Domination and Independent Domination Numbers of Graphs

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Abstract. Let X be a graph and let $\alpha(X)$ and $\alpha'(X)$ denote the domination number and independent domination number of X, respectively. We show that for every triple (m, k, n), $m \geq 5, 2 \leq k \leq m, n > 1$, there exist m-regular k-connected graphs X with $\alpha'(X) - \alpha(X) > n$. The same also holds for $m = 4$ and $k \in \{2, 4\}$.

1.Terminology and Introduction

By $X(V, E)$ we denote a graph with vertex-set $V(X)$ and edge-set $E(X)$. Graphs considered in this paper are undirected and contain neither loops nor multiple edges. We say that two paths are *disjoint* if they have at most their end-vertices in common. A subset $D \subset V(X)$ is a *dominating set* if every vertex $w \in V(X) \backslash D$ is adjacent to at least one vertex $v \in D$. The *domination number*, $\alpha(X)$, is the smallest number of vertices in a dominating set of X. If a dominating set I is independent we call I an *independent dominating set*. The *independent domination number*, $\alpha'(X)$, is the smallest number of vertices in an independent dominating set of X. Obviously $\alpha' \geq \alpha$ holds for all graphs. By $H_{k,m}$, $m \geq 3$, $1 \leq k \leq m$, we denote the class of k-connected, m-regular graphs. We say that $\alpha' - \alpha$ is *unbounded for* $H_{k,m}$ if for every integer $n \geq 1$ there is a graph $X \in H_{k,m}$ such that $\alpha'(X) - \alpha(X) > n$ holds.

It is well known that $\alpha' - \alpha$ is unbounded for $H_{1,3}$ and $H_{2,3}$ (see e.g. [1]). In [1] it was also conjectured that $\alpha' - \alpha$ is bounded for $H_{3,3}$ which was shown to be wrong in [4]. In this paper we prove that $\alpha' - \alpha$ is unbounded for all $H_{k,m}$, $m \geq 5$, $2 \leq k \leq m$, for $H_{2,4}$ and for $H_{4,4}$.

The method we use involves the concept of covering graphs which we explain in the sequel:

Let X_1 and X_2 be graphs and let f denote a homomorphism from X_1 onto X_2. By $S(v)$, $v \in V(X_1)$, we denote the star consisting of v and all edges incident to v. If $f(S(v))$ is isomorphic to $S(v)$ for all $v \in V(X_1)$, we call f a *covering map* and X_1 a *covering graph* of X_2.

In [2], p. 127, the following construction of a covering graph of a graph X with respect to a group G is given: Each edge $[u, v] \in E(X)$ gives rise to two 1-arcs, $[u, v]$ and $[v, u]$. By $A(X)$ we denote the set of 1-arcs and by $\varphi : A(X) \to G$ we denote a mapping such that $\varphi([u, v]) = (\varphi([v, u]))^{-1}$ for all $[u, v] \in A(X)$. The *covering graph* $\tilde{X} = \tilde{X}(G, \varphi)$ *of* X *with respect to* G is defined on the vertex-set $V(\tilde{X}) = G \times V(X)$ and two vertices $(g_1, u), (g_2, v) \in V(\tilde{X})$ are adjacent in $\tilde{X}$ if and only if $[u, v] \in A(X)$ and $g_2 = g_1 \varphi([u, v])$.

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2.The result

Besides covering graphs Menger's Theorem (see e. g. [3]) is crucial for the proof of our result:

Theorem 2.1. *The minimum number of vertices separating two nonadjacent vertices x and y is the maximum number of disjoint paths connecting x and y.*

Theorem 2.2. *The difference $\alpha' - \alpha$ is unbounded for every $H_{k,m}$, $m \geq 5$, $2 \leq k \leq m$, and for $H_{2,4}$ and $H_{4,4}$.*

Proof: Let $X = K_{m,m}$ for some $m \geq 4$ and let $V(X) = \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_m\}$ and $E(X) = \{[u_j, v_l] \mid 1 \leq j \leq m, 1 \leq l \leq m\}$. We now construct covering graphs of X which are contained in $H_{k,m}, 2 \leq k \leq m$, respectively, such that the difference between their independent domination numbers are domination numbers is arbitrarily large (with the exception of $H_{3,4}$). Thereby we distinguish two cases:

Case 1: k is even.
In this case we construct a covering graph $\tilde{X}$ of X with respect to the cyclic group $\mathbb{Z}_q = \langle a \mid a^q = e \rangle, q > 2$, where φ is given by

$$\varphi([u,v]) = \varphi([v,u]) = e \text{ if } [u,v] \in \{[u_r, v_s] \mid r \leq m - \frac{k}{2} \text{ or } s \leq m - \frac{k}{2}\}$$

and

$$\varphi([u,v]) = a, \varphi([v,u]) = a^{-1} \text{ if } [u,v] \in \{[u_r, v_s] \mid r > m - \frac{k}{2} \text{ and } s > m - \frac{k}{2}\}.$$

If Y, $V(Y) = V(X)$, now denotes that subgraph of X which contains all those edges of X which are mapped onto e by φ, then $\tilde{X}$ consists of q pairwise disjoint copies of Y which are connected by edges corresponding to a and a^{-1}. Clearly $\tilde{X}$ is m-regular. We now show that $\tilde{X}$ is also k-connected.

To do that we first mention that the $a^l, 0 \leq l \leq q-1$, act as automorphism on $\tilde{X}$ as follows:

$$a^l(h, x) = (a^l h, x)$$

for all $(h, x) \in V(\tilde{X})$. This implies that $\mathbb{Z}_q$ acts transitively on the set of copies of Y in $\tilde{X}$. Hence we denote the copies of Y in $\tilde{X}$ by Y_l, $0 \leq l \leq q-1$, where $Y_l = a^l(Y_0)$.

Let $p = \frac{k}{2}$. The graph $\tilde{X}$ contains p disjoint cycles K_j, $m - p + 1 \leq j \leq m$, which are given by

$$K_j = [(e, u_j), (a, v_j)] \cup P_j^1 \cup [(a, u_j), (a^2, v_j)] \cup \cdots \cup P_j^{q-1} \\ \cup [(a^{q-1}, u_j), (e, v_j)] \cup P_j^0$$

where

$$P_j^l = [(a^l, v_j), (a^l, u_{j-m+p}), (a^l, v_{j-m+p}), (a^l, u_j)]$$

for $0 \le l \le q-1$.

This immediately implies that between any pair $Y_r, Y_s, r, s \in \{0, ..., q-1\}$, there are k disjoint paths with endvertices $(a^r, u_{m-p+1}), \ldots, (a^r, u_m)$, $(a^r, v_{m-p+1}), \ldots, (a^r, v_m)$ and $(a^s, u_{m-p+1}), \ldots, (a^s, u_m)$, $(a^s, v_{m-p+1}), \ldots,$ (a^s, v_m), respectively.

In the sequel we show that no pair of nonadjacent vertices (a^l, x), $(a^l, y) \in V(Y_l)$, $0 \le l \le q-1$, can be separated by less than k vertices. Because of Theorem 2.1 and the above defined action of the a^l on $\widetilde{X}$ it is sufficient to show that every pair of nonadjacent vertices $(e, x), (e, y) \in V(Y_0)$ is connected by at least k disjoint paths. In fact it turns out that those vertices are always connected by m disjoint paths, not depending on k.

By $K_j[(e, u_j), (a^r, x)] \subset K_j, m-p+1 \le j \le m$, we denote that subpath of K_j which connects (e, u_j) and (a^r, x), $r \in \{0, 1, \ldots, q-1\}$, and contains (a, v_j) as its second vertex. $K_j[(e, v_j), (a^r, x)] \subset K_j$ denotes that subpath of K_j which connects (e, v_j) and (a^r, x) but contains (a^{q-1}, u_j) as its second vertex.

Case 1.1: $x, y \in \{u_1, \ldots, u_{m-p}\}$ or $x, y \in \{v_1, \ldots, v_{m-p}\}$.
Without loss of generality we can assume that $x = u_1$ and $y = u_2$. The disjoint paths B_i, $1 \le i \le m$, are given by

$$B_i = [(e, u_1), (e, v_i), (e, u_2)]$$

Case 1.2: $x \in \{u_1, \ldots, u_{m-p}\}$, $y \in \{u_{m-p+1}, \ldots, u_m\}$ or $x \in \{v_1, \ldots, v_{m-p}\}$, $y \in \{v_{m-p+1}, \ldots, v_m\}$.
Without loss of generality we assume that $x = u_1$ and $y = u_m$. The m disjoint are now given by

$$B_i = [(e, u_1), (e, v_i)(e, u_m)] \text{ for } 1 \le i \le m-p$$

and by

$$B_i = [(e, u_1), (e, v_i)] \cup K_i[(e, v_i), (a, v_i)] \cup [(a, v_i), (e, u_m)]$$

for $m-p+1 \le i \le m$.

Case 1.3: $x \in \{u_{m-p+1}, \ldots, u_m\}$, $y \in \{v_{m-p+1}, \ldots, v_m\}$.
Without loss of generality we assume that $x = u_m$ and $y = v_m$. The m disjoint paths are now given by

$$B_i = [(e, u_m), (e, v_i), (e, u_i), (e, v_m)] \text{ if } 1 \le i \le m-p$$

and by

$$B_i = K_i[(e,u_i),(a^{q-1},u_i)] \cup [(a^{q-1},u_i),(e,v_m)]$$

for $m-p+1 \le i \le m$.

Case 1.4: $x, y \in \{u_{m-p+1},\dots,u_m\}$ or $x, y \in \{v_{m-p+1},\dots,v_m\}$.
Without loss of generality we assume that $x = u_{m-p+1}$ and $y = u_m$. The m disjoint paths are given by

$$B_i = [(e,u_{m-p+1}),(e,v_i),(e,u_m)]$$

for $1 \le i \le m-p$ and by

$$B_i = [(e,u_{m-p+1}),(a,v_i),(e,u_m)]$$

for $m-p+1 \le i \le m$.

We finally show that no pair of vertices $(e,x) \in V(Y_0)$ and $(a^r,y) \in V(Y_r)$, $r \in \{1,\dots,q-1\}$ can be separated by less than k vertices. Let $S \subset V(\tilde{X})$ be a set of at most $k-1$ vertices. Then we know from the above that there are vertices $(e,w) \in V(Y_0)$ and $(a^r,v) \in V(Y_r)$ which are connected by a path in $\tilde{X}\backslash S$, since at least one of the k paths contained in the cycles K_j, $m-p+1 \le j \le m$, still exists in $\tilde{X}\backslash S$. Cases 1.1–1.4, together with Theorem 2.1, show that (e,x) as (a^r,y) and (a^r,v), are also connected in $\tilde{X}\backslash S$. Hence (e,x) and (a^r,y) are connected in $\tilde{X}\backslash S$.

From the action of the elements of $\mathbb{Z}_q$ on $\tilde{X}$ it now immediately follows no pair of vertices of $\tilde{X}$ can be separated by less than k vertices.

Let $D = \{(a^l,u_1),(a^l,v_1) \mid 0 \le l \le q-1\}$. Clearly D is a dominating set of $\tilde{X}$. Hence $\alpha(\tilde{X}) \le 2q$. Each Y_l also contains two subsets, namely $V_l^u = \{(a^l,u_1),\dots,(a^l,u_{m-p})\}$ and $V_l^v = \{(a^l,v_1),\dots,(a^l,v_{m-p})\}$, whose vertices are only adjacent to vertices of Y_l. Since $m-p \ge 2$ each dominating set of $\tilde{X}$ must contain at least two vertices of every Y_l. So $\alpha(\tilde{X}) = 2q$.

To show that $\alpha'(\tilde{X}) - \alpha(\tilde{X})$ grows if we enlarge q, we first consider the case $m \ge 5$. We also mention that superscripts of a and subscripts of Y are taken modulo q in the sequel.

Since k is even, $m-p \ge 3$ holds if $m \ge 5$. If an independent dominating set I now contains a vertex of V_l^u or V_l^v, then it clearly contains all vertices of V_l^u or V_l^v, respectively. Hence, if I contains only two vertices of Y_r, $r \in \{0,\dots,q-1\}$, then it must contain a vertex (a^r,u_i) and a vertex (a^r,v_i), $i \in \{m-p+1,\dots,m\}$. Suppose $(a^r,v_m) \in I$. Then none of the vertices (a^{r-1},u_i) can be contained in I. Hence the vertices of V_{r-1}^v are either contained in I or they are dominated by a vertex of V_{r-1}^u. In both cases $|\, I \cap V(Y_{r-1}) \,| \ge 3$ holds. So $|\, (V(Y_{r-1}) \cup V(Y_r)) \cap I \,| \ge 5$ holds for every $r \in \{0,\dots,q-1\}$ and every independent dominating set I. This immediately implies that $\alpha'(\tilde{X}) - \alpha(\tilde{X})$ grows if we enlarge q.

Let $m = 4$. If $k = 2$ the same arguments as above can be used, since $m - p = m - 1 = 3$ in this case.

Let $m = k = 4$. Again we consider a $Y_r, r \in \{0, \dots, q-1\}$. Assume that I now contains at least one of the vertices (a^r, u_3), (a^r, u_4), (a^r, v_3) or (a^r, v_4), say (a^r, v_4). Since (a^r, v_4) and (a^r, v_3) are adjacent to the same vertices, the set I then contains both of these vertices. But then (a^r, v_1) and (a^r, v_2) are still not dominated by I. So I also contains the vertices (a^r, v_1) and (a^r, v_2) or (a^r, u_3) and (a^r, u_4), respectively.

Hence $|I \cap V(Y_r)| = 2$ can only hold if (a^r, u_1) and (a^r, u_2) or (a^r, v_1) and (a^r, v_2) are contained in I, respectively. Assume that $(a^r, u_1), (a^r, u_2) \in I$. If now $|I \cap V(Y_r)| = 2$ holds, then (a^r, u_3) and (a^r, u_4) must be dominated by (a^{r+1}, v_3) and (a^{r+1}, v_4). But as we have seen above, this implies that $|I \cap V(Y_{r+1})| = 4$ holds. Hence $\alpha'(\tilde{X}) - \alpha(\tilde{X})$ again grows if we enlarge q.

Case 2: k is odd.

In this case the proof is a little bit more involved since we have to construct our covering graphs in a different way to obtain k-connected graphs. Here we construct $\tilde{X}$ with respect to the group $G = \langle a_1 \rangle \times \langle a_2 \rangle$ where $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are cyclic groups of order $q > m$ with generators a_1 and a_2, respectively. Also $p = \frac{k-1}{2}$. The mapping φ is now given by

$$\varphi([u,v]) = \varphi([v,u]) = e \text{ for all } [u,v] \in \{[u_r, v_s] \mid r \leq m-p-1 \text{ or } s \leq m-p\}$$

$$\varphi([u,v]) = a_1, \varphi([v,u]) = a_1^{-1} \text{ if } [u,v] \in \{[u_r, v_s] \mid r > m-p \text{ and } s > m-p\}$$

and

$$\varphi([u,v]) = a_2, \varphi([v,u]) = a_2^{-1} \text{ if } [u,v] \in \{[u_r, v_s] \mid r = m-p \text{ and } s > m-p\}.$$

Let Y, $V(Y) = V(X)$, now again denote the graph which contains all those edges of X which are mapped onto e by φ. In this case $\tilde{X}$ consists of q^2 disjoint copies of Y and all $g \in G$ act as automorphisms on $\tilde{X}$ where

$$g(h,x) = (gh,x)$$

for all vertices $(h,x) \in V(\tilde{X})$. This again implies that G acts transitively on the copies of Y in $\tilde{X}$. Hence we denote those copies by Y_g, $g \in G$, and first again show that no two subgraphs $Y_{g_1}, Y_{g_2}, g_1, g_2 \in G$ can be separated by less than k vertices. Because of the action of the $g \in G$ on $\tilde{X}$ it is again sufficient to show that Y_e can not be separated from any Y_h, $h \in G$, by less than k vertices. Again we have p disjoint cycles K_j, $m-p+1 \leq j \leq m$, defined as in Case 1 with a_1 instead of a. In addition we have a cycle K_{m-p},

$$K_{m-p} = [(e, u_{m-p}), (a_2, v_m)] \cup P_1 \cup [(a_2, u_{m-p}), (a_2^2, v_m)] \cup \ldots \\ \cup P_{q-1} \cup [(a_2^{q-1}, u_{m-p}), (e, v_m)] \cup P_0$$

where the $P_l, 0 \leq l \leq q-1$, denote the paths

$$P_l = [(a_2^l, v_m), (a_2^l, u_1), (a_2^l, v_1), (a_2^l, u_{m-p})]$$

respectively.

Let $W_l, 0 \leq l \leq q$, denote the subgraphs

$$W_l = \bigcup_{z=0}^{q-1} Y_{a_2^l a_l^z}.$$

Since we have chosen $q > m$ no pair of subgraphs W_r, W_s, $r, s \in \{0, \ldots, q-1\}$ can be separated by less than k vertices, as they are connected by the disjoint cycles $a_1(K_{m-p})$.

Let $S \subset V(X)$ now denote an arbitrary set which contains $k-1$ vertices. We first show that no Y_g, $g \in \{a_1, \ldots, a_1^{q-1}\}$ can be separated from Y_e by S. If the vertices of S are not all contained in the cycles K_j, $m-p+1 \leq j \leq m$, then clearly $\tilde{X} \backslash S$ contains a subpath of at least one of the K_j which connects vertices of Y_e and Y_g. If all vertices of S are contained in the K_j then it can happen that all connections between Y_e and Y_g inside W_0 are interrupted in $\tilde{X} \backslash S$. But this implies that S does not contain any of the vertices $(b, u_{m-p}), b \in \{e, a_1, \ldots, a_1^{q-1}\}$. Hence a path from (e, u_{m-p}) via W_1 to $(g, u_{m-p}) \in Y_g$ still exists in $\tilde{X} \backslash S$.

Using essentially the same arguments it is now obvious that Y_e is connected to each Y_h, $h \in G$, in $\tilde{X} \backslash S$. Because of the action of the elements of G on $\tilde{X}$ it is then clear that no pair of different copies of Y in $\tilde{X}$ can be separated by less than k vertices. On the other hand it is also clear that e.g. $\tilde{X} \backslash \{(e, u_{m-p}), \ldots, (e, u_m), (e, v_{m-p+1}, \ldots, (e, v_m)\}$ is disconnected, which means that the graphs $\tilde{X}$ are at most k-connected.

Hence it remains to show that also each pair of nonadjacent vertices (g, x), $(g, y) \in V(\tilde{X})$ is connected by at least k disjoint paths. As in Case 1 we can restrict this part of the proof to nonadjacent vertices $(e, x), (e, y) \in V(Y_e)$ and it again turns out that these vertices are always connected by m disjoint paths, not depending on k.

By $K_j^l, 0 \leq l \leq q-1$, we denote the cycles $a_2^l(K_j)$, $m-p+1 \leq j \leq m$. Now $K_j^l[(a_2^l a_1^r, u_j), (a_2^l a_1^s, x)]$ denotes that subpath of K_j^l which connects $(a_2^l a_1^r, u_j)$

and $(a_2^l a_1^s, x)$, $r, s \in \{0, 1, \ldots, q-1\}$ and contains $(a_2^l a_1^{r+1}, v_j)$ as its second vertex. $K_j^l[(a_2^l a_1^r, v_j), (a_2^l a_1, x)]$ is that subpath of K_j^l, connecting $(a_2^l a_1^r, v_j)$ and $(a_2^l a_1^s, x)$, which contains $(a_2^l a_1^{r-1}, u_j)$ as its second vertex. By K_{m-p}^l we denote the cycles $a_1^l(K_{m-p})$, $0 \leq l \leq q-1$, and the subpaths we use in the sequel are defined analogously to the above.

Case 2.1: $x, y \in \{u_1, \ldots, u_{m-p-1}\}$ or $x, y \in \{v_1, \ldots, v_{m-p}\}$.
We obtain the m disjoint paths as in Case 1.1.

Case 2.2: $x = u_{m-p}, y \in \{u_1, \ldots, u_{m-p-1}\}$
Without loss of generality we assume that $y = u_1$. Then

$$B_i = [(e, u_{m-p}), (e, v_i), (e, u_1)]$$

for $1 \leq i \leq m-p$ and

$$\begin{aligned} B_i = [(e, u_{m-p}), (a_2, v_i)] \cup K_i^1[(a_2, v_i), (a_2 a_1^i, v_i)] \\ \cup [(a_2 a_1^i, v_i), (a_1^i, u_{m-p}), (a_1^i, v_{m-p}), (a_1^i, u_i)] \\ \cup K_i[(a_1^i, u_i), (e, v_i)] \cup [(e, v_i), (e, u_1)]. \end{aligned}$$

for $m-p+1 \leq i \leq m$.

Case 2.3: $x \in \{u_1, \ldots, u_{m-p-1}\}, y \in \{u_{m-p+1}, \ldots, u_m\}$.
Without loss of generality we assume that $x = u_1$ and $y = u_m$. Then

$$B_i = [(e, u_1), (e, v_i), (e, u_m)]$$

for $1 \leq i \leq m-p$ and

$$B_i = [(e, u_1), (e, v_i)] \cup K_i^0[(e, v_i), (a_1, v_i)] \cup [(a_1, v_i), (e, u_m)]$$

for $m-p+1 \leq i \leq m$.

Case 2.4: $x = u_{m-p}, y \in \{u_{m-p+1}, \ldots, u_m\}$.
We set $y = u_m$. Then

$$B_i = [(e, u_{m-p}), (e, v_i), (e, u_m)]$$

for $1 \leq i \leq m-p$ and

$$\begin{aligned} B_i = [(e, u_{m-p}), (a_2, v_i)] \cup K_i^1[(a_2, v_i), (a_2 a_1^i, v_i)] \\ \cup [(a_2 a_1^i, v_i), (a_1^i, u_{m-p}), \\ (a_1^i, v_{i-m+p}), (a_1^i, u_{i-m+p}), (a_1^i, v_i)] \\ \cup K_i^0[(a_1^i, v_i), (a_1, v_i)] \cup [(a_1, v_i), (e, u_m)] \end{aligned}$$

for $m-p+1 \leq i \leq m$.

Case 2.5: $x = u_{m-p}$, $y \in \{v_{m-p+1}, \ldots, v_m\}$.
Without loss of generality we set $y = v_m$. Then

$$B_i = [(e, u_{m-p}), (e, v_i), (e, u_i), (e, v_m)]$$

for $1 \leq i \leq m-p-1$,

$$\begin{aligned} B_{m-p} =& [(e, u_{m-p}), (e, v_{m-p}), (e, u_m)] \cup K_m^0[(e, u_m), (a_1^{q-1}, u_m)] \\ & \cup [(a_1^{q-1}, u_m), (e, v_m)], \\ B_i =& [(e, u_{m-p}), (a_2, v_i)] \cup K_i^1[(a_2, v_i), (a_2 a_1^i, v_i)] \\ & \cup [(a_2 a_1^i, v_i), (a_1^i, u_{m-p}), (a_1^i, v_{m-p}), (a_1^i, u_i)] \\ & \cup K_i^0[(a_1^i, u_i), (a_1^{q-1}, u_i)] \cup [(a_1^{q-1}, u_i), (e, v_m)] \end{aligned}$$

for $m-p+1 \leq i \leq m-1$ and

$$B_m = K_{m-p}^0[(e, u_{m-p}), (e, v_m)]$$

Case 2.6: $x, y \in \{u_{m-p+1}, \ldots, u_m\}$.
Let $x = u_{m-1}$ and $y = u_m$. Then

$$B_i = [(e, u_{m-1}), (e, v_i), (e, u_m)]$$

for $1 \leq i \leq m-p$ and

$$B_i = [(e, u_{m-1}), (a_1, v_i), (e, u_m)]$$

for $m-p+1 \leq i \leq m$.

Case 2.7: $x \in \{u_{m-p+1}, \ldots, u_m\}$, $y \in \{v_{m-p+1}, \ldots, v_m\}$.
We set $x = u_m$ and $y = v_m$. Then

$$B_i = [(e, u_m), (e, v_i), (e, u_i), (e, v_m)]$$

for $1 \leq i \leq m-p-1$,

$$B_{m-p} = [(e, u_m), (e, v_{m-p}), (e, u_{m-p}] \cup K_{m-p}^0[(e, u_{m-p}), (e, v_m)]$$

and

$$B_i = [(e, u_m), (a_1, v_i)] \cup K_i^0[(a_1, v_i), (a_1^{q-1}, u_i)] \cup [(a_1^{q-1}, u_i), (e, v_m)]$$

for $m-p+1 \leq i \leq m$.

Case 2.8: $x \in \{v_1, \dots, v_{m-p}\}, y \in \{v_{m-p+1}, \dots, v_m\}$.
We set $x = v_1$ and $y = v_m$. Then

$$B_i = [(e, v_1), (e, u_i), (e, v_m)]$$

for $1 \le i \le m - p - 1$,

$$B_{m-p} = [(e, v_1), (e, u_{m-p})] \cup K^0_{m-p}[(e, u_{m-p}), (e, v_m)]$$

and

$$B_i = [(e, v_1), (e, u_i)] \cup K^0_i[(e, u_i), (a_1^{q-1}, u_i)] \cup [(a_1^{q-1}, u_i), (e, v_m)]$$

for $m - p + 1 \le i \le m$.
Case 2.9: $x, y \in \{v_{m-p+1}, \dots, v_m\}$.
We set $x = v_{m-1}$ and $y = v_m$. Then

$$B_i = [(e, v_{m-1}), (e, u_i), (e, v_m)]$$

for $1 \le i \le m - p - 1$,

$$B_{m-p} = [(e, v_{m-1}), (a_2^{q-1}, u_{m-p}), (e, v_m)]$$

and

$$B_i = [(e, v_{m-1}), (a_1^{q-1}, u_i), (e, v_m)]$$

for $m - p + 1 \le i \le m$.

The proof that no pair of nonadjacent vertices of $\widetilde{X}$ can be separated by less than k vertices can now be done analogously to Case 1.

To show that the difference between the independent domination number and the domination number of $\widetilde{X}$ again grows with q, we first mention that $\widetilde{X}$ consists of q^2 copies of Y in this case. Analogously to Case 1 this implies that $\alpha(\widetilde{X}) = 2q^2$.

Let $m \ge 6$. Since k is odd it is clear that $m-p-1 \ge 3$ holds in this case. Hence each $Y_g, g \in G$ contains two disjoint subsets, namely $\{(g, u_1), \dots, (g, u_{m-p-1})\}$ and $\{(g, v_1), \dots, (g, v_{m-p})\}$, with cardinality at least three, which are only adjacent to vertices of Y_g. Hence, if an independent dominating set I of $\widetilde{X}$ contains one of those vertices, then it contains at least 3 vertices of Y_g. If I contains only 2 vertices of some Y_g then these vertices must therefore be contained in $\{(g, u_{m-p}), \dots, (g, u_m)\}$ and $\{(g, v_{m-p+1}), \dots, (g, v_m)\}$, respectively. But if I contains at least one vertex $(g, x) \in \{(g, u_{m-p}), \dots, (g, u_m)\}$ then it cannot contain the vertices $(h, v_{m-p+1}), \dots, (h, v_m)$, where $h = ga_1$ and/or $h = ga_2$. Hence

I contains at least 3 vertices of Y_h. As in Case 1 it is now clear that the difference between $\alpha'(\tilde{X})$ and $\alpha(\tilde{X})$ grows if we enlarge q.

Let $m = 5$. If $k = 3$ then we have the same situation as above. Let now $k = m = 5$. If I contains a vertex of $\{(g, v_1), (g, v_2), (g, v_3)\}$ then we again know that it contains more than 2 vertices of Y_g. If I contains one of the vertices $(g, v_4), (g, v_5)$ then it must contain both of them, since they are adjacent to the same vertices of $\tilde{X}$. But if I contains these two vertices, then the vertices $(g, v_1), (g, v_2)$ and (g, v_3) are still not dominated. Hence I must in addition contain at least one of the vertices $(g, u_3), \ldots, (g, u_5)$ or the vertices (g, v_1), $(g, v_2), (g, v_3)$ themselves. In both cases I again contains more than two vertices of Y_g. So, if I contains only 2 vertices of Y_g, then it can only contain vertices of $\{(g, u_1), \ldots, (g, u_m)\}$. Since the vertices (g, u_1) and (g, u_2) are not dominated in this case, these two vertices must be contained in I. If I now also contains one of the vertices $(g, u_3), (g, u_4), (g, u_5)$, then I again contains at least three vertices of Y_g. If it does not contain any of these vertices, then they must be adjacent to a vertex of I, which means that I contains the vertices (h, v_4) and (h, v_5) for $h = ga_1$ and $h = ga_2$. But from the above we know that then I contains at least three vertices of each Y_h. This again implies that the difference between $\alpha'(\tilde{X})$ and $\alpha(\tilde{X})$ grows if we enlarge q. ∎

We emphasize that also in the case $m = 4, k = 3$ the method of the above proof is sufficient to show that $\alpha'(\tilde{X}) > \alpha(\tilde{X})$ but not to show that $\alpha' - \alpha$ is unbounded for $H_{3,4}$. Clearly we also cannot obtain 1-connected m-regular graphs by constructing covering graphs as we did in this proof. Nevertheless we do not think that $H_{1,m}$ and $H_{3,4}$ are classes of graphs for which $\alpha' - \alpha$ is bounded.

Conjecture 2.3. *$\alpha' - \alpha$ is also unbounded for all $H_{1,m}$, $m \geq 4$, and for $H_{3,4}$.*

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