Decompositions of Directed Graphs with Loops and Related Algebras

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Abstract. A correspondence between decompositions of complete directed graphs with loops into collections of closed trails which partition the edge set of the graph and the variety of all column latin groupoids is established. Subvarieties which consist of column latin groupoids arising from decompositions where only certain trail lengths occur are examined. For all positive integers m, the set of values of n for which the complete directed graph with loops on a vertex set of cardinality n can be decomposed in this manner such that all the closed trails have length m is shown to be the set of all n for which m divides n^2

1. Introduction

In recent years a great deal of work has been done on decompositions of complete undirected graphs such that the edge set of the graph is partitioned. Under certain conditions it is possible to associate various types of algebras with these decompositions; for example see [4], [6], [7] or [8].

Necessary and sufficient conditions for the existence of an edge partitioning decomposition, into closed trails of length m, of the complete undirected graph have been given in [5]. Such decompositions with the added restriction that there are no repeated vertices in the closed trails are called m-cycle systems and have been studied extensively, see [7]. Analogous decompositions of directed graphs (usually referred to as Mendelsohn designs) have also attracted much attention, see [2].

In this paper we examine decompositions of complete directed graphs with loops into collections of closed trails which partition the edge set of the graph and the variety of algebras associated with them. In the case of graphs with loops we must decompose into closed trails (and not cycles) because repeated vertices cannot be avoided. This problem has been considered previously in [1] and [3] but in both these papers closed trails of length three only are considered.

We write closed trails as cyclically ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of vertices and refer to these m-tuples as *m-circuits*. An m-circuit $(x_1, x_2, \ldots, x_m)$ equivalently consists of the cyclically ordered m-tuple $(x_1x_2, x_2x_3, \ldots, x_mx_1)$ of edges. Of course, the m-circuit, $(x_1, x_2, \ldots, x_m)$ is equal to any cyclic permutation of itself.

Definition 1.1. *An L-circuit directed looped system (or L-circuit DLS) of order n is a pair (V, C) where V $(|V| = n)$ is the vertex set of the complete directed graph with loops D_n and C is a collection of circuits whose lengths make up the set L, with the property that each edge of D_n occurs exactly once in C.*

Example 1.2:

(1) If $V = \{1,2\}$ and $C = \{(1),(2),(1,2)\}$ then (V, C) is a $\{1,2\}$-circuit DLS of order 2.
(2) If $V = \{1,2,3\}$ and $C = \{(1,1,2),(2,2,3),(3,3,1)\}$ then (V, C) is a $\{3\}$-circuit DLS of order 3.
(3) If $V = \{1,2,3\}$ and $C = \{(1,1,2,2,3,3,1,3,2)\}$ then (V, C) is a $\{9\}$-circuit DLS of order 3.

We note at this point that infinite order L-circuit DLS's are well defined and indeed an infinite L-circuit DLS may contain circuits of infinite length, in which case $\infty \in L$.

2. The Algebra of a DLS

In this section we will examine the correspondence between the class of an DLS's and the variety $\mathcal{V}$ of algebras $\mathbf{A} = \langle A, *, \circ \rangle$ satisfying the identities

(1) $(a * b) \circ b = a$ and
(2) $(a \circ b) * b = a$.

Given any L-circuit DLS (V, C) we define binary operations $*$ and $\circ$ by $a*b = c$ if and only if the edge ab is immediately followed by the edge bc in C and $a \circ b = c$ if and only if the edge ba is immediately preceeded by the edge cb in C.

Conversely, if we are given an algebra satisfying these two identities we can reconstruct the DLS by stipulating that the edge ab is followed by the edge $b(a*b)$. Since $(b \circ a) * a = b$, the edge ab is preceeded by the edge $(b \circ a)a$.

If we consider $*$ to be "multiplication" and $\circ$ to be "right division" then it is clear that the variety $\mathcal{V}$ is the class of all column latin groupoids. Given any b and c in A there exists an a, namely $a = c \circ b$, such that $a*b = (c \circ b)*b = c$, by identity (2). Also this a is unique since if $a_1 * b = a_2 * b$ then $(a_1 * b) \circ b = (a_2 * b) \circ b$ and so by identity (1), $a_1 = a_2$. We always take $*$ to be "multiplication" and $\circ$ to be "right division" though clearly this choice is arbitrary.

We call the algebra corresponding to an L-circuit DLS an L-clgpd (short for L-column latin groupoid) and we refer to the circuits of the DLS corresponding to a clgpd simply as the circuits of the clgpd. If $|L| = 1$, say $L = \{m\}$ then we usually write just m-clgpd instead of $\{m\}$-clgpd.

Example 2.1:

(1) The clgpd

*	1	2
1	2	2
2	1	1

is a 4-clgpd and contains the single 4-circuit (1,1,2,2).

(2) The clgpd

*	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

is the 3-clgpd corresponding to the $\{3\}$-circuit DLS of Example 1.2 (2).

We introduce some convenient notation. Define a sequence of words $W_i(x, y)$ by

$$\begin{aligned} W_0(x,y) &= x \\ W_1(x,y) &= y \\ W_2(x,y) &= x * y \\ W_3(x,y) &= y * (x * y) \end{aligned}$$

and inductively define $W_i(x,y) = W_{i-2}(x,y) * W_{i-1}(x,y)$ so that if you start at x of the edge xy and count t vertices to the right in the circuit containing xy you arrive at the vertex $W_t(x,y)$. Clearly, if there is a t such that $W_t(x,y) = x$ and $W_{t+1}(x,y) = y$ then the circuit containing y has finite length equal to the smallest such t and moreover its length must divide any such t.

3. Methods of Constructing m-CLGPDS

Definition 3.1: *Let $\mathbf{G}_n$ be the groupoid we get by defining a binary operation $*$ on the set of elements of $\mathbf{Z}_n$ by $a * b = 2b - a + 1 \pmod{n}$.*

Lemma 3.2. *If n is odd then $\mathbf{G}_n$ is an n-clgpd and if n is even then $\mathbf{G}_n$ is a $2n$-clgpd.*

Proof:

$$\begin{aligned} & a_1 * b = a_2 * b \pmod{n} \\ \rightarrow & 2b - a_1 + 1 = 2b - a_2 + 1 \pmod{n} \\ \rightarrow & a_1 = a_2 \pmod{n} \end{aligned}$$

Hence, $\mathbf{G}_n$ is a clgpd of order n.

Now let c be any circuit in $\mathbf{G}_n$, let ab be any edge in c and suppose c has length t. Then $W_t(a,b) = a$ and $W_{t+1}(a,b) = b$. Now, it is easy to show (by induction) that $W_t(a,b) = a + t(b-a) + \frac{t(t-1)}{2} \pmod{n}$.

$$\begin{aligned} W_t(a,b) = a \rightarrow a + t(b-a) + \frac{t(t-1)}{2} &= a \pmod{n} \\ \rightarrow t(b-a) + \frac{t(t-1)}{2} &= 0 \pmod{n} \\ \rightarrow 2t(b-a) + t(t-1) &= 0 \pmod{2n} \end{aligned}$$

$$W_{t+2}(a,b) = b \rightarrow a + (t+1)(b-a) + \frac{t(t+1)}{2} = b \pmod{n}$$
$$\rightarrow b + t(b-a) + \frac{t(t+1)}{2} = b \pmod{n}$$
$$\rightarrow t(b-a) + \frac{t(t+1)}{2} = 0 \pmod{n}$$
$$\rightarrow 2t(b-a) + t(t+1) = 0 \pmod{2n}$$

Hence,

$$t(t-1) = t(t+1) \pmod{2n} \rightarrow t^2 - t = t^2 + t \pmod{2n}$$
$$\rightarrow 2t = 0 \pmod{2n}$$
$$\rightarrow t = 0 \pmod{n}$$

Now, for any a and b, $W_n(a,b) = a + n(b-a) + \frac{n(n-1)}{2} = a + \frac{n(n-1)}{2} \pmod{n}$. Hence, if n is odd then for any a and b, $W_n(a,b) = a$ and so $\mathbf{G}_n$ is an n-clgpd.

If n is even then we have, for any a and b, $W_{2n}(a,b) = a + 2n(b-a) + \frac{2n(2n-1)}{2} = a \pmod{n}$. Hence, t divides $2n$ and so we must have $t = n$ or $t = 2n$. If $t = n$ then $W_n(a,b) = a$. That is, $a + \frac{n(n-1)}{2} = a \pmod{n} \rightarrow \frac{n(n-1)}{2} = 0 \pmod{n} \rightarrow n - 1 = 0 \pmod{2}$, which is not true and so there are no circuits of length n. Hence, $\mathbf{G}_n$ is an $2n$-clgpd. ∎

Lemma 3.3. *If $(x, a_2, a_3, \ldots, a_{m_1})$ and $(x, b_2, b_3, \ldots, b_{m_2})$ are two circuits of a clgpd then we can replace these two circuits with $(x, a_2, a_3, \ldots, a_{m_1}, x, b_2, b_3, \ldots, b_{m_2})$ and what we get is still a clgpd.*

Proof: This is clear since exactly the same edges are covered. ∎

We call the process in the above lemma *linking* the two circuits together. Hence, any two circuits can be linked provided they are not vertex-wise disjoint. Conversely, we can separate any circuit which has a repeated vertex into two circuits and we still have a clgpd.

Lemma 3.4. *The direct product $\mathbf{G} \times \mathbf{H}$ of an L_1-clgpd $\mathbf{G} = \langle G, *, \circ \rangle$ with an L_2-clgpd $\mathbf{H} = \langle H, *, \circ \rangle$ is an L-clgpd where $L = \{lcm(l_1, l_2) | l_1 \in L_1, l_2 \in L_2\}$.*

Proof: Let (g_1, h_1) and (g_2, h_2) be any two elements in $G \times H$. Then the circuit containing the edge $(g_1, h_1)(g_2, h_2)$ has length t where t is the smallest positive integer such that $W_t((g_1, h_1), (g_2, h_2)) = (g_1, h_1)$ and $W_{t+1}((g_1, h_1), (g_2, h_2)) = (g_2, h_2)$. That is, $W_t(g_1, g_2) = g_1$, $W_{t+1}(g_1, g_2) = g_2$, $W_t(h_1, h_2) = h_1$ and $W_{t+1}(h_1 h_2) = h_2$. Clearly these conditions are satisfied simultaneously if and only if t is a multiple of both l_1 and l_2 where l_1 is the length of the circuit of $\mathbf{G}$ containing $g_1 g_2$ and l_2 is the length of the circuit of $\mathbf{H}$ containing $h_1 h_2$. Hence, $t = lcm(l_1, l_2)$ where $l_1 \in L_1$ and $l_2 \in L_2$.

Conversely, let $t = lcm(l_1, l_2)$ where $l_1 \in L_1$ and $l_2 \in L_2$ and let $g_1 g_2$ be an edge in a circuit of **G** of length l_1 and $h_1 h_2$ be an edge in a circuit of **H** of length l_2. Then the circuit of **G** $\times$ **H** containing the edge $(g_1, h_1)(g_2, h_2)$ will be of length t. ∎

4. The Spectra of m-CLGPDS

For given m, the spectrum of m-clgpds is defined to be the set of values of n for which there exists an m-clgpd of order n. A necessary condition for the existence of an m-clgpd of order n is that m divides n^2, since the total number of edges in the graph is n^2 and each m-circuit consists of m edges. For any given integer m we define m' to be the smallest positive integer whose square is divisible by m. Hence, the necessary condition for the existence of an m-clgpd of order n becomes $n = 0 \pmod{m'}$.

For any positive integer m $(m \geq 3)$ let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t} q_1^{\beta_1} q_2^{\beta_2} \ldots q_u^{\beta_u}$ where $\{p_1, p_2, \ldots, p_t, q_1, q_2, \ldots, q_u\}$ is a set of $t + u$ distinct primes, $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ is a set of t even integers greater than or equal to 2 and $\{\beta_l, \beta_2, \ldots, \beta_u\}$ is a set of u odd integers greater than or equal to 1. Then it is clear that $m' = p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \ldots p_t^{\frac{\alpha_t}{2}} q_1^{\frac{\beta_1+1}{2}} q_2^{\frac{\beta_2+1}{2}} \ldots q_u^{\frac{\beta_u+1}{2}}$.

In this section we will show that for $m \geq 3$ the above necessary condition is sufficient. That is we will show that for an $m \geq 3$ and all $n = 0 \pmod{m'}$ there exists an m-clgpd of order n. Clearly, the only 1-clgpd is of order 1 and there are no 2-clgpds since there can not be a 2-circuit containing the edge aa. The circuit (a, a) is a 1-circuit (a), and $(a, a, b, \ldots)$ has length at least 3.

We first construct for each $m \geq 3$, an m-clgpd of order m'. We break the constructions into the four cases

(1) m is odd,
(2) $m = 2^r$,
(3) $m = 2^k m_0$ with m_0 odd and $k > 1$ and
(4) $m = 2m_0$ with m_0 odd.

(1) Let m be odd. In this case m' is odd. The m'-clgpd $G_{m'}$ (see Lemma 3.2) consists of the m', m'-circuits $(0, 0, 1, \ldots), (1, 1, 2, \ldots), \ldots, (m' - 1, m' - 1, 0, \ldots)$.

If we write m as a product of primes as above then it is easy to see that $\frac{m}{m'} = p_1^{\frac{\alpha_1}{2}} \ldots p_t^{\frac{\alpha_t}{2}} q_1^{\frac{\beta_1-1}{2}} \ldots q_u^{\frac{\beta_u-1}{2}}$ and that $m' = \frac{m q_1 q_2 \cdots q_t}{m'}$ and so if we link together (as per Lemma 3.3) the circuits of $G_{m'}$, $\frac{m}{m'}$ at a time we will get a collection of $q_1 q_2 \ldots q_t$ m-circuits which hence form an m-clgpd of order m'. It is clear that we do not need to link disjoint circuits during this process if we link the first $\frac{m}{m'}$ together and then the second $\frac{m}{m'}$, and so on (the first entry in any circuit is the same as the third entry in the next when they are written in the natural order as above). Example 1.2 (3) is obtained from Example 1.2 (2) by linking its circuits together in this manner.

(2) Let $m = 2^r$. If $r = 2s$ then $m' = 2^s$ and if $r = 2s + 1$ then $m' = 2^{s+1}$. We show that given a 2^{2s}-clgpd of order 2^s we can construct a 2^{2s+1}-clgpd of order 2^{s+1}, and that given a 2^{2s+1}-clgpd of order 2^{s+1} we can construct a 2^{2s+2}-clgpd of order 2^{s+1}.

A 2^{2s}-clgpd of order 2^s defined on the set $S = \{1, 2, \ldots, 2^s\}$ consists of a single 2^{2s}-circuit. Let this circuit be $(x_1, x_2, \ldots, x_m)$ and consider the collection C consisting of the two $2m$-circuits with entries chosen from $S \times \{a, b\}$ shown below

$$((x_1, a), (x_2, a), (x_3, a), \ldots, (x_{m-1}, a), (x_m, a),$$
$$(x_1, b), (x_2, b), (x_3, b), \ldots, (x_{m-1}, b), (x_m, b))$$

$$((x_1, a), (x_2, b), (x_3, a), \ldots, (x_{m-1}, a), (x_m, b),$$
$$(x_1, b), (x_2, a), (x_3, b), \ldots, (x_{m-1}, b), (x_m, a))$$

It is straight forward to check that every ordered pair of elements occurs exactly once at distance 1 in C. Hence, C gives us a 2^{2s+1}-clgpd of order 2^{s+1}.

A 2^{2s+1}-clgpd of order 2^{s+1} consists of two 2^{2s+1}-circuits. Since these two circuits can not be vertex-wise disjoint (if they were disjoint then either one of the circuits would form a 2^{2s+1}-clgpd of order less than 2^{s+1} by itself which is impossible) we can link them together to form a 2^{2s+2}-clgpd of order 2^{s+1}.

Since there exists a 4-clgpd of order 2 (see Example 1.3 (1)) we can deduce from the above results that for all $m = 2^r$ with $r > 1$ there exists an m-clgpd of order m'.

(3) Let $m = 2^k m_0$ with m_0 odd and $k > 1$. In this case $m' = (2^k)' m_0'$. But we know there exists a 2^k-clgpd of order 2^k and an m_0-clgpd of order m_0' and so the direct product of these two clgpds is a m-clgpd of order m'.

(4) Let $m = 2m_0$ with m_0 odd. In this case $m' = 2m_0'$. Let G be an m_0-clgpd of order m_0' with underlying set $G = \{1, 2, \ldots, m_0'\}$. We define a collection C of $2m_0$-circuits on $G \times \{a, b\}$ as follows. For each m_0-circuit $(x_1, x_2, \ldots, x_{m_0})$ of G let the two circuits shown below be in C.

$$((x_1, a), (x_2, a), (x_3, a), \ldots, (x_{m_0-1}, a), (x_{m_0}, a),$$
$$(x_1, b), (x_2, b), (x_3, b), \ldots, (x_{m_0-1}, b), (x_{m_0}, a))$$

$$((x_1, a), (x_2, b), (x_3, a), \ldots, (x_{m_0-1}, b), (x_{m_0}, b),$$
$$(x_1, b), (x_2, a), (x_3, b), \ldots, (x_{m_0-1}, a), (x_{m_0}, b))$$

It is straight forward to check that each ordered pair occurs exactly once at distance 1 in C. Hence C gives us an m-clgpd of order m'.

This completes the construction for all four cases and hence for all positive integers $m \geq 3$. We now show that the existence of an m-clgpd of order m' ensures the existence of an m-clgpd of order n for all $n = 0 \pmod{m'}$.

Lemma 4.1. *If there exists an m-clgpd of order n then for all positive integers k there exists a m-clgpd of order kn.*

Proof: Let G be an m-clgpd of order n with underlying set G. We construct a set C of m-circuits with entries chosen from the set $G \times K$ where $K = \{1, 2, \ldots, k\}$. First, suppose m is even. For each m-circuit $(x_1, x_2, \ldots, x_m)$ of G and for each i and j in K let

$$((x_1, i)(x_2, j), (x_3, i), \ldots, (x_{m-1}, i), (x_m, j)) \in C$$

Now, consider any edge $(p, a)(q, b)$. The edge $(p, a)(q, b)$ occurs only in the circuit $((x_1, i), (x_2, j), \ldots, (x_m, j))$ where $(x_1, x_2, \ldots, x_m)$ is the circuit containing the edge pq and if $p = x_t$ where t is odd then $i = a$ and $j = b$ and if $p = x_t$ where t is even then $i = b$ and $j = a$.

Now, suppose m is odd. Let $\cdot$ be a binary operation such that $(K, \cdot)$ is a quasigroup. For each m-circuit $(x_1, x_2, \ldots, x_m)$ of G and for each i and j in K let

$$((x_1, i), (x_2, j), (x_3, i), \ldots, (x_{m-1}, j), (x_m, i \cdot j)) \in C$$

Again, consider any edge $(p, a)(q, b)$. Let $(x_1, x_2, \ldots, x_m)$ be the circuit containing the edge pq. If $p \neq x_{m-1}$ or x_m then, as for the m even case, $(p, a)(q, b)$ occurs only once. If $p = x_{m-1}$ then $(p, a)(q, b)$ occurs only in $((x_1, i), (x_2, j), \ldots, (x_{m-1}, j), (x_m, i \cdot j))$ where $j = a$ and i is the unique member of K such that $i \cdot j = b$. If $p = x_m$ then $(p, a)(q, b)$ occurs only in $((x_1, i), (x_2, j), \ldots, (x_{m-1}, j), (x_m, i \cdot j))$ where $i = b$ and j is the unique member of K such that $i \cdot j = a$. Hence, C gives us an m-clgpd of order kn. ∎

We can deduce the following theorem from the above constructions.

Theorem 4.2. *For all positive integers $m \geq 3$ the spectrum of m-clgpds is the set of all n such that m divides n^2. Equivalently, the complete directed graph with loops on a vertex set of n elements can be decomposed into a set of closed trails of length m which partitions the edge set of the graph if and only if m divides n^2.*

5. Subvarieties and Restrictions on L

In this section we examine subvarieties of $\mathcal{V}$ whose members contain circuits all of whose lengths divide a positive integer $m \geq 3$.

Definition 5.1. *For each positive integer m we define the subvariety $\mathcal{V}_m$ of $\mathcal{V}$ by the identity $W_m(x, y) = x$.*

Lemma 5.2. *If G is an L-clgpd where each $l \in L$ divides m then G $\in \mathcal{V}_m$.*

Proof: Let x_0 and x_1 be any two elements in G and let $(x_0, x_2, \ldots, x_l)$ be the l-circuit of G which contains the edge $x_0 x_1$. Then, since l divides m, we have $W_m(x_0, x_1) = x_0$. Hence, G $\in \mathcal{V}_m$. ∎

Lemma 5.3. *If* G $\in \mathcal{V}_m$ *then* G *is an* L*-clgpd where each* $l \in L$ *divides* m.

Proof: Let $x_0 x_1$ be an edge in any circuit of **G** and suppose this circuit has length l. Let $m = tl + r$ where $0 \leq r \leq l-1$ then $x_0 = W_m(x_0, x_1) = W_r(x_0, x_1)$. Also, $x_1 = W_m(x_1, x_0 * x_1) = W_r(x_1, x_0 * x_1) = W_{r+1}(x_0, x_1)$. Hence, if $r \neq 0$ then $x_0 x_1$ must occur twice in this circuit which is impossible and so we must have $r = 0$ and l divides m. ∎

These last two theorems tell us that $\mathcal{V}_m$ consists precisely of those clgpds in which each member of L divides m. We note that the variety $\mathcal{V}_m$ is generated by the class $\mathcal{C}_m$ of all m-clgpds. In fact if **H** $\in \mathcal{V}_m$ and **G** $\in \mathcal{C}_m$ then **G** $\times$ **H** $\in \mathcal{C}_m$ by Lemma 3.4.

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