Near-Rotational Directed Triple Systems

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Abstract. A directed triple system of order v, denoted $DTS(v)$, is said to be k-near-rotational if it admits an automorphism consisting of 3 fixed points and k cycles of length $\frac{v-3}{k}$. In this paper, we give necessary and sufficient conditions for the existence of k-near-rotational $DTS(v)$s.

1. Introduction

A *directed triple system* of order v, denoted $DTS(v)$, is a v-element set, X, of points, together with a set, β, of ordered triples of elements of X, called *blocks*, such that any ordered pair of points of X occur in exactly one block of β. The notation $[x, y, z]$ will be used for the block containing the ordered pairs (x, y), (x, z), and (y, z). Hung and Mendelsohn [7] introduced directed triple systems as a generalization of Steiner triple systems and showed that a $DTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$. An *automorphism* of a $DTS(v)$ is a permutation of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, \ldots, p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, B, is said to be a *set of base blocks for a $DTS(v)$ under the permutation π* if the orbits of the blocks of B produce the $DTS(v)$ and exactly one block of B occurs in each orbit.

Several types of automorphisms have been explored in connection with the problem of determining the values v for which there are certain types of block designs of order v admitting the automorphism. In particular, a *cyclic* $DTS(v)$ admits an automorphism of type $[0, 0, \ldots, 1]$ and exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [3]. A $DTS(v)$ admitting an automorphism of type $[1, 0, \ldots, 0, k, \ldots, 0]$ is said to be *k-rotational*. A k-rotational $DTS(v)$ exists if and only if $kv \equiv 0 \pmod 3$ and $v \equiv 1 \pmod k$ [2]. Steiner triple systems, denoted STS, have been extensively explored in connection with this question. For a survey of results, see [4]. A cyclic $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [6, 8, 11]. The case of k-rotational STSs has been solved for $k = 1, 2, 3, 4$, and 6 [1, 9]. A triple system admitting an automorphism of type $[3, 0, \ldots, 0, k, 0, \ldots, 0]$ is said to be *k-near-rotational*. The case of k-near-rotational STSs has been solved for k divisible by 2 or 3 [5]. The purpose of this paper is to address the problem of existence for k-near-rotational directed triple systems.

2. Near-Rotational Directed Triple Systems

We have the following necessary conditions:

Lemma 2.1. *If a k-near-rotational $DTS(v)$ exists, then $k(v+2) \equiv 0$ (mod 3), $v \equiv 3$ (mod k), and $v \equiv 0$ or 1 (mod 3).*

Proof: A k-near-rotational $DTS(v)$ on the set $X = \{\infty_1, \infty_2, \infty_3\} \bigcup \{\mathbf{Z}_N \times \mathbf{Z}_k\}$ where $N = \frac{v-3}{k}$ admitting $\pi = (\infty_1)(\infty_2)(\infty_3)(0_0, 1_0, \ldots, (N-1)_0) \cdots (0_{k-1}, 1_{k-1}, \ldots, (N-1)_{k-1})$ as an automorphism may contain blocks of the following forms only:

1. $[\infty_i, \infty_j, \infty_m]$ where $i \neq j \neq m \neq i$ and $i, j, m \in \{1, 2, 3\}$,
2. $[x_i, \infty_m, y_j]$ where $m \in \{1, 2, 3\}$ and $x_i, y_j \in \mathbf{Z}_N \times \mathbf{Z}_k$,
3. $[\infty_m, x_i, y_j]$ or $[x_i, y_j, \infty_m]$ where $m \in \{1, 2, 3\}$, $i \neq j$, and $x_i, y_j \in \mathbf{Z}_N \times \mathbf{Z}_k$, and
4. $[x_i, y_j, z_m]$ where $x_i, y_j, z_m \in \mathbf{Z}_N \times \mathbf{Z}_k$.

There are two blocks of the first type, both of which are fixed under π. The orbits of blocks of the second, third and fourth types are of length N. The number of blocks in a $DTS(v)$ is $\frac{v(v-1)}{3}$ so a requirement for a k-near-rotational $DTS(v)$ is $\frac{v(v-1)}{3} - 2 \equiv 0$ (mod N). That is, $k(v+2) \equiv 0$ (mod 3). The other two conditions follow trivially. ∎

Lemma 2.1 says that a necessary condition for a k-near-rotational $DTS(v)$ is that

1. if $k \equiv 0$ (mod 3) then $v \equiv 3$ (mod k), or
2. if $k \equiv 1$ or 2 (mod 3) then $v \equiv 1$ (mod 3) and $v \equiv 3$ (mod k).

If π is an automorphism on a v-element set and is of type $[3, 0, \ldots, 0, k, 0, \ldots, 0]$, then π^n is of type $[3, 0, \ldots, 0, nk, 0, \ldots, 0]$ provided $n \mid N$ where $N = \frac{v-3}{k}$. So it would be sufficient to show the existence of k-near-rotational $DTS(v)$s for

1. $k = 1$ and $v \equiv 1$ (mod 3) and
2. $k = 3$ and $v \equiv 0$ (mod 3).

In each of the following lemmas, k-near-rotational $DTS(v)$s will be constructed on the set X with the automorphism π, where X and π are as described in Lemma 2.1.

We address the case for $k = 1$ in the next two lemmas.

Lemma 2.2. *A 1-near-rotational $DTS(v)$ exists for $v \equiv 1$ (mod 6).*

Proof:
case 1. If $v = 7$ then consider the blocks:

$$[\infty_1, \infty_2, \infty_3],\ [\infty_3, \infty_2, \infty_1],\ [0_0, \infty_1, 1_0],\ [0_0, \infty_2, 2_0], \text{ and } [0_0, \infty_3, 3_0].$$

case 2. If $v = 13$ then consider the blocks:

$$[\infty_1, \infty_2, \infty_3],\ [\infty_3, \infty_2, \infty_1],\ [0_0, \infty_1, 3_0],\ [0_0, \infty_2, 4_0],$$

$$[0_0, \infty_3, 6_0],\ [0_0, 1_0, 9_0], \text{ and } [0_0, 2_0, 7_0].$$

case 3. If $v \equiv 1 \pmod 6$, $v \geq 19$, say $v = 6t + 1$ where $t \geq 3$, then consider the blocks:

$$[\infty_1, \infty_2, \infty_3],\ [\infty_3, \infty_2, \infty_1],$$

$$[0_0, \infty_1, (3t-1)_0],\ [0_0, \infty_2, (2t-2)_0],\ [0_0, \infty_3, (2t)_0],$$

$$[0_0, (2r)_0, (3t-1+r)_0] \text{ for } r = 1, 2, \ldots, t-2, \text{ and}$$

$$[0_0, (2r-1)_0, (5t-3+r)_0] \text{ for } r = 1, 2, \ldots, t.$$

In each case, these are collections of base blocks for a 1-near-rotational $DTS(v)$ under the automorphism π. ▮

Lemma 2.3. *A 1-near-rotational $DTS(v)$ exists for $v \equiv 4 \pmod 6$, $v \geq 10$.*

Proof: Suppose $v \equiv 4 \pmod 6$, say $v = 6t + 4$. Consider the blocks:

$$[\infty_1, \infty_2, \infty_3],\ [\infty_3, \infty_2, \infty_1],$$

$$[0_0, \infty_1, (2t)_0],\ [0_0, \infty_2, (5t)_0],\ [0_0, \infty_3, (6t)_0],$$

$$[0_0, (2r-1)_0, (3t+r)_0] \text{ for } r = 1, 2, \ldots, t, \text{ and}$$

$$[0_0, (2r)_0, (5t+r)_0] \text{ for } r = 1, 2, \ldots, t-1 \text{ (omit if } t = 1).$$

These are the base blocks for a 1-near-rotational $DTS(v)$ under π. ▮

Lemmas 2.1-2.3 combine to give us:

Theorem 2.1. *A k-near-rotational $DTS(v)$ where $k \equiv 1$ or $2 \pmod 3$ exists if and only if $v \equiv 1 \pmod 3$, $v \geq 7$ and $v \equiv 3 \pmod k$.*

We now turn our attention to the case $k = 3$. In each of the following lemmas, the subscripts are reduced modulo 3.

Lemma 2.4. *If $v \equiv 0 \pmod{18}$, then there exists a 3-near-rotational $DTS(v)$.*

Proof: Suppose $v \equiv 0 \pmod{18}$, say $v = 18t$. Consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$, $[0_i, \infty_1, (3t-1)_i]$ for $i \in \mathbf{Z}_3$,

$[\infty_2, 0_0, 0_1]$, $[0_0, \infty_2, 0_2]$, $[0_1, 0_2, \infty_2]$, $[0_1, 0_0, \infty_3]$, $[0_2, \infty_3, 0_0]$, $[\infty_3, 0_2, 0_1]$,

$[0_0, r_1, (2r)_2]$ and $[(2r)_2, r_1, 0_0]$ for $r = 1, 2, \ldots, 6t-2$,

$[0_i, (2r-1)_i, (5t-2+r)_i]$ for $r = 1, 2, \ldots, t$ and for $i \in \mathbf{Z}_3$, and

$[0_i, (2r)_i, (3t-1+r)_i]$ for $r = 1, 2, \ldots, t-1$ (omit if $t = 1$) and for $i \in \mathbf{Z}_3$.

These are the base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

Lemma 2.5. *If $v \equiv 6$ (mod 18) and $v \geq 24$, then there exists a 3-near-rotational $DTS(v)$.*

Proof: Suppose $v \equiv 6$ (mod 18), say $v = 18t + 6$. Consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$,

$[0_i, \infty_1, (2t)_i]$ for $i \in \mathbf{Z}_3$, $[0_i, \infty_2, (2t+1)_i]$ for $i \in \mathbf{Z}_3$,

$[0_i, \infty_3, (3t+1)_i]$ for $i \in \mathbf{Z}_3$,

$[0_0, r_1, (2r)_2]$ and $[(2r)_2, r_1, 0_0]$ for $r = 0, 1, \ldots, 6t$,

$[0_i, (2r-1)_i, (5t+r)_i]$ for $r = 1, 2, \ldots, t$ and for $i \in \mathbf{Z}_3$,

$[0_i, (2r)_i, (3t+1+r)_i]$ for $r = 1, 2, \ldots, t-1$ (omit for $t = 1$) and for $i \in \mathbf{Z}_3$.

These are the base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

Lemma 2.6. *If $v \equiv 12$ (mod 18), then there exists a 3-near-rotational $DTS(v)$.*

Proof:
<u>case 1.</u> If $v = 30$ then consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_3]$, $[0_i, \infty_1, 3_i]$ and $[0_i, \infty_2, 5_i]$ for $i \in \mathbf{Z}_3$,

$[\infty_3, 0_0, 0_1]$, $[0_0, \infty_3, 0_2]$, $[0_1, 0_2, \infty_3]$, $[0_2, 0_1, 0_0]$,

$[0_0, r_1, (2r)_2]$ and $[(2r)_2, r_1, 0_0]$ for $r = 1, 2, \ldots, 8$,

$[0_i, 1_i, 8_i]$ and $[0_i, 2_i, 6_i]$ for $i \in \mathbf{Z}_3$.

<u>case 2.</u> If $v \equiv 12$ (mod 18), $v \neq 30$, say $v = 18t + 12$ where $t \neq 1$, then consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$ $[0_i, \infty_1, (2t+1)_i]$ for $i \in \mathbf{Z}_3$, $[0_i, \infty_2, (4t+2)_i]$ for $i \in \mathbf{Z}_3$,

$[\infty_3, 0_0, 0_1]$, $[0_0, \infty_3, 0_2]$, $[0_1, 0_2, \infty_3]$, $[0_2, 0_1, 0_0]$,

$[0_0, r_1, (2r)_2]$ and $[(2r)_2, r_1, 0_0]$ for $r = 1, 2, \ldots, 6t+2$,

$[x_i, y_i, z_i]$ and $[z_i, y_i, x_i]$ where (x_i, y_i, z_i) is a base block of a cyclic $STS(6t+3)$ on $\mathbf{Z}_{6t+3} \times \{i\}$ under the automorphism $(0_i, 1_i, \ldots, (6t+2)_i)$ for $i \in \mathbf{Z}_3$, with the exception of the base block in the orbit of the block $(0_i, (2t+1)_i, (4t+2)_i)$ (omit these blocks if $t = 0$).

In both cases, these are the base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

The following lemma will make use of a particular structure. A *(C, k)-system* is a set of ordered pairs $\{(a_r, b_r)$ for $r = 1, 2, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \ldots, k$ and $\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 2, \ldots, k, k+2, \ldots, 2k+1\}$. A (C, k)-system exists if and only if $k \equiv 0$ or 3 (mod 4) [10].

Lemma 2.7. *If $v \equiv 3$ or 9 (mod 24) $v \geq 9$, then there exists a 3-near-rotational $DTS(v)$.*

Proof: Consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$, $[0_i, \infty_1, (\frac{v-3}{6})_i]$ for $i \in \mathbf{Z}_3$,

$[0_1, \infty_2, (\frac{v-3}{6})_0]$, $[0_2, \infty_2, (\frac{v-3}{6})_1]$, $[0_0, \infty_2, (\frac{v-3}{6})_2]$,

$[0_0, \infty_3, (\frac{v-3}{6})_1]$, $[0_1, \infty_3, (\frac{v-3}{6})_2]$, $[0_2, \infty_3, (\frac{v-3}{6})_0]$,

$[0_0, 0_1, 0_2]$, $[0_2, 0_1, 0_0]$, and

$[0_i, r_i, (b_r)_{i+1}]$ and $[(b_r)_{i+1}, r_i, 0_i]$ for $r = 1, 2, \ldots, \frac{v-9}{6}$ and $i \in \mathbf{Z}_3$ where $\{(a_r, b_r)$ for $r = 1, 2, \ldots, \frac{v-9}{6}\}$ is a $(C, \frac{v-9}{6})$-system.

These are the base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

Lemma 2.8. *If $v \equiv 15$ (mod 24), then there exists a 3-near-rotational $DTS(v)$.*

Proof: Suppose $v \equiv 15$ (mod 24), say $v = 24t + 15$. Consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$, $[0_0, 0_1, 0_2]$, $[0_2, 0_1, 0_0]$, $[0_i, \infty_1, (4t+2)_i]$ for $i \in \mathbf{Z}_3$,

$[0_0, \infty_2, (2t+1)_1]$, $[0_1, \infty_2, (2t+1)_2]$, $[0_2, \infty_2, (2t+1)_0]$,

$[0_1, \infty_3, (6t+3)_0]$, $[0_2, \infty_3, (6t+3)_1]$, $[0_0, \infty_3, (6t+3)_2]$,

$[0_i, (2r-1)_i, (6t+2+r)_{i+1}]$ and $[(6t+2+r)_{i+1}, (2r-1)_i, 0_i]$ for $r = 1, 2, \ldots, 2t+1$ and for $i \in \mathbf{Z}_3$,

$[0_i, (2r)_i, (2t+1+r)_{i+1}]$ and $[(2t+1+r)_{i+1}, (2r)_i, 0_i]$ for $r = 1, 2, \ldots, 2t$ and for $i \in \mathbf{Z}_3$.

These are base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

Lemma 2.9. *If $v \equiv 21$ (mod 24), then there exists a 3-near-rotational $DTS(v)$.*

Proof: Suppose $v \equiv 21 \pmod{24}$, say $v = 24t + 21$. Consider the blocks:

$[\infty_1, \infty_2, \infty_3]$, $[\infty_3, \infty_2, \infty_1]$, $[0_0, 0_1, 0_2]$, $[0_2, 0_1, 0_0]$,

$[0_i, \infty_1, (4t+3)_i]$ for $i \in \mathbf{Z}_3$,$[0_0, \infty_2, (2t+2)_1]$, $[0_1, \infty_2, (2t+2)_2]$,

$[0_2, \infty_2, (2t+2)_0]$, $[0_1, \infty_3, (6t+4)_0]$, $[0_2, \infty_3, (6t+4)_1]$, $[0_0, \infty_3, (6t+4)_2]$,

$[0_i, (2r-1)_i, (6t+4+r)_{i+1}]$ and $[(6t+4+r)_{i+1}, (2r-1)_i, 0_i]$ for $r = 1, 2, \ldots, 2t+1$ and for $i \in \mathbf{Z}_3$, and

$[0_i, (2r)_i, (2t+2+r)_{i+1}]$ and $[(2t+2+r)_{i+1}, (2r)_i, 0_i]$ for $r = 1, 2, \ldots, 2t+1$ and for $i \in \mathbf{Z}_3$.

These are the base blocks for a 3-near-rotational $DTS(v)$ under π. ∎

Combining the results of Lemmas 2.2-2.9, we see that the necessary conditions of Lemma 2.1 are also sufficient. We therefore have:

Theorem 2.2. *A k-near-rotational $DTS(v)$ exists if and only if $k(v+2) \equiv 0$ (mod 3), $v \equiv 3$ (mod k), and $v \equiv 0$ or 1 (mod 3), $v \geq 7$.*

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