices in each block indicate vertices that are not covered by (i.e., not adjacent to) any stone in that block. Note however that the blocks AB_r, $r = 1,2,3,4,5$, have no uncovered vertices since the uncovered vertices in A are covered by stones in B_r.

We let $(BA)^s = BABA\ldots BA$ denote the concatenation of BA with itself $s \geq 0$ times. Note that $(BA)^s$ is a $5 \times 10s$ block with $6 \cdot 2s$ stones with only two vertices that are not covered by these stones.

Now let $n = 5q + r$, $1 \leq r \leq 5$, as above. If q is even then $q = 2s$, $s \geq 0$. One easily verifies that the stones in $(BA)^s B_r$ give a dominating set S for $P_5 \times P_n$ with $6q + b_r = \phi_5(n)$ elements where b_r is as given in (3). If q is odd, write $q = 2s + 1$; then the stones in $A(BA)^s B_r$ give a dominating set with $6q + b_r = \phi_5(n)$ elements.

3 Dominating Sets for $6 \times n$ grid graphs, $n \geq 6$

Here we will construct a dominating set S for $P_6 \times P_n$ of cardinality

$$\phi_6(n) = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor, & n \geq 6 \text{ and } n \equiv 1 \bmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor, & \text{otherwise if } n \geq 6 \end{cases}$$

Let $n = 7q + r$, $1 \leq r \leq 7$, then from (2)

$$\phi_6(n) = 10q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6	7
b_r	2	4	6	7	8	10	11

Table 2:

Now use the blocks in Figure 5, to construct S. Note that A and B each have 10 stones and B_r has b_r stones. We concatenate these blocks as we did in the $5 \times n$ case. For $n = 7q + r$, $1 \le r \le 7$, we again have two cases:

If $q = 2s$, then $(BA)^s B_r$ gives a dominating set with the desired number $10q + b_r$ of stones.

If $q = 2s + 1$, then $A(BA)^s B_r$ gives a dominating set with the desired number $10q + b_r$ of stones.

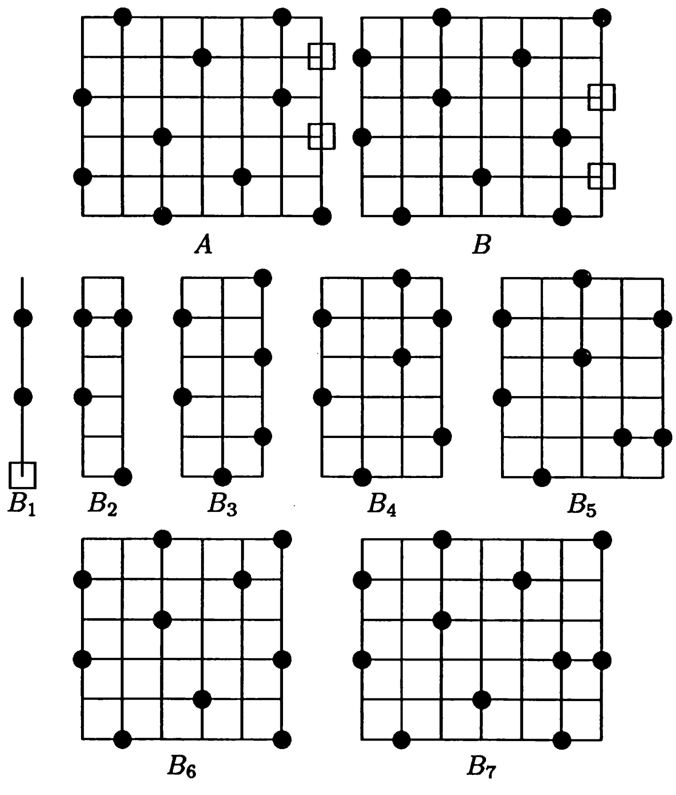


Figure 5: Blocks for constructing dominating sets for $P_6 \times P_n$.

4 Dominating Sets for $7 \times n$ grid graphs, $n \ge 7$

Here we will construct a dominating set S for $P_7 \times P_n$ of cardinality

$$\phi_7(n) = \lfloor \frac{5n+3}{3} \rfloor.$$

Let $n = 6q + r$, $1 \le r \le 6$, then

$$\phi_7(n) = 10q + c_r,$$

where the values of c_r are given by Table 3.

r	1	2	3	4	5	6
c_r	2	4	6	7	9	11

Table 3:

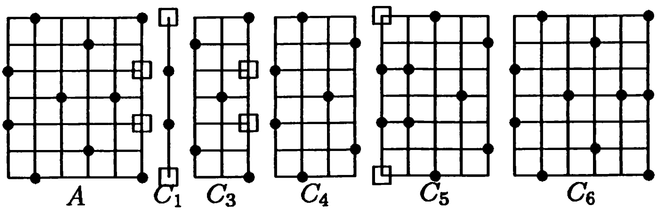


Figure 6: Blocks for constructing dominating sets for $P_7 \times P_n$.

We use the blocks in Figure 6, to construct S. Note that A has 10 stones and C_r has c_r stones for $r = 1, 4, 5, 6$. We must handle the cases $r = 2$ and 3 separately from the other values of r. For each case we exhibit below a particular concatenation of the blocks in Figure 6 in which the stones form a dominating set of $P_7 \times P_n$ of cardinality $10q + c_r = \phi_7(n)$ where $n = 6q + r$, $1 \le r \le 6$. Note that since $n \ge 7$, we have $q \ge 1$.

$r = 2$: $A^{q-1}C_4C_4$ has $10(q-1) + 14 = 10q + 4$ stones.
$r = 3$: $C_3A^{q-1}C_6$ has $5 + 10(q-1) + 11 = 10q + 6$ stones.
$r = 1, 4, 5, 6$: A^qC_r has $10q + c_r$ stones.

5 Dominating Sets for $8 \times n$ grid graphs, $n \ge 8$

Here we will construct a dominating set S for $P_8 \times P_n$, $n \ge 8$ of cardinality

$$\phi_8(n) = \lfloor \frac{15n + 14}{8} \rfloor.$$

If $n = 8q + r$, $1 \le r \le 8$, then

$$\phi_8(n) = 15q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6	7	8
b_r	3	5	7	9	11	13	14	16

Table 4:

We use the blocks in Figure 7. Note that b_r is the number of stones in the block B_r, the blocks A and B each have 15 stones and the blocks A',

B' have 16 each. As above we simply exhibit an appropriate concatenation in each case which has $\phi_8(n) = 15q + b_r$ stones given $n = 8q + r$, $1 \le r \le 8$.

For $1 \le r \le 6$, we take

$$\begin{cases} (AB)^s B_r & \text{if } q = 2s \\ A(AB)^s B_r & \text{if } q = 2s+1. \end{cases}$$

For $r = 7$, we take

$$\begin{cases} B_7(BA)^s B' & \text{if } q = 2s+1 \\ B_7(BA)^{s-1} BA' & \text{if } q = 2s. \end{cases}$$

For $r = 8$, we take

$$\begin{cases} A(BA)^s B' & \text{if } q = 2s+1 \\ (BA)^s B' & \text{if } q = 2s. \end{cases}$$

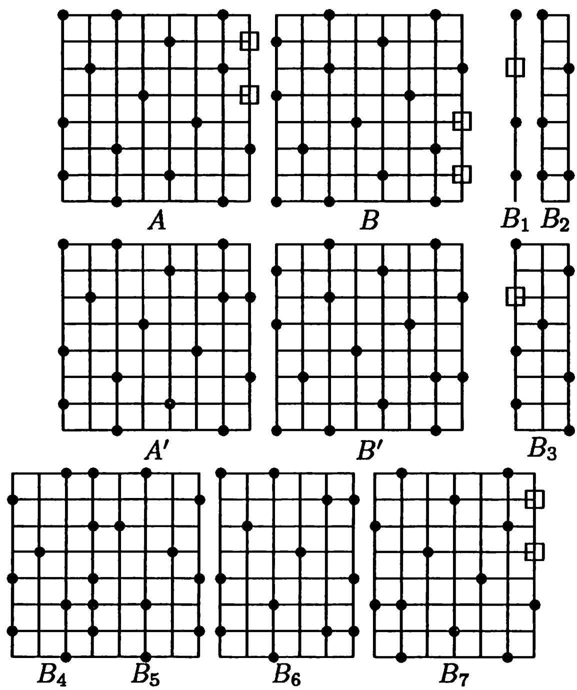


Figure 7: Blocks for constructing dominating sets for $P_8 \times P_n$.

6 Dominating Sets for $9 \times n$ grid graphs, $n \geq 9$

We now construct a dominating set S for $P_9 \times P_n$ for $n \geq 9$ of cardinality

$$\phi_9(n) = \lfloor \frac{23n+20}{11} \rfloor.$$

If $n = 11q + r$, $1 \leq r \leq 11$, then

$$\phi_9(n) = 23q + b_r,$$

where b_r is given by Table 5.

r	1	2	3	4	5	6	7	8	9	10	11
b_r	3	6	8	10	12	14	16	18	20	22	24

Table 5:

We use the blocks in Figure 8. For $r \neq 6, 7$, the blocks named B_r have b_r stones and the blocks A and B have 23 stones each. It follows that if $r \neq 6, 7$, the desired dominating sets are given by

$$\begin{cases} (AB)^s B_r & \text{if } q = 2s \\ B(AB)^s B_r & \text{if } q = 2s+1. \end{cases}$$

To handle the cases $r = 6$ and 7 we use the remaining blocks P and Q. In these cases we take

$$\begin{cases} P(AB)^s B_8 & \text{if } r = 6 \text{ and } q = 2s+1 \\ P(AB)^s B_9 & \text{if } r = 7 \text{ and } q = 2s+1 \\ QB(AB)^{s-1} B_8 & \text{if } r = 6 \text{ and } q = 2s \\ QB(AB)^{s-1} B_9 & \text{if } r = 7 \text{ and } q = 2s. \end{cases}$$

7 Dominating Sets for $10 \times n$ grid graphs, $n \geq 10$

We construct dominating sets S for $P_{10} \times P_n$, $n \geq 10$, of cardinality

$$\phi_{10}(n) = \begin{cases} \lfloor \frac{30n+37}{13} \rfloor, & \text{for } n \equiv 0 \text{ or } 3 \bmod 13,\ n \neq 13,\ 16 \\ \lfloor \frac{30n+24}{13} \rfloor, & \text{otherwise.} \end{cases}$$

If $n = 13q + r$, $1 \leq r \leq 13$, then except for $n = 13$ and 16 we have

$$\phi_{10}(n) = 30q + b_r,$$

where b_r is given by the Table 6.

We now refer to the blocks in Figures 9(a) and 9(b).

For the special cases $n = 13$ and 16 the blocks D_{13} and D_{16} in Figure 9(b) give, respectively, dominating sets with $\phi_{10}(13) = 31$ and $\phi_{10}(16) = 38$

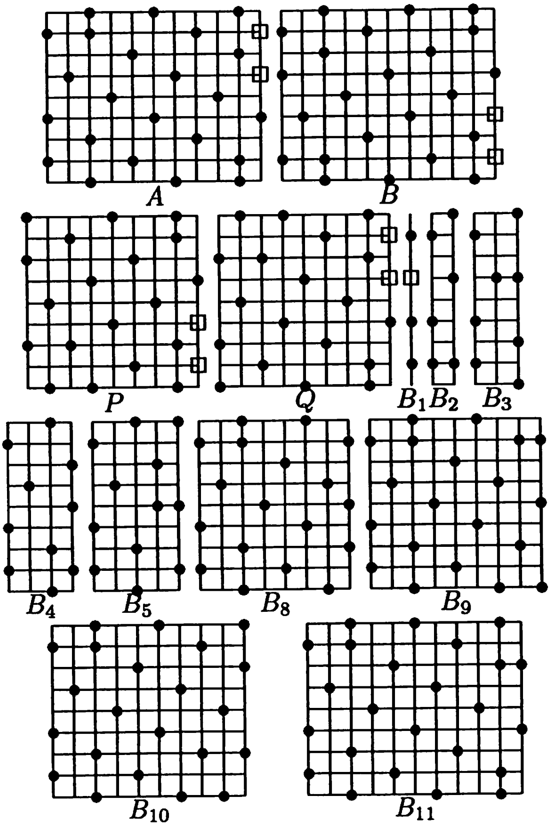


Figure 8: Blocks for constructing dominating sets for $P_9 \times P_n$.

stones. If $q = 0$, we have only the remaining cases $n = 10$, 11 and 12. In these cases one may obtain a dominating set by replacing the first two columns of B_r, $r = 10$, 11 and 12, by E_2.

We may now assume that $q \geq 1$ and if $q = 1$ then $r \neq 3$ since we have already handled the exceptional case $n = 16$. The blocks A, A', B, B' have 30 stones each and the blocks B_r, for $r \neq 4$, 6, have b_r stones each. So for

r	1	2	3	4	5	6	7	8	9	10	11	12	13
b_r	4	6	9	11	13	15	18	20	22	24	27	29	32

Table 6:

$r \neq 4$, 6 we obtain the required $30q + b_r$ stones by taking

$$\begin{cases} A'(BA)^s B_r & \text{if } q = 2s+1,\ s \geq 0 \\ B'A(BA)^{s-1} B_r & \text{if } q = 2s,\ s \geq 1. \end{cases}$$

For $r = 4$ note that D_2 has 5 stones which with the 6 stones in B_2 gives $11 = b_4$ stones. So we take

$$\begin{cases} F_2(BA)^s B_2 & \text{if } q = 2s,\ s \geq 1 \\ D_2 A(BA)^{s-1} B_2 & \text{if } q = 2s+1,\ s \geq 0. \end{cases}$$

For $r = 6$, we note that B_{19} has $45 = 30 + 15 = 30 + b_6$ stones. So if $q > 1$ we take

$$\begin{cases} A'(BA)^{s-1} B_{19} & \text{if } q = 2s,\ s \geq 1 \\ B'A(BA)^{s-1} B_{19} & \text{if } q = 2s+1,\ s \geq 1. \end{cases}$$

If $q = 1$, then $n = 19$. In this case we replace the first two columns of B_{19} by E_2. This produces a block with a dominating set consisting of requisite $30 + b_6 = 45$ stones.

8 Remarks

Hare has also obtained the following results by computer for the $11 \times n$ and $12 \times n$ grid graph domination numbers.

$$\gamma_{11,n} = \begin{cases} 29 & \text{for } n = 11 \\ \lfloor \frac{38n+33}{15} \rfloor & \text{for } 12 \leq n \leq 33 \\ \lfloor \frac{38n+36}{15} \rfloor & \text{for } 34 \leq n \leq 122 \end{cases}$$

$$\gamma_{12,n} = \lfloor \frac{36n + 26}{13} \rfloor \text{for } 6 \leq n \leq 33$$

We plan to deal with $k \times n$ grid graphs, $k \geq 11$, in a forthcoming paper.

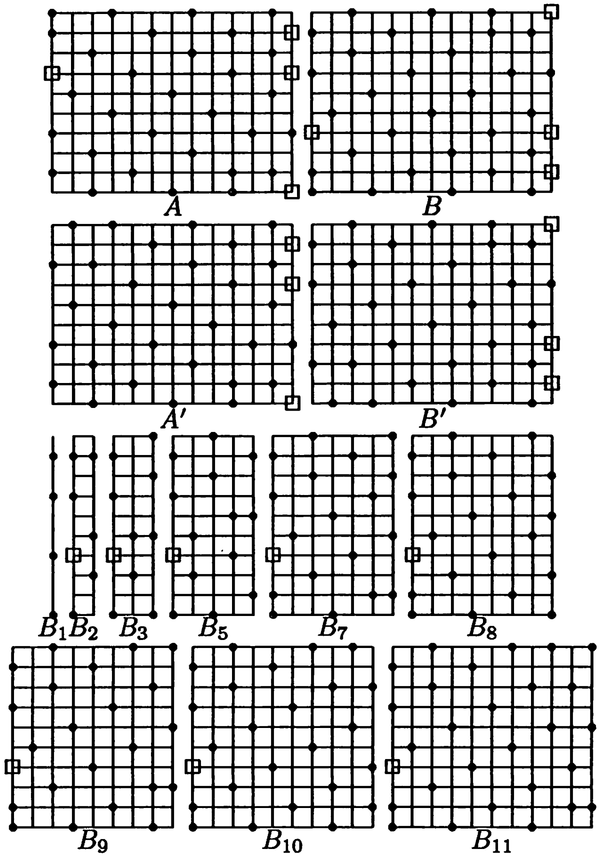


Figure 9: (a) Blocks for constructing dominating sets for $P_{10} \times P_n$.

References

[1] T. Y. Chang and W. E. Clark, The domination numbers of the $5 \times n$ and the $6 \times n$ grid graphs, Journal of Graph Theory, vol. 17, no. 1, (1993), 81-107.

[2] B. N. Clark, C. J. Colbourn and D. S. Johnson, Unit disk graphs, Discrete Math., 86,1990,165-177.

[3] E. J. Cockayne, E. O. Hare, S. T. Hedetniemi, T. V. Wimer, Bounds for the domination number of grid graphs, Congressus Numerantium, 47, 1985, 217-228.

[4] E. O. Hare, Algorithms for grid and grid-like graphs, Ph. D. Thesis, Dept. Computer Sci., Clemson University, 1989.

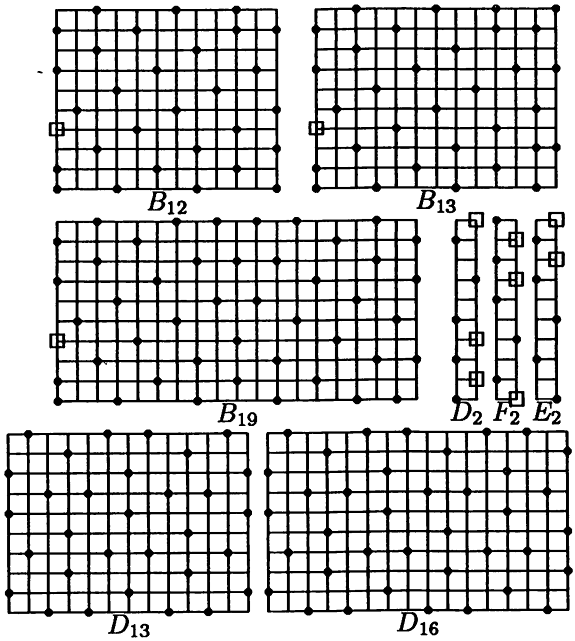


Figure 9: (b) Blocks for constructing dominating sets for $P_{10} \times P_n$.

[5] E. O. Hare, S. T. Hedetniemi, W. R. Hare, Algorithms for computing the domination number of $k \times n$ complete grid graphs, Congressus Numerantium, 55, 1986, 81-92.

[6] S. T. Hedetniemi and R. C. Laskar. Topics on domination, Discrete Math, vol 86, number 1-3, Dec. 14, 1990.

[7] M. S. Jacobson and L. F. Kinch. On the domination number of products of graphs:I, Ars Combinatoria, vol 18, 1983, 33-44.