Maximal k-Multiple-Free Sets of Integers

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Abstract. Let k be an integer greater than one. A set S of integers is called k-multiple-free (or k-MF for short) if $x \in S$ implies $kx \notin S$. Let $f_k(n) = \max\{|A| : A \subseteq [1,n]$ is k-MF$\}$. A subset A of $[1,n]$ with $|A| = f_k(n)$ is called a maximal k-MF subset of $[1,n]$. In this paper, we give a recurrence relation and a formula for $f_k(n)$. In addition, we give a method for constructing a maximal k-MF subset of $[1,n]$.

A set S of integers is called double-free if $x \in S$ implies $2x \notin S$. Let $[a,b]$ be the set of integers between a and b, both inclusive, and let

$$f(n) = \max\{|A| : A \subseteq [1,n] \text{ is double-free}\}.$$

E.T.H. Wang [3] gave a recurrence relation and a formula for $f(n)$. The former is

$$f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lfloor \frac{n}{4} \right\rfloor\right) \text{ and } f(1) = 1, \tag{1}$$

where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are the ceiling and floor functions, respectively. The latter is

$$f(n) = \frac{2n}{3} + \frac{1}{3}\sum_{i \geq 0} b_i - d, \tag{2}$$

where b_i are the coefficients in the base 4 expansion of n:

$$n = \sum_{i \geq 0} b_i 4^i, \quad 0 \leq b_i \leq 3, \tag{3}$$

and

$$d = \text{ the number of } b_i \text{ that are equal to 2 or 3.} \tag{4}$$

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As pointed out by Wang [3], a double-free set is related to a sum-free set [2].

Naturally, the concept of a double-free set can be generalized to a k-multiple-free set. Let k be an integer greater than one. A set of integers is called k-multiple-free (or k-MF for short) if $x \in S$ implies $kx \notin S$. Let

$$f_k(n) = \max\{|A| : A \subseteq [1, n] \text{ is } k\text{-multiple-free}\}.$$

A subset A of $[1, n]$ with $|A| = f_k(n)$ is called a maximal k-MF subset of $[1, n]$. In this paper, we give a recurrence relation and a formula for $f_k(n)$ as well as a method for constructing a maximal k-MF subset of $[1, n]$. As it turns out, we were not aware of the work of Lai [1] who has also dealt with the same topic. But the approach and results here are different from the ones in [1] and [3].

We first prove the following lemma.

Lemma 1. *There exists a maximal k-MF subset of $[1, n]$, A, such that*

$$\left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right] \subseteq A. \tag{5}$$

Proof: Let B be a maximal k-MF subset of $[1, n]$. If $\left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right] \subseteq B$, then we are done. Thus, we may assume that there is an $x \in \left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right]$, but $x \notin B$. Observe that $kx > n$ and, hence, $kx \notin B$. There are two cases to consider.

Case 1: $x \not\equiv 0 \pmod{k}$.

In this case, $x \neq ka$ for any $a \in B$. Thus, $\hat{B} = B \cup \{x\}$ is a k-MF set. But $|\hat{B}| = |B| + 1 > |B|$, a contradiction to the maximality of B.

Case 2: $x \equiv 0 \pmod{k}$.

In this case, we have $x = k^c b$ for some $c \geq 1$ and $b \not\equiv 0 \pmod{k}$. Consider the set $\hat{B} = B - \{k^{c-1} b\} \cup \{x\}$. $\hat{B}$ is a k-MF set and $|\hat{B}| \geq B$. Thus, $\hat{B}$ is a maximal k-MF subset of $[1, n]$ with $x \in \hat{B}$.

Repeating the above procedure, we will eventually obtain a maximal k-MF subset of $[1, n]$ with $\left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right]$ being its subset. ∎

Based on the above lemma, we can give a recurrence relation for $f_k(n)$, as shown in the next theorem.

Theorem 1. *For any positive integers n and $k > 1$, we have*

$$f_k(n) = n - \left\lfloor \frac{n}{k} \right\rfloor + f_k\left(\left\lfloor \frac{n}{k^2} \right\rfloor\right) \text{ and } f_k(1) = 1. \tag{6}$$

Proof: Let A be a maximal k-MF subset of $[1, n]$. By Lemma 1, we may assume that $\left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right] \subseteq A$. Since

$$k\left(\left\lfloor \frac{n}{k^2} \right\rfloor + 1\right) \geq \left\lfloor \frac{n}{k} \right\rfloor + 1 \quad \text{and} \quad k\left\lfloor \frac{n}{k} \right\rfloor \leq n,$$

we know that $\left[\left\lfloor \frac{n}{k^2} \right\rfloor + 1, \left\lfloor \frac{n}{k} \right\rfloor\right] \cap A = \emptyset$, and, hence, $A - \left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right]$ is a maximal k-MF subset of $\left[1, \left\lfloor \frac{n}{k^2} \right\rfloor\right]$. Therefore, we have

$$|A| = \left|\left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right]\right| + f_k\left(\left\lfloor \frac{n}{k^2} \right\rfloor\right) = n - \left\lfloor \frac{n}{k} \right\rfloor + f_k\left(\left\lfloor \frac{n}{k^2} \right\rfloor\right).$$

∎

The next theorem gives a method for constructing a maximal k-MF subset of $[1, n]$. For the purpose of stating the next theorem, we introduce the following notation: Let

$$\hat{A} = \bigcup_{i \geq 0} \left[\left\lfloor \frac{n}{k^{2i+1}} \right\rfloor + 1, \left\lfloor \frac{n}{k^{2i}} \right\rfloor\right]. \tag{7}$$

Note that $[a, 0]$ is the empty set when $a > 0$, and that the union in (7) is finite.

Theorem 2. *For any positive integers n and $k > 1$, $\hat{A}$, as defined in (7), is a maximal k-MF subset of* $[1, n]$.

Proof: The proof is by induction on n. By definition, the maximal k-MF subset of $[1, 1]$ is $\{1\}$ which can also be expressed as $\left[\left\lfloor \frac{1}{k} \right\rfloor + 1, \left\lfloor \frac{1}{1} \right\rfloor\right]$. Thus, the basis of the induction holds. Assuming that the induction hypothesis holds for all positive integers less than n, where $n > 1$, we wish to prove that it also holds for n.

Let A be a maximal k-MF subset of $[1, n]$. By Lemma 1, we may assume that A satisfies (5). Moreover, from the proof of Theorem 1, we may assume that

$$A = \left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right] \cup A_1, \tag{8}$$

where A_1 is a maximal k-MF subset of $\left[1, \left\lfloor \frac{n}{k^2} \right\rfloor\right]$. By the induction hypothesis, we may assume that

$$A_1 = \bigcup_{i \geq 0} \left[\left\lfloor \frac{\left\lfloor \frac{n}{k^2} \right\rfloor}{k^{2i+1}} \right\rfloor + 1, \left\lfloor \frac{\left\lfloor \frac{n}{k^2} \right\rfloor}{k^{2i}} \right\rfloor\right].$$

Since $\left\lfloor \frac{\left\lfloor \frac{n}{k^i} \right\rfloor}{k^j} \right\rfloor = \left\lfloor \frac{n}{k^{j+i}} \right\rfloor$, we have

$$A_1 = \bigcup_{i \geq 0} \left[\left\lfloor \frac{n}{k^{2i+3}} \right\rfloor + 1, \left\lfloor \frac{n}{k^{2i+2}} \right\rfloor\right]. \tag{9}$$

Combining (8) and (9), we obtain

$$\begin{aligned} A &= \left[\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right] \cup \left\{\bigcup_{i \geq 0} \left[\left\lfloor \frac{n}{k^{2i+3}} \right\rfloor + 1, \left\lfloor \frac{n}{k^{2i+2}} \right\rfloor\right]\right\} \\ &= \bigcup_{i \geq 0} \left[\left\lfloor \frac{n}{k^{2i+1}} \right\rfloor + 1, \left\lfloor \frac{n}{k^{2i}} \right\rfloor\right] = \hat{A}. \end{aligned}$$

∎

Theorem 2 gives a construction for a maximal k-MF subset of $[1, n]$. The following theorem gives the cardinality of such a subset. Let the base k expansion of n be

$$n = a_m k^m + a_{m-1} k^{m-1} + \ldots + a_1 k + a_0, \tag{10}$$

where $0 \le a_i < k$ for each $0 \le i \le m-1$, and $0 < a_m < k$.

Theorem 3. *For any positive integer n with (10) as its base k expansion, we have*

$$f_k(n) = \frac{1}{k+1}\left(kn + \sum_{i \ge 0}(-1)^i a_i\right). \tag{11}$$

Proof: By Theorem 2, we have

$$\begin{aligned} f_k(n) = |\hat{A}| &= \sum_{i \ge 0}\left(\left\lfloor \frac{n}{k^{2i}} \right\rfloor - \left\lfloor \frac{n}{k^{2i+1}} \right\rfloor\right) \\ &= \sum_{i \ge 0}(-1)^i \sum_{j \ge i} a_j k^{j-i} \\ &= \sum_{j \ge 0} a_j k^j \sum_{i \le j}\left(\frac{-1}{k}\right)^i \\ &= \sum_{j \ge 0} a_j \frac{k^{j+1} + (-1)^j}{k+1} \\ &= \frac{1}{k+1}\left(kn + \sum_{i \ge 0}(-1)^i a_i\right). \end{aligned}$$

∎

Corollary. *When $k = 2$, we have*

$$f_2(n) = \frac{2}{3}n + \frac{1}{3}\sum_{i \ge 0}(-1)^i a_i. \tag{12}$$

It is not difficult to see that Wang's formula (2) can be simplified to (12). The following theorem, which is an immediate consequence of (11), shows the validity of a conjecture proposed by W. Janous in a letter of June 29, 1988, to E. Wang.

Theorem 4.

$$f_k(n) = \frac{k}{k+1}n + O(\log n), \quad n \to \infty.$$

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