Degree Factors of Line Graphs

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Abstract. Let G be a simple graph, a and b integers and $f\colon E(G) \to \{a, a+1, \ldots, b\}$ an integer-valued function with $\sum_{e\in E(G)} f(e) \equiv 0 \pmod 2$. We prove the following results: (1) If $b \geq a \geq 2$, G is connected and $\delta(G) \geq \max[b/2+2, (a+b+2)^2/(8a)]$, then the line graph $L(G)$ of G has an f-factor; (2) If $b \geq a \geq 2$, G is connected and $\delta(L(G)) \geq (a+2b+2)^2/(8a)$, then $L(G)$ has an f-factor.

1. Introduction.

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The line graph $L(G)$ of G is a graph defined by $V(L(G)) = E(G)$, $E(L(G)) = \{(e, f)\colon e, f \in E(G),\ e \neq f$, and e, f have an endvertex in common$\}$. The number of edges in G incident with a vertex v is called the degree of v and denoted by $\deg_G(v)$. In particular, for a vertex of a subgraph H of G, we denote the degree of v by $\deg_H(v)$. Further, $\delta(G)$ denotes the minimum degree of G. For a proper subset A of $V(G)$, $G-A$ denotes the subgraph of G obtained from G by deleting the vertices in A together with the edges incident with them. If A and B are disjoint subsets of $V(G)$, then $e_G(A, B)$ denotes the number of edges that join a vertex in A and a vertex in B. A subset $A \subset V(G)$ is often identified with the subgraph of G induced by A. Definitions and notations not defined here will be found in [1].

Let f be an integer-valued function defined on $V(G)$, and a and b integers such that $1 \leq a \leq b$. An f-factor of G is a spanning subgraph F_1 of G such that $\deg_{F_1}(v)$ for all $v \in V(G)$. Further, a spanning subgraph F_2 of G such that $a \leq \deg_{F_2}(v) \leq b$ for all $v \in V(G)$ is called an $[a, b]$-factor of G.

In this paper, we consider f-factors and $[a, b]$-factors, and give sufficient conditions for the existence of such factors in the line graph of a graph G.

We prove

Theorem 1. *Let G be a connected graph, a and b integers such that $2 \leq a \leq b$ and $f\colon E(G) \to \{a, a+1, \ldots, b\}$ a function such that $\sum_{e\in E(G)} f(e) \equiv 0$ $(\mathrm{mod}\ 2)$. Suppose that*

$$\delta(G) \geq \max\left[\frac{b}{2}+2, \frac{(a+b+2)^2}{8a}\right].$$

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Then $L(G)$ has an f-factor.

Theorem 2. *Let G be a connected graph, a and b integers such that $2 \le a \le b$ and $f\colon E(G) \to \{a, a+1, \dots, b\}$ a function such that $\sum_{e \in E(G)} f(v) \equiv 0 \pmod 2$. Suppose that the minimum degree $\delta(L(G))$ of the line graph $L(G)$ of G satisfies*

$$\delta(L(G)) \ge \frac{(a+2b+2)^2}{8a}.$$

Then $L(G)$ has an f-factor.

As for $[a,b]$-factors, we have

Theorem 3. *Let G be a graph and a and b integers such that $1 \le a < b$. If $\delta(G) \ge a/2 + 1$, then $L(G)$ has an $[a,b]$-factor.*

Theorem 4. *Let G be a graph and a and b integers such that $1 \le a < b$. Suppose that*

$$\delta(L(G)) \ge \begin{cases} a & \text{if } b \ge 2a, \\ \frac{(2a+b+2)^2}{8b} - 1 & \text{if } b \le 2a - 1. \end{cases}$$

Then $L(G)$ has an $[a,b]$-factor.

In proving Theorem 1 and Theorem 2, we use the following well-known criterion for the existence of an f-factor:

Theorem A. *(Tutte [6]). Let G be a graph and $f\colon V(G) \to N$ (the set of natural numbers) an integer-valued function such that $\sum_{v \in V(G)} f(v) \equiv 0 \pmod 2$. Then G has an f-factor if and only if*

$$\theta_G(S,T) := \sum_{v \in S} f(v) + \sum_{v \in T} (\deg_{G-S}(v) - f(v)) - h_G(S,T) \ge 0$$

for all disjoint subsets S and T of $V(G)$, where $h_G(S,T)$ denotes the number of components C of $G-(S \cup T)$ such that $\sum_{v \in C} f(v) + e_G(C,T) \equiv 1 \pmod 2$. Moreover, whether G has an f-factor or not, we have $\theta_G(S,T) \equiv 0 \pmod 2$ for any disjoint subsets S, T of $V(G)$.

Likewise, proofs of Theorem 3 and Theorem 4 depend upon the following criterion for the existence of an $[a,b]$-factor:

Theorem B. *Lovász [3]). Let a and b be integers such that $1 \le a < b$. Then a graph G has an $[a,b]$-factor if and only if*

$$\gamma_G(S,T) := b|S| + \sum_{v \in T} \deg_{G-S}(v) - a|T| \ge 0$$

for any disjoint subsets S, T of $V(G)$.

2. Numerical results.

In this section, we prove two numerical results.

Lemma 5. *Let a and b be integers such that $2 \leq a \leq b$ and x, y, and z nonnegative integers. Let $A = (a/2)x + y(y+z-b/2-1)$. Suppose $x+y+z \geq M := \max\left[\frac{b}{2}+2, \frac{(a+b+2)^2}{8a}\right]$. Then we have $A \geq 0$. Further, if in addition $(x+y)z \neq 0$, then we have $A \geq 1$.*

Proof: Let a, b, x, y, and z be integers satisfying the hypotheses of the lemma. If $y \leq a/2$, then since $M \geq b/2+2$, we have

$$\begin{aligned} A &= \frac{a}{2}x + y\left(y+z-\frac{b}{2}-1\right) \geq yx + y\left(y+z-\frac{b}{2}-1\right) \\ &\geq y\left(M - \frac{b}{2} - 1\right) \geq y \\ &\geq 0. \end{aligned} \tag{1}$$

If $y > a/2$, then since $x \geq M - y - z$ and $M \geq (a+b+2)^2/(8a)$, we have

$$\begin{aligned} A &\geq \frac{a}{2}(M-y-z) + y\left(y+z-\frac{b}{2}-1\right) \\ &= \left(y - \frac{a}{2}\right)z + \left[y - \frac{(a+b+2)}{4}\right]^2 \\ &\geq 0. \end{aligned} \tag{2}$$

Suppose that $(x+y)z \neq 0$. If $y = 0$, then we have $x \neq 0$ and $z \neq 0$. Hence, we have $A \geq (a/2)x \geq 1$. Therefore, we may assume $y \geq 1$ and $z \geq 1$. If $y \leq a/2$, then, by (1), we clearly have $A \geq 1$. Suppose $y = (a+1)/2$. If $b > a$, then we have $A \geq z/2 + (b-a)^2/16 > 1/2$ by (2). And if $b = a$, then $A \geq y(y+z-b/2-1) = y(y+z-a/2-1) \geq (a+1)/4 > 1/2$. Now, since $2A$ is an integer, these mean $A \geq 1$. Therefore, we may assume that $y \geq a/2+1$. Then we get $A \geq z + [y-(a+b+2)/4]^2 \geq z \geq 1$. ∎

Lemma 6. *Let a and b be integers such that $2 \leq a \leq b$ and x_1, x_2, z_1, and z_2 nonnegative integer, and y_1 and y_2 positive integers. Let $A = (a/2)x_1 + y_1(y_1+z_1-b/2-1)$ and $B = (a/2)x_2 + y_2(y_2+z_2-b/2-1)$. Suppose that*

$$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 \geq \frac{(a+2b+2)^2}{8a} + 2.$$

Then the following inequalities hold:

$$(A-1)y_2 + (B-1)y_1 \geq 0 \text{ if } z_1 \neq 0 \text{ and } z_2 \neq 0, \tag{3}$$

$$(A-1)y_2 + By_1 \geq 0 \text{ if } z_1 \neq 0 \text{ and } z_2 = 0, \tag{4}$$

$$Ay_2 + (B-1)y_1 \geq 0 \text{ if } z_1 = 0 \text{ and } z_2 \neq 0, \tag{5}$$

$$Ay_2 + By_1 \geq 0 \text{ if } z_1 = 0 \text{ and } z_2 = 0. \tag{6}$$

Proof: Let $a, b, x_1, x_2, y_1, y_2, z_1$ and z_2 be integers satisfying the hypothses of the lemma. We regard y_1 and y_2 as constants and x_1, x_2, z_1 and z_2 as variables. Thus, we regard A, B and the left-hand sides of the desired inequalities as polynomials of degree 1 in x_1, x_2, z_1 and z_2.

Case 1: $y_1 \leq a/2$ and $y_2 \leq a/2$.

In A, the coefficient of x_1, which is $a/2$, is greater than or equal to that of z_1, which is y_1. Therefore, by replacing x_1 by 0 and y_1 by $x_1 + y_1$, we may assume $x_1 = 0$. Similarly, we can assume $x_2 = 0$ in B. Then, whether z_1 or z_2 is 0 or not, we have

$$\begin{aligned}(A-1)y_2+(B-1)y_1 &= y_1 y_2(y_1 + y_2 + z_1 + z_2 - b - 2) - (y_1 + y_2)\\ &\geq \frac{y_1 y_2}{8a}\left[(a+2b+2)^2 - 8a(b+1)\right] - 1 + (y_1 - 1)(y_2 - 1)\\ &= \frac{y_1 y_2}{8a}(a - 2b - 2)^2 - 1 + (y_1 - 1)(y_2 - 1)\\ &\geq \frac{1}{8a}\left[(a - 2b - 2)^2 - 8a\right] \geq \frac{1}{8a}\left[(a + 2)^2 - 8a\right]\\ &\geq 0.\end{aligned}$$

Case 2: $y_1 \leq a/2$ and $y_2 \geq (a + 1)/2$ (or $y_1 \geq (a + 1)/2$ and $y_2 \leq a/2$).

In this case, by the symmetry of A and B, we may only consider the case where $y_1 \leq a/2$ and $y_2 \geq (a + 1)/2$. In any of the desired inequalities, the coefficient of x_2 is smaller than those of x_1, z_1, and z_2. Consequently, we may assume $x_1 = 0$ and $z_1 = z_2 = 1$ in proving (3), $x_1 = z_2 = 0$ and $z_1 = 1$ in proving (4), $x_1 = z_1 = 0$ $z_2 = 1$ in proving (5), $x_1 = z_1 = z_2 = 0$ in proving (6). We first prove (3) and (5) under these new assumptions. Using the trivial inequality $A - 1 \geq A - y_1$, we find that the values of the left-hand sides of (3) and (5) are at least

$$y_1 y_2\left(y_1 - \frac{b}{2} - 1\right) + y_1\left[\frac{a}{2}x_2 + y_2\left(y_2 - \frac{b}{2}\right) - 1\right].$$

Now, by the assumption, we have $x_2 \geq (a+2b+2)^2/(8a) - y_1 - y_2$. Therefore, the value of the above expression is at least $y_1\phi(y_1, y_2)$, where

$$\begin{aligned}&\phi(y_1, y_2)\\ &= y_2\left(y_1 - \frac{b}{2} - 1\right) + \frac{a}{2}\left[\frac{(a + 2b + 2)^2}{8a} - y_1 - y_2\right] + y_2\left(y_2 - \frac{b}{2}\right) - 1\\ &= y_1\left(y_2 - \frac{a}{2}\right) + y_2^2 - y_2\left(b + \frac{a}{2} + 1\right) + \frac{(a + 2b + 2)^2}{16} - 1.\end{aligned}$$

Also

$$
\begin{aligned}
&\phi(y_1, y_2)\\
&\geq \phi(1, y_2) = y_2^2 - y_2\left(b + \frac{a}{2}\right) + \frac{(a+2b+2)^2}{16} - \frac{a}{2} - 1\\
&\geq \frac{(a+2b+2)^2}{16} - \frac{a}{2} - 1 - \frac{1}{4}\left(b + \frac{a}{2}\right)^2\\
&= \frac{1}{4}(2b - a - 3).
\end{aligned}
$$

If $b \geq a + 1$ or $a \geq 3$, then $2b - a - 3 \geq 0$. Further, if $a = b = 2$, then $\phi(1, y_2) \geq (y_2 - 2)\ (y_2 - 1) \geq 0$. Thus, $\phi(y_1, y_2) \geq 0$. This proves (3) and (5). Similarly, we find that the values of the left-hand sides of (4) and (6) are at least

$$
y_1 y_2\left(y_1 - \frac{b}{2} - 1\right) + y_1\left[\frac{a}{2}x_2 + y_2\left(y_2 - \frac{b}{2} - 1\right)\right].
$$

Since $x_2 \geq (a + 2b + 2)^2/(8a) + 1 - y_1 - y_2$ in (4) and (6), the value of the above expression is at least $y_1\ \phi(y_1, y_2)$, where

$$
\phi(y_1, y_2) = y_1\left(y_2 - \frac{a}{2}\right) + y_2^2 - y_2\left(b + \frac{a}{2} + 2\right) + \frac{(a+2b+2)^2}{16} + \frac{a}{2}.
$$

Also, we have $\phi(y_1, y_2) \geq \phi(1, y_2) \geq (a+2b+2)^2/16 - (b+a/2+1)^2/4 = 0$. This proves (4) and (6).

Case 3: $y_1 \geq (a + 1)/2$ and $y_2 \geq (a + 1)/2$.

In this case, without loss of generality, we may assume $y_1 \leq y_2$ by the symmetry of A and B. Then, in any of the desired inequalities, the coefficient of x_2 is smaller than or equal to those of x_1, z_1, and z_2, and the rest of the proof goes exactly the same way as in Case 2.

3. Proofs.

In this section, we prove Theorems. But, we omit the proofs of Theorem 3 and Theorem 4 because they are essentially the same as, and much easier than, those of Theorem 1 and Theorem 2, respectively.

Proof of Theorem 1: Let a, b, G, and f be as in Theorem 1. Let S and T be disjoint subsets of $V(L(G))$ $(= E(G))$, and set $U = L(G) - (S \cup T)$. What we want to show is $\theta_{L(G)}(S,T) \geq 0$, where $\theta_{L(G)}(S,T)$ is as defined in Theorem A. If $S \cup T = \emptyset$, then we have $\theta_{L(G)}(S,T) = 0$. Therefore, we may assume $S \cup T \neq \emptyset$. Now, note that

$$
\theta_{L(G)}(S,T) \geq a|S| + \sum_{e \in T} \deg_{L(G)}(e) - b|T| - e_{L(G)}(S,T) - h_{L(G)}(S,T). \quad (7)
$$

We denote the sets of edges incident with $v \in V(G)$ and contained in S, T and U by S_v, T_v and U_v, respectively. Then we clearly have the following:

$$|S_v \cup T_v \cup U_v| = \deg_G(v) \text{ for } v \in V(G), \tag{8}$$

$$\sum_{v \in V(G)} |S_v| = 2|S|, \sum_{v \in V(G)} |T_v| = 2|T|, \sum_{v \in V(G)} |U_v| = 2|U|, \tag{9}$$

$$\sum_{e \in T} \deg_{L(G)}(e) = \sum_{v \in V(G)} [\deg_G(v) - 1] |T_v|, \tag{10}$$

$$e_{L(G)}(S,T) = \sum_{v \in V(G)} |S_v| |T_v|. \tag{11}$$

Inserting (8), (9), (10), and (11), into (7), we obtain

$$\begin{aligned} &\phi_{L(G)}(S,T) \\ &\geq \sum_{v \in V(G)} \left[\frac{a}{2}|S_v| + \left(\deg_G(v) - \frac{b}{2} - 1\right) |T_v| - |T_v||S_v|\right] - h_{L(G)}(S,T) \\ &\geq \sum_{v \in V(G)} \left[\frac{a}{2}|S_v| + |T_v| \left(|T_v| + |U_v| - \frac{b}{2} - 1\right)\right] - h_{L(G)}(S,T). \end{aligned}$$

Set

$$\lambda(v) := \frac{a}{2}|S_v| + |T_v| \left[|T_v| + |U_v| - \frac{b}{2} - 1\right], \; v \in V(G).$$

For each component C of U, let R_C be the set of vertices $v \in V(G)$ such that $C \cap U_v \neq \emptyset$. Then for any two distinct components C_1, C_2, we have $R_{C_1} \cap R_{C_2} = \emptyset$. Suppose that there exists a component C of U such that $S_v \cup T_v = \emptyset$ for all $v \in R_C$. Then R_C forms a component of G. Since G is connected, this means $R_C = V(G)$, so $U = L(G)$, which contradicts the assumption that $S \cup T \neq \emptyset$. Thus, for each component C of U, R_C contains a vertex v with $|S_v \cup T_v|\ |U_v| \neq \emptyset$. From these observations, it follows that the number k of vertices v of G with $|S_v \cup T_v|\ |U_v| \neq \emptyset$ is at least $h_{L(G)}(S,T)$. Hence, we have $\theta_{L(G)}(S,T) \geq \sum_{v \in V(G)} \lambda(v) - k$. Therefore, in order to prove $\theta_{L(G)}(S,T) \geq 0$, it suffices to show $\lambda(v) \geq 1$ (resp. 0) for all vertices $v \in V(G)$ such that $|S_v \cup T_v|\ |U_v| \neq 0$ (resp. $|S_v \cup T_v|\ |U_v| = 0$). But this readily follows if we apply Lemma 5 with $x = |S_v|$, $y = |T_v|$ and $z = |U_v|$. This completes the proof of Theorem 1. ∎

Proof of Theorem 2: Let a and b be integers such that $2 \leq a \leq b$ and G a connected graph satisfying the hypotheses of Theorem 2. Then, the condition on the minimum degree of $L(G)$ is equivalent to requiring

$$\deg_G(u) + \deg_G(v) \geq \frac{(a+2b+2)^2}{8a} + 2 \tag{12}$$

for all vertices $u, v \in V(G)$ with $uv \in E(G)$.

Let S and T be disjoint subsets of $V(L(G))$ $(= E(G))$ and $U, S_v, T_v, U_v, \lambda(v)$ and R_G as in the proof of Theorem 1. We want to show $\theta_{L(G)}(S,T) = \sum_{v \in V(G)} \lambda(v) - h_{L(G)}(S,T) \geq 0$. We may assume $S \cup T \neq \emptyset$ by the same reason in the proof of Theorem 1. For convenience, we set

$$\lambda_0(v) = \begin{cases} \lambda(v) - 1 & \text{if } |S_v \cup T_v|\,|U_v| \neq 0, \\ \lambda(v) & \text{otherwise.} \end{cases}$$

Now, let k be the number of vertices v of G with $|S_v \cup T_v|\,|U_v| \neq 0$. Then, k is at least $h_{L(G)}(S,T)$ from the properties of R_G observed in the proof of Theorem 1. Therefore, in order to prove $\theta_{L(G)}(S,T) \geq 0$, it suffices to show $\sum_{v \in V(G)} \lambda(v) - k \geq 0$, which is equivalent to $\sum_{v \in V(G)} \lambda_0(v) \geq 0$.

Let us consider the following subsets of $V(G)$:

$$P := \left\{ v \in V(G) : |T_v| \neq 0 \text{ and } \deg_G(v) < \max\left[\frac{(a+b+2)^2}{8a}, \frac{b}{2} + 2\right]\right\},$$

$$Q := \left\{ v \in V(G) : |T_v| \neq 0 \text{ and } \deg_G(v) \geq \max\left[\frac{(a+b+2)^2}{8a}, \frac{b}{2} + 2\right]\right\},$$

Since easy calculation shows $(a+2b+2)^2/(8a)+2 \geq 2\max[(a+b+2)^2/(8a), b/2+2]$, P is an independent set by (12). Applying Lemma 5 with $x = |S_v|$, $y = |T_v|$, and $z = |U_v|$, we have $\lambda_0(v) \geq 0$ for all vertices $v \in Q$. Further, we also have $\lambda_0(v) \geq \lambda(v) - 1 \geq 0$ (resp. $\lambda_0(v) = \lambda(v) \geq 0$) for all $v \in V(G) - (P \cup Q)$ such that $|S_v| \neq 0$ (resp. $|S_v| = 0$). For each $u \in Q$, let T'_u denote the set of those edges in T_u whose endvertex which is different from u lies in P. Then we have

$$\begin{aligned} &\theta_{L(G)}(S,T) \\ &\quad \geq \sum_{v \in V(G)} \lambda_0(v) \geq \sum_{v \in P \cup Q} \lambda_0(v) \geq \sum_{v \in P} \lambda_0(v) \\ &\quad + \sum_{u \in Q} \frac{|T'_u|\lambda_0(u)}{|T_u|} = \sum_{\substack{v \in P, u \in Q \\ uv \in T}} \left[\frac{\lambda_0(v)}{|T_v|} + \frac{\lambda_0(u)}{|T_u|}\right]. \end{aligned}$$

Applying Lemma 6 with $x_1 = |S_v|$, $y_1 = |T_v|$, $z_1 = |U_v|$, $x_2 = |S_u|$, $y_2 = |T_u|$, and $z_2 = |U_u|$, we see that each term in this last expression is nonnegative. This completes the proof of Theorem 2. ∎

4. Examples.

Finally, we show that the condition in Theorem 1 is almost the weakest possible (Example 1 and Example 2). Further, we construct an example which shows that the condition on the value of the minimum degree of $L(G)$ in Theorem 2 cannot be weakened (Example 3).

Example 1: Let a be an odd integer, b an even integer such that $a < b$ and $b/2 + 2 \geq (a+b+2)^2/(8a)$. Let G be a $(b/2+1)$-regular graph of order $4n$ (n is a sufficiently large integer) such that G has a 1-factor F. Define a function f: $E(G) \to \{a, a+1, \ldots, b\}$ by

$$f(e) = \begin{cases} a & \text{if } e \in E(F), \\ b & \text{if } e \in E(G - E(F)). \end{cases}$$

Then we have $\sum_{e \in E(G)} f(e) = 2na + nb^2 \equiv 0 \pmod 2$.

Now, we consider the line graph $L(G)$ of G. Set $S = \emptyset$ and $T = E(G - E(F))$. Then, by the definition of an f-odd component for $L(G)$, we have $h_{L(G)}(S,T) = 4n/2 = 2n$. Further, we have

$$\theta_{L(G)}(S,T) = 4n\left[\frac{b}{2}\left(\frac{b}{2} + 1 - \frac{b}{2} - 1\right)\right] - 2n = -2n (\leq -2).$$

Therefore, $L(G)$ has no f-factor by the f-factor theorem, but we have $\deg_G(v) \geq b/2 + 1$ for all $v \in V(G)$.

Example 2: Let a and b be nonnegative integers such that a is even, $a \leq b$, $a + b \equiv 2 \pmod 4$ and $(a+b+2)^2/(8a) \geq b/2 + 2$, and set $\ell = \lceil (a+b+2)^2/8a \rceil$ $p = (a+b+2)/4$. Let G be a connected $(\ell - 1)$-regular graph with a sufficiently large even order $4n$ such that G can be decomposed into one p-factor H and $(\ell - p - 1)$ 1-factors F_i. We define an integer-valued function f as follows:

$$f(e) = \begin{cases} a & \text{if } e \in E(\cup F_i), \\ b & \text{if } e \in E(H). \end{cases}$$

Then we have $\sum_{e \in E(G)} f(e) = 2n(\ell - p - 1)a + 2npb \equiv 0 \pmod 2$.

Consider the line graph $L(G)$ of G. Let $S = E(\cup F_i)$, and $T = E(H)$. Then we have

$$\begin{aligned} &\theta_{L(G)}(S,T) \\ &= 4n\left[(\ell - p - 1)\frac{a}{2} + p\left(p - \frac{b}{2} - 1\right)\right] \\ &\leq 4n\left[\left(\left\lceil\frac{(a+b+2)^2}{8a}\right\rceil - 1\right)\frac{a}{2} - \frac{(a+b+2)^2}{16}\right] \\ &\leq 4n\left[\left(\frac{(a+b+2)^2}{16}\frac{2}{a} - \frac{1}{(a/2)}\right)\frac{a}{2} - \frac{(a+b+2)^2}{16}\right] \\ &= -4n. \end{aligned}$$

Therefore, $L(G)$ has no f-factor, but we clearly have $\deg_G(v) \geq \lceil (a+b+2)^2/(8a) \rceil - 1$ for all $v \in V(G)$.

Example 3: Let a and b be positive even integers such that $a \equiv 2 \pmod 4$ and $a < b$, and let $q = (a + 2b + 2)/4$ and set $p = \lceil (a+2b+2)^2/8a \rceil - q - 1$. Further, let r be an integer such that $r \geq p + q$ and $r \equiv 1$ or $2 \pmod 4$. We define graphs $G(p,q,r)$. The vertex set of $G = G(p,q,r)$ is defined as follows:

$$\left. \begin{array}{c} V(G) = X \cup Y \cup Z, \\ Y = \bigcup_{i=1}^{p+1} Y_i, \quad Z = \bigcup_{i=1}^{p+1} \left(\bigcup_{j=1}^{q} C_{ij} \right) \end{array} \right\} \text{ (disjoint union)}$$

where

$$X = \{a_1, \ldots, a_{p+1}\}, \quad Y_i = \{b_{i1}, \ldots, b_{iq}\},$$
$$C_{ij} = \{c_{ij}^{(1)}, \ldots, c_{ij}^{(r+1)}\}.$$

The adjacency in G is defined as follows:

$$N_G(a_i) = (X - \{a_i\}) \cup Y_i, \quad N_G(b_{ij}) = \{a_i\} \cup \{c_{ij}^{(1)}\},$$
$$\langle C_{ij} \rangle_G = K_{r+1}, \quad N_G(c_{ij}^{(1)}) = \{C_{ij} - c_{ij}^{(1)}\} \cup \{b_{ij}\},$$

where $N_G(v)$ is the set of neighbors of a vertex v in G, $\langle S \rangle_G$ is the subgraph of G induced by $S \subset V(G)$ and K_n denotes the complete graph with n vertices.

Moreover, we set

$$S = \{a_i a_j : 1 \leq i < j \leq p+1\}$$
$$T = \{a_i b_{ij} : 1 \leq i \leq p+1, \ 1 \leq j \leq q\}.$$

Define an integer-valued function f as follows:

$$f(e) = \begin{cases} a & \text{if } e \in S, \\ b & \text{otherwise.} \end{cases}$$

Note that $h_{L(G)}(S,T) = (p+1)q$ and

$$\sum_{e \in V(L(G))} f(e) = a\frac{p(p+1)}{2} + b(p+1)q\left[\frac{r(r+1)}{2} + 2\right] \equiv 0 \pmod 2.$$

Then we have $\min(\deg_G(u) + \deg_G(v);\, u, v \in V(G), uv \in E(G)) = \lceil (a + 2b + 2)^2/8a \rceil + 1$, but

$$\begin{aligned}
&\theta_{L(G)}(S,T) \\
&\quad \leq \sum_{v\in X} \lambda(v) + \sum_{v\in Y} \lambda(v) - h_{L(G)}(S,T) \\
&\quad = (p+1)\left[\frac{a}{2}p + q\left(q - \frac{b}{2} - 1\right)\right] + (p+1)q\left(1 - \frac{b}{2}\right) - (p+1)q \\
&\quad = (p+1)\left[\frac{a}{2}\left(\left\lceil\frac{(a+2b+2)^2}{8a}\right\rceil - q - 1\right) + q(q-b-1)\right] \\
&\quad \leq (p+1)\left(\frac{a}{2}\left(\frac{(a+2b+2)^2}{8a} - \frac{1}{a/2}\right) - \frac{(a+2b+2)^2}{16}\right) \leq -(p+1).
\end{aligned}$$

This shows that $G(p,q,r)$ has no f-factor.

For values of a and b not considered in Example 1 and Example 2 (resp. Example 3), we do not know whether the condition of Theorem 1 (resp. Theorem 2) is the best or not. However, for each pair of integers a, b with $2 \leq a < b$, similar constructions yield infinitely many examples which show that if we replace the condition for the minimum degree of G (resp. $L(G)$) by the condition

$$\delta(G) \geq \max\left[\frac{b}{2}, \frac{(a+b+2)^2}{8a} - 2\right]$$

$$\left(\text{resp. } \delta(L(G)) \geq \frac{(a+2b+2)^2}{8a} - 2\right),$$

the theorem is no longer true.

We have a similar situation concerning the sharpness of the condition in Theorem 4, but we shall not go into details.

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References

1. M. Behzad, G. Chartrand, and L. Lesniak-Foster, "Graphs and Digraphs", Prindle Weber and Schmids, Boston, 1979.
2. M. Kano and N. Tokushige, *Binding numbers and f-factors of graphs*, J. Combin. Theory. Ser. B **64** (1992), 213–221.
3. L. Lovász, *Subgraphs with prescribed valencies*, J. Combin. Theory **8** (1970), 391–416.
4. T. Nishimura, *Regular factors of line graphs*, Discrete Math. **85** (1990), 215–219.
5. T. Nishimura, *Regular factors of line graphs II*, Math. Japonica **36** (1991), 1033–1040.
6. W.T. Tutte, *The factor of graphs*, Canad. J. Math. **4** (1952), 314–328.
7. W.T. Tutte, *A short proof of the factor theorem for finite graphs*, Canad. J. Math. **6** (1954), 347–352.