A Characterization of $\frac{3}{2}$-Tough Cubic Graphs

Bill Jackson
Department of Mathematical Studies
Goldsmiths' College
London SE14 6NW, England

P. Katerinis
Department of Informatics
Athens University of Economics and Business
Athens, 10434 Greece

Abstract. We show that a cubic graph is $\frac{3}{2}$-tough if and only if it is equal to K_4 or $K_2 \times K_3$ or else is the inflation of a 3-connected cubic graph.

1. Introduction

All graphs considered will be finite and without loops or multiple edges. We shall use $c(G)$ to denote the number of components of a graph G. In [C], Chvátal defined a graph G to be t-tough for some positive real number t, if $tc(G-S) \leq |S|$ for all vertex cut-sets S of G. In addition he defined the *toughness* of G, $t(G)$, to be the largest value of t such that G is t-tough, putting $t(K_n) = \infty$. He went on to relate $t(G)$ to various other graph invariants and to discuss relationships between $t(G)$ and the existence of Hamilton cycles and k-factors. In particular in [C, Section 6] he considered the toughness of regular graphs. He showed that if G is k-regular and not complete then $t(G) \leq k/2$ and asked for which values if k and n there exists a $k/2$-tough k-regular on n vertices. He showed that if k is even then such a graph exists for all $n \geq k + 1$. On the other hand for k odd and n large he suggested that such graphs may exist only for $n \equiv 0 \pmod k$. He verified this for $k = 3$ by proving:

Theorem 1.1. [C, Corollary 6.2] *A necessary and sufficient condition for the existence of a $\frac{3}{2}$-tough cubic graph with n vertices is either $n = 4$ or $n \equiv 0$* $\pmod 6$.

The main purpose of this note is to extend Theorem 1.1 by characterizing the $\frac{3}{2}$-tough cubic graphs. We shall also show that Theorem 1.1 does not extend to odd $k \geq 5$ by constructing an infinite family of $\frac{k}{2}$-tough k-regular graphs with n vertices and $n \not\equiv 0 \pmod k$. A similar construction was independently discovered by L.L. Doty [D].

The complexity of determining the toughness of an arbitrary graph was considered by Bauer, Hakimi and Schmeichel [BHS] who showed that for any fixed positive rational t, it is an NP-hard problem to determine if a graph is t-tough. In contrast we note that our characterization easily gives rise to a polynomial algorithm for determining whether a cubic graph is $\frac{3}{2}$-tough.

2. Characterization of $\frac{3}{2}$-tough cubic graphs

Our characterization uses the concept of the *inflation* of a graph G. This was defined in [C] as the graph G^* such that $V(G^*)$ is the set of all ordered pairs (v, e) where $v \in V(G)$ and e is an edge of G incident with v, and (v_1, e_1) is adjacent to (v_2, e_2) in G^* if they differ in exactly one coordinate. Inflations can be used to construct graphs of known toughness by using:

Theorem 2.1. [C, Theorem 5.1] *Let $G \neq K_2$ be a graph without isolated vertices. Then $t(G^*)$ is equal to one half the edge-connectivity of G.*

We shall also need a result Chvátal used to prove Theorem 1.1 concerning vertex colouring. A vertex colouring of a graph is *balanced* if each colour class has the same size.

Theorem 2.2. [C, Theorem 6.1] *No $\frac{3}{2}$-tough cubic graph has an unbalanced 3-colouring.*

We are now ready to give our characterization.

Theorem 2.3. *Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if $G = K_4$, $K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph.*

Proof: It follows from Theorem 2.1 that if $G = K_4, K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph then G is $\frac{3}{2}$-tough. Hence suppose that G is a $\frac{3}{2}$-tough graph other than K_4 or $K_2 \times K_3$. We shall use Theorem 2.2 to show that each vertex of G must belong to a triangle.

Suppose to the contrary that some vertex v_1 belongs to no triangle of G. Let $N(v_1) = \{v_2, v_3, v_4\}$ and let H be the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by adding a new vertex u and joining u to each vertex of $N(v_i)$ for $2 \leq i \leq 4$. Thus $d_H(u) \leq 6$, and since G is $\frac{3}{2}$-tough, $G - \{v_2, v_3, v_4\}$ has at most two components. Thus $H - u$ is connected. We shall adopt the proof of Brook's Theorem given by Lovasz in [BM] to show that H is 3-colourable. We first notice that we can find $v, w \in V(H) - u$ such that $uv, vw \in E(H)$ and $uw \notin E(H)$. If this were not the case then, since $H - u$ is connected, we must have $V(H) = \{u\} \cup N_H(u)$ and thus $|V(H)| \leq 7$. Hence $|V(H)| \leq 10$. Using Theorem 1.1 we deduce that $|V(G)| = 6$, and since v_1 belongs to no triangles in G, that $G = K_{3,3}$. This contradicts the hypothesis that G is $\frac{3}{2}$-tough and hence the vertices v and w exist as required. If $H - \{u, w\}$ were disconnected then $G - \{v_2, v_3, v_4, w\}$ would have three components, contradicting the hypothesis that G is $\frac{3}{2}$-tough. Thus $H - \{u, w\}$ is connected and we may order the vertices of H as $x_1, x_2, \ldots, x_m$ where $x_1 = u, x_2 = w, x_m = v$ and each vertex x_i is adjacent to a vertex x_j for $1 \leq i < j \leq m$ (this can be done by ordering $V(H) - \{u, w\}$ in non-increasing order of distance from v).

Now we may 3-colour H by colouring x_1 and x_2 with colour 1, then colouring the remaining x_i in order using any available colour from $\{1, 2, 3\}$, noting that each x_i for $i \leq m-1$ has at most two previously coloured neighbours so may be coloured and that $x_m = v$ has two neighbours x_1 and x_2 of the same colour so may also be coloured. This 3-colouring of $H-u$ can be extended to two different 3-colourings of G by colouring v_2, v_3 and v_4 with colour 1 and then colouring v_1 with either colour 2 or colour 3. Clearly these two 3-colourings of G cannot both be balanced, contradicting Theorem 2.2.

Thus, each vertex of G belongs to a triangle. Since G is cubic and $\frac{3}{2}$-tough (and hence 3-connected) and $G \neq K_4$, it follows that any two triangles must be disjoint and thus the set of triangles of G are a 2-factor of G. Contracting each triangle to a single vertex we obtain a cubic graph F such that G is the inflation of F. To complete the proof we note that by Theorem 2.1 the toughness of G is one half the edge-connectivity of F. Thus F is 3-edge connected and since F is cubic it must also be 3-connected. ∎

3. Construction of $\frac{3}{2}$-tough, k-regular graphs

To construct $\frac{3}{2}$-tough, k-regular graphs on n vertices with $n \not\equiv 0 \pmod{k}$ we shall use:

Theorem 3.1. [MS, Theorem 10] *Let G be a non-complete graph containing no induced $K_{1,3}$. Then $t(G)$ is equal to one half the connectivity of G.*

We shall say that a graph is essentially k-edge connected if it has no edge cut containing fewer than k edges and leaving at least one edge on both sides of the cut.

Construction 3.2: Let k be an odd integer greater than three and put $k = 2m+1$. Let G be an essentially k-edge connected bipartite graph with bipartition $V(G) = A \cup B$ where each vertex of A has degree $m+1$ and each vertex of B has degree $m+2$. Thus $|A| = (m+2)s$ and $|B| = (m+1)s$ for some integer s. The line graph of G is k-connected and $K_{1,3}$-free so by Theorem 3.1 is $\frac{k}{2}$-tough and k-regular with $n = (m+1)(m+2)s$ vertices. Choosing s coprime to k gives $n \not\equiv 0 \pmod{k}$. As an explicit construction for the bipartite graph G we may proceed as follows. Suppose m is odd, $m = 2p+1$. Let $A = \bigcup_{i=1}^{2s} A_i$, $B = \bigcup_{i=1}^{2s} B_i$ where $|B_i| = p+1$ for $1 \leq i \leq 2s$, $|A_{2i}| = p+1$ for $1 \leq i \leq s$ and $|A_{2i-1}| = p+2$ for $1 \leq i \leq s$. Join $x \in A_i$ to $y \in B_j$ if and only if $j = i-1$ or i where subscripts are read modulo $2s$.

A similar construction to 3.2 has been given by Doty [D].

4. Problems

W.D. Goddard and H.C. Swart [GS, Conjecture 3.3] have conjectured that a k-regular graph is $\frac{k}{2}$-tough if and only if it is k-connected and $K_{1,3}$-free. This conjecture is true for $k = 3$ by Theorem 2.3. Also note that k-connected, $K_{1,3}$-free graphs are $\frac{k}{2}$-tough by Theorem 3.1.

We suggest the following weaker problem.

4.1 Is it true that every $\frac{k}{2}$-tough k-regular graph contains a triangle?

We also raise the following problem for graphs which are not necessarily regular.

4.2 Does there exist a constant t such that every t-tough graph contains a triangle?

Note that in [C, Conjecture 2.6] Chvátal conjectures the existence of a constant t such that every t-tough graph is pancyclic.

References

[BHS] D. Bauer, E.L. Hakimi, and E. Schmeichel, *Recognizing tough graphs is NP-hard*, Discrete Applied Math **28** (1990), 191–195.

[BM] J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", North-Holland, Amsterdam, 1976.

[C] V. Chvátal, *Tough graphs and Hamiltonian cycles*, Discrete Math **5** (1973), 215–228.

[D] L.L. Doty, *A large class of maximally tough graphs*, OR Spectrum **13** (1991), 147–151.

[GS] W.D. Goddard and H.C. Swart, *On some extremal problems in connectivity*, in "Graph Theory", Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs (Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk, eds., Western Michigan University, 1991), pp. 535–551.

[MS] M.M. Mathews and D.P. Sumner, *Hamiltonian results in $K_{[1,3]}$-free graphs*, J. Graph Theory **8** (1984), 139–146.