Clique Pseudographs and Pseudo Duals*

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ABSTRACT. We introduce a concept of "pseudo dual" pseudographs which can be thought of as generalizing some of the recent work on iterated clique graphs. In particular, we characterize those pseudographs which have pseudo duals and show that they encompass several natural families of intersection pseudographs.

1 Clique Pseudographs

The *clique graph* $K(G)$ of any graph G is the intersection graph of all the maximal cliques (i.e., inclusion maximal complete subgraphs, which we call *maxcliques*) of G. Clique graphs were introduced by Hamelink [10] and characterized by Roberts and Spencer [25]. Starting with Escalante [5], several authors have looked at classes of graphs preserved under the clique graph operator and at iterating this operator, determining classes of graphs G for which there exist parameters n and p such that $K^{n+p}(G) \cong K^n(G)$. Bandelt and Prisner [2] is an excellent survey and consolidation of the literature.

The *clique multigraph* of a graph G is the intersection multigraph of the maxcliques of G, with the multiplicity of a multiple edge (*multiedge*) equal to the number of vertices common to the corresponding maxcliques. These have been investigated and used in several papers [15,16,17,18,19,20]. Typically, loops have been ignored as adding nothing useful, or at least nothing essential. In this section we introduce a process of iterating clique multigraphs in which the loops are essential. Hence we use *pseudographs*: multigraphs with parallel loops allowed (constituting special multiedges called *multiloops*).

*Research supported in part by Office of Naval Research grant N00014-91-J-1210.

Suppose M is any pseudograph. For vertices v_i, $v_j \in V(M)$ (allowing $i = j$) let $\mu(v_i, v_j)$ denote the multiplicity of the multiedge joining v_i with v_j. By a *maxclique* of a pseudograph we mean a maxclique, including one loop at each vertex, in the *underlying looped graph* gotten by replacing each multiedge with a simple edge. Thus a maxclique of a multigraph C contains exactly one edge (or loop) from each bundle of parallel edges with the same endpoints in C.

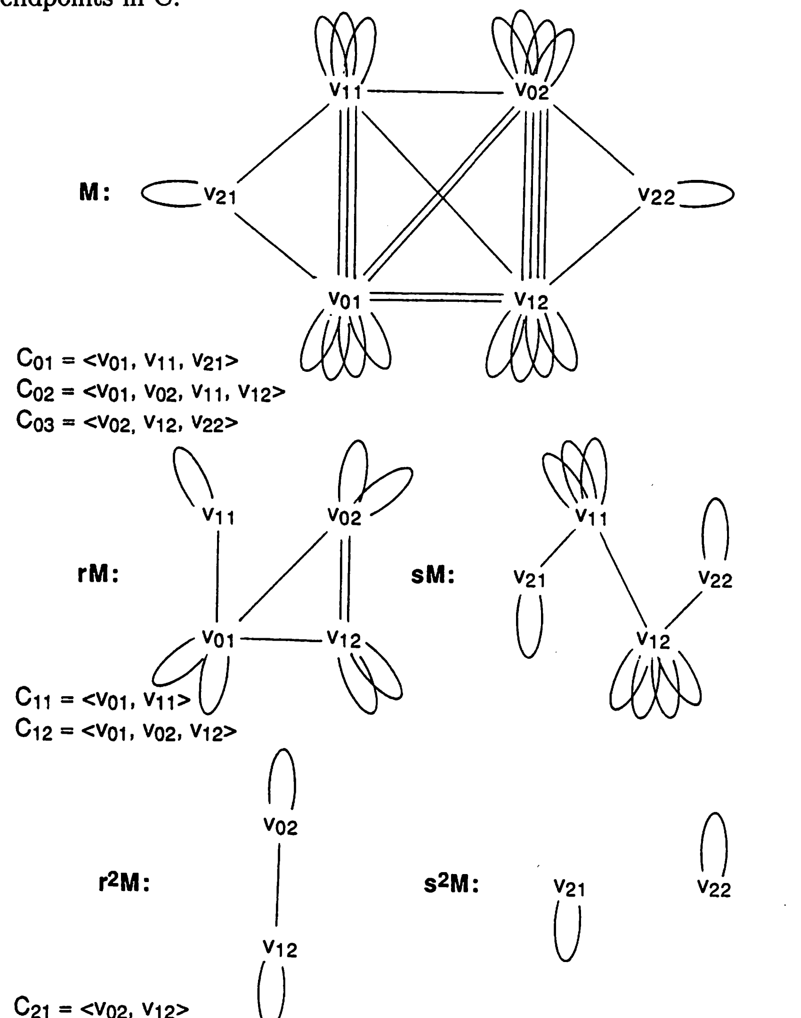


Figure 1

Let $C_{01}, \ldots, C_{0m_0}$ be the maxcliques of M. For each v_i, $v_j \in V(M)$ (always allowing $i = j$), put $\kappa(v_i, v_j) = |\{C_{0k} \colon v_i, v_j \in C_{0k}\}|$. If M everywhere satisfies $\mu(v_i, v_j) \geq \kappa(v_i, v_j)$, then we obtain rM (called the *residual pseudograph* of M) from M by decreasing each multiedge multiplicity by $\kappa(v_i, v_j)$, i.e., simultaneously removing one copy of each of $C_{01}, \ldots, C_{0m_0}$ from M and removing any isolated, loopless vertices created. (See Figure 1.)

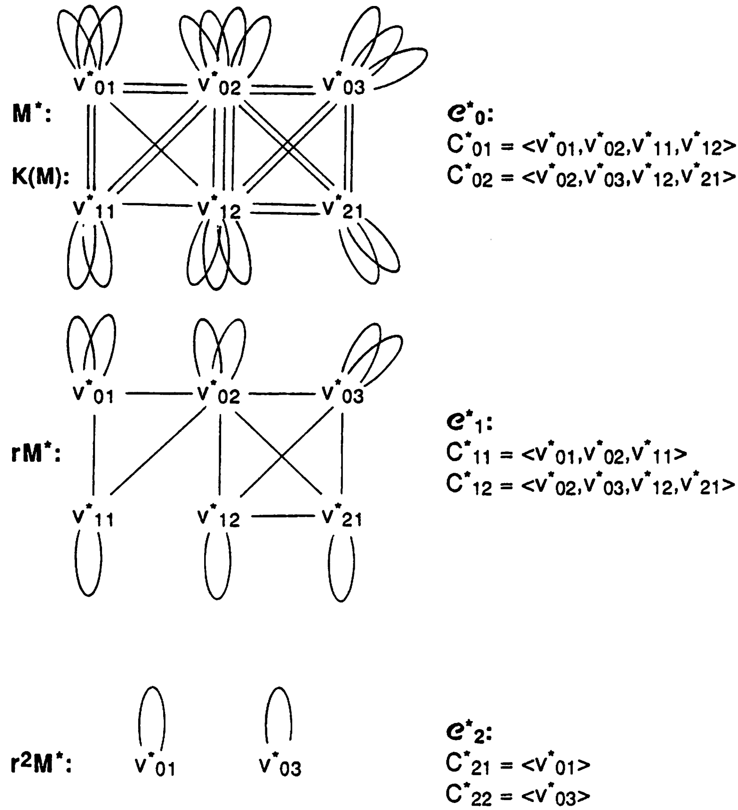


Figure 2

Let $C_{11}, \ldots, C_{1m_1}$ be the maxcliques of rM. If rM everywhere satisfies $\mu(v_i, v_j) \geq \kappa(v_i, v_j)$, we can again decrease each multiplicity by $\kappa(v_i, v_j)$ and drop isolated, loopless vertices to obtain the pseudograph $r(rM) = r^2M$. If this can be repeated until no vertices remain (obtaining m_0 maxcliques C_{0i} of $M = r^0M$, m_1 maxcliques C_{1i} of $rM = r^1M$, m_2 maxcliques C_{2i} of $r(rM) = r^2M$, etc.), then we call M a *reducible pseudograph* and call each C_{ij} a *residual clique* of M. For a reducible pseudograph M, let C be the multiset of all the residual cliques C_{ij} $(0 \leq i, 1 \leq j \leq m_i)$ of M. Figure 1 shows a reducible pseudograph M, all six of its residual cliques, and its residual pseduographs rM and r^2M; r^3M is empty. (The double subscripting on the vertices will be explained later.) Figure 2 shows another example.

The *clique pseudograph* $K(M)$ of a reducible pseudograph M is the intersection pseudograph of the family $\mathcal{C}$ of all the residual cliques of M, $K^2(M)$ is $K(K(M))$, etc. We let $K(M)$ have vertex set $\{v^*_{ij}: 0 \leq i$ and $1 \leq j \leq m_i\}$, where each v^*_{ij} corresponds to $C_{ij} \in \mathcal{C}$ from M. Figure 2 shows the clique pseudograph of the pseudograph of Figure 1. Figure 3 shows another reducible pseudograph M and its clique pseudograph $K(M)$; notice that this $K(M)$ is not itself reducible (since, in $rK(M)$, $\mu(v^*_{02}, v^*_{02}) = 2 < \kappa(v^*_{02}, v^*_{02}) = 3$). Therefore the clique pseudograph operator cannot necessarily be iterated.

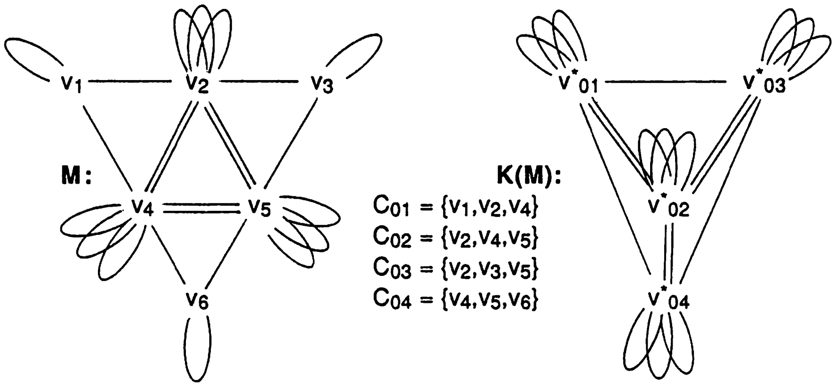


Figure 3

2 Pseudo Duals

In studying iterated clique graphs, it is common to restrict attention to what are sometimes called *clique-Helly graphs*: graphs whose families of maxcliques satisfy the *Helly property* (i.e., given any subfamily of maxcliques such that every two members of the subfamily have pairwise nonempty intersection, there must be some element simultaneously common to all members of the subfamily). As shown in [6] (see also [2]), the family of all clique-Helly graphs is preserved under the clique graph operator, every clique-Helly graph is itself the clique graph of some clique-Helly graph, and every clique-Helly graph has parameters n and $p \leq 2$ for which $K^{n+p}(G) \cong K^n(G)$. For instance, the four-spoked wheel W_5 is clique-Helly and $K(W_5) \cong K_4$, $K^2(W_5) = K(K_4) \cong K_1 \cong K^3(W_5) \cong K^4(W_5) = \ldots$; i.e., the parameters are $n = 2$ and $p = 1$. The octahedron $K_{2,2,2} = K_{3(2)}$, on the other hand, is not clique-Helly, and iterating the clique graph operator produces increasingly large graphs; in fact $K(K_{3(2)}) \cong K_{2,2,2,2} = K_{4(2)}$, and in general $K(K_{n(2)}) \cong K_{t(2)}$ where $t = 2^{n-1}$ (see [22]).

In this section we introduce an analog of clique-Helly for pseudographs for which the clique-pseudograph operator can be iterated with parameters $n = 1$ and $p \leq 2$ (and so there will be a notion of "dual" as explained below). Theorem 1 will characterize which pseudographs have such duals. Corollaries 1, 2, 3 and 4 will show natural families of clique pseudographs which have duals. (We do not consider the general question of when parameters n and p exist for a pseudograph, nor how they are determined.)

Additional structure is needed for our pseudograph analog of clique-Helly. For a reducible pseudograph M, write $v_i \lhd v_j$ whenever, for every $C \in \mathcal{C}$, $v_i \in C \Rightarrow v_j \in C$. This is easily recognized in the pseudograph since $v_i \lhd v_j$ is equivalent to $\mu(v_i, v_i) = \mu(v_i, v_j)$. (Moreover, $v_i \lhd v_j$ is equivalent to $\mu(v_i, v) \leq \mu(v_j, v)$ for every $v \in V(M)$, and so agrees with the notion of v_j "dominating" v_i in the underlying graph, as in [2]). Write $v_i \bowtie v_j$ if both $v_i \lhd v_j$ and $v_j \lhd v_i$. (So $v_i \bowtie v_j$ is equivalent to $\mu(v_i, v) = \mu(v_j, v)$ for every $v \in V(M)$). For instance, in the pseudograph in Figure 1, each $v_{2i} \lhd v_{ji}$, $v_{11} \lhd v_{01}$, and $v_{02} \bowtie v_{12}$. Let $V_0 = \{v_{01}, \ldots, v_{0n_0}\}$ contain one representative vertex from each $\bowtie$-equivalence class of maximal elements with respect to the order $(V(M), \lhd)$. Let sM be the induced subpseudograph $\langle V(M) \setminus V_0 \rangle$ of M. (In Figure 1, v_{01}, v_{02} and v_{12} are maximal elements; from the $\bowtie$-equivalent vertices v_{02} and v_{12}, we chose v_{02} to be in V_0, and so v_{12} occurs in sM.) Observe that the choice of V_0 does not affect whether sM is reducible.

If sM is also reducible, repeat the procedure of the preceding paragraph for sM. Say this produces $V_1 = \{v_{11}, \ldots, v_{1n_1}\}$, containing one vertex from each $\bowtie$-equivalence class of maximal elements with respect to the order $(V(sM), \lhd)$. Also, $s^2M = s(sM) = \langle V(M) \setminus \{v_{ij} \colon 0 \leq i \leq 1 \text{ and } 1 \leq j \leq n_i\} \rangle$. As long as s^iM is reducible, repeat this to form $s^{i+1}M$, and take $s^0M = M$. (In Figure 1, s^iM is shown for $i \leq 2$; s^iM is empty for $i \geq 3$.)

Call a reducible pseudograph M *absolutely clique-Helly* whenever every s^iM is reducible and the family of residual cliques of every s^iM satisfies the Helly property. If M is absolutely clique-Helly, we write $K(M) = M^*$. When $M \cong M^{**}$, we call M^* the *dual pseudograph* (or the *pseudo dual*) of M.

The following theorem characterizes which pseudographs have pseudo duals and considers a sense in which every r^iM and s^iM are "duals".

Theorem 1. *A pseudograph has a pseudo dual if and only if it is absolutely clique-Helly.*

Proof: Suppose M is absolutely clique-Helly with $V(M) = \{v_{ij} \colon 0 \leq i \text{ and } 1 \leq j \leq n_i\}$, the family $\mathcal{C} = \{C_{ij} \colon 0 \leq i \text{ and } 1 \leq j \leq m_i\}$ of residual cliques, and $V(M^*) = \{v^*_{ij} \colon 0 \leq i \text{ and } 1 \leq j \leq m_i\}$ as above. For $0 \leq i$ and $1 \leq j \leq$

n_i, put $C^*_{ij} = \langle\{v^*_{ab}: v_{ij} \in C_{ab}\}\rangle$ in M^*. Therefore $v_{ij} \in C_{ab} \iff v^*_{ab} \in C^*_{ij}$. (This makes the family of all such C^*_{ij} be the "hypergraph dual" of the family of all the C_{ab}. We discuss this connection with hypergraph duality further in Section 4.) By definition of $M^* = K(M)$, these C^*_{ij} are cliques which (with a loop at each vertex) partition the edge set of $K(M) = M^*$. Put $C^*_k = \{C^*_{kj}: 1 \leq j \leq n_k\}$ and let C^* be the multiset of all C^*_{ij}, $0 \leq i$ and $1 \leq j \leq n_i$. (All this notation is illustrated in Figures 1 and 2.) We first show that these cliques in C^* are precisely the residual cliques of M^*.

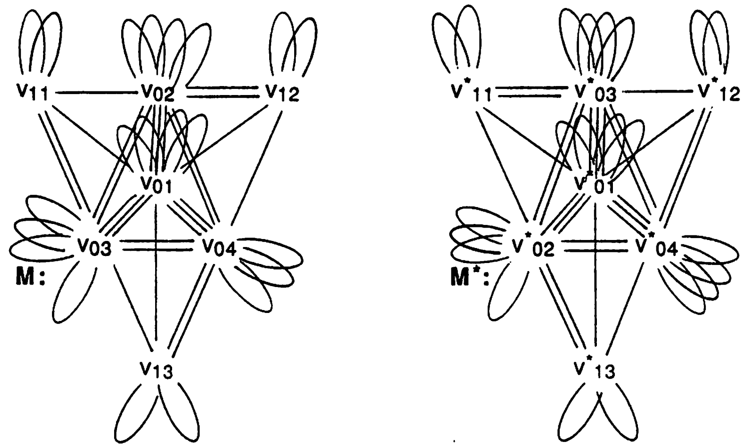
Observe that using multiloops insures that the residual cliques of each s^kM are precisely the restrictions of members of C to $V(s^kM) = \{v_{ij}: k \leq i \text{ and } 1 \leq j \leq n_i\}$. Therefore $v_{a_ib_i} \lhd v_{a_jb_j}$ in s^kM if and only if both $k \leq a_i, b_i$ and $v_{a_ib_i} \lhd v_{a_jb_j}$ in M, and this is in turn equivalent to $C^*_{a_ib_i} \subseteq C^*_{a_jb_j}$ in $K(M)$. Thus each v_{kj} being maximal in the order $(V(s^kM), \lhd)$ corresponds to C^*_{kj} being an inclusion-maximal member of C^*_k. Whenever $v^*_{a_1b_1}, v^*_{a_2b_2}, \ldots$ form a maxclique of $r^k(M^*)$, then $C_{a_1b_1}, C_{a_2b_2}, \ldots$ will have pairwise nonempty intersections in M and (since M being absolutely clique-Helly implies that C satisfies the Helly property) there will be some vertex of M common to every $C_{a_ib_i}$. Since we are starting from a maxclique of $r^k(M^*)$, this common vertex will be maximal in the order $V(s^kM, \lhd)$ and so of the form v_{kj}. Therefore, C^*_{kj} contains every $v^*_{a_ib_i}$, and so C^*_k consists precisely of the maxcliques of $r^k(M^*)$ and C^* is the multiset of residual cliques of M^*.

From here it follows easily that $M^{**} = K(M^*) \cong M$ using the isomorphism carrying each $C^*_{ij} \in V(K(M^*))$ to $v_{ij} \in V(M)$, checking that each $\mu(C^*_{ab}, C^*_{cd})$ in $K(M^*)$ equals $|C^*_{ab} \cap C^*_{cd}|$ in M^* which equals $|\{i: v^*_{0i} \in C^*_{ab}$ and $v^*_{0i} \in C^*_{cd}\}| = |\{i: v_{ab} \in C_{0i} \text{ and } v_{cd} \in C_{0i}\}| = \mu(v_{ab}, v_{cd})$ in M. □

A similar argument shows that, for clique-Helly M, each $K(s^kM) \cong r^kK(M)$, i.e., that each $(s^kM)^* \cong r^k(M^*)$. Figure 4 also illustrates the isomorphism $C^*_{ij} \mapsto v_{ij}$ showing $M^{**} \cong M$. Notice that, in Figure 4, $M \cong M^*$ and so M is *self-pseudo-dual*. But this example is special in that the mapping $v_{ij} \mapsto v^*_{ij}$ is not an isomorphism: $\mu(v_{02}, v_{12}) = 2$ in M but $\mu(v^*_{02}, v^*_{12}) = 0$ in M^*. (This corresponds to there not being an involutory vertex-to-residual-clique mapping, paralleling [9] and [1] for traditional graph duality; e.g., $v_{11} \in C_{13}$ but $v_{13} \notin C_{11}$ $(v^*_{11} \notin C^*_{13})$.)

Theorem 1 allows us to parallel other properties of clique-Helly graphs mentioned at the beginning of this section. If M is absolutely clique-Helly, then so is $K(M)$ (since $K(M)$ has a pseudo dual, namely M). Also, M being absolutely clique-Helly implies that M is itself the clique psuedograph of an absolutely clique-Helly pseudograph (namely $M \cong K(M^*)$). Also notice that, for absolutely clique-Helly M, each $(r^kM)^* \cong s^k(M^*)$.

Observe that we have only defined clique pseudographs $K(M)$ where M is a reducible pseudograph. But it is also natural to consider the intersection



$\mathcal{C}$: $C_{01} = \{v_{01}, v_{02}, v_{03}, v_{04}\}$
$C_{02} = \{v_{01}, v_{02}, v_{03}, v_{11}\}$
$C_{03} = \{v_{01}, v_{02}, v_{04}, v_{12}\}$
$C_{04} = \{v_{01}, v_{03}, v_{04}, v_{13}\}$
$C_{11} = \{v_{02}, v_{12}\}$
$C_{12} = \{v_{04}, v_{13}\}$
$C_{13} = \{v_{03}, v_{11}\}$

$\mathcal{C}^*$: $C^*_{01} = \{v^*_{01}, v^*_{02}, v^*_{03}, v^*_{04}\}$
$C^*_{02} = \{v^*_{01}, v^*_{02}, v^*_{03}, v^*_{11}\}$
$C^*_{03} = \{v^*_{01}, v^*_{02}, v^*_{04}, v^*_{13}\}$
$C^*_{04} = \{v^*_{01}, v^*_{03}, v^*_{04}, v^*_{12}\}$
$C^*_{11} = \{v^*_{02}, v^*_{13}\}$
$C^*_{12} = \{v^*_{03}, v^*_{11}\}$
$C^*_{13} = \{v^*_{04}, v^*_{12}\}$

Figure 4

pseudograph of the family of all the maxcliques of any graph G; we abuse notation somewhat in the remainder of this section by also denoting this pseudograph as $K(G)$. If G is not clique-Helly, then $K(G)$ need not even be reducible: for instance, take G to be the underlying loopless graph of the multigraph M of Figure 3, making $K(M) = K(G)$.

Corollary 1. *The maxclique intersection pseudograph of any clique-Helly graph has a pseudo dual.*

Proof: If G is clique-Helly, then $K(G)$ will be reducible with residual cliques $\{C_v : v \in V(G)\}$, where each C_v consists of those maxcliques of G which contain v. These residual cliques automatically satisfy the Helly property. Since the maxcliques of G are incomparable, $\mu(v_i, v_i) > \mu(v_i, v_j)$ holds throughout $K(G)$, making every v_i maximal in $(V(K(G)), \triangleleft)$ and so $sK(G)$ empty. Therefore $K(G)$ is absolutely clique-Helly and so has a pseudo dual. In fact, the pseudo dual is the pseudograph built from G by making each $\mu(u, v)$ equal to the number of maxcliques of G which contain $\{u, v\}$). □

3 Examples of Pseudo Duals

We now show that many previously-studied intersection pseudographs do in fact have pseudo duals. (Actually, the references cited below are to intersection multigraphs, but by allowing loops they can be easily modified for the corresponding pseudographs. Only the proofs of the corollaries in this section require details from the references cited.)

An *interval pseudograph* is the intersection pseudograph of a family of subpaths of a path. These are natural modifications of interval graphs as surveyed in Chapter 8 of [8]; see [18] for specific discussion of interval multigraphs.

Corollary 2. *Every interval pseudograph has a pseudo dual.*

Proof: Interval pseudographs are reducible by Theorem 2(1) of [18]. The family $\mathcal{C}$ satisfies the Helly property since the members of correspond to simplicial vertices of $M+$ in the proof of Theorem 2 of [18] and so to subpaths of a path. We also know that sM is an interval pseudograph since the subpaths-of-a-path representation of sM comes from that of M by deleting occurrences of each v_{0i}. Hence we can again show that sM is reducible and its family of residual cliques satisfies the Helly property. Repeating this as needed shows that M is absolutely clique-Helly. □

Observe that subtrees of a tree also satisfy the Helly property, but that their intersection pseudographs (corresponding to the "chordal multigraphs" in [18]) need not have pseudo duals; they need not even be reducible. For instance, consider $V(M) = \{v_1, v_2, v_3, v_4\}$ and $\mu(v_i, v_j)$ as follows:

$$\mu(v_i, v_j) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2 \\ 3 & \text{if } i = j \in \{1, 2\} \\ 4 & \text{if } i = j \in \{3, 4\} \\ 2 & \text{otherwise} \end{cases}$$

Then M is chordal but not reducible, since $\mu(v_3, v_4) = 1$ in rM, yet $\{v_3, v_4\}$ is in two maxcliques of rM.

The distinctive feature of **intervals** (or subpaths of a path) is that they satisfy both the Helly property and its hypergraph dual *conformal property* (i.e., given any set of points and any set of distinguished intervals, if every two points are in a common distinguished interval, then they must all be in a common distinguished interval). See Section 4 for more on the connection with hypergraph duality.

Given any property $\mathcal{P}$ of pseudographs (e.g., being an interval pseudograph), call an absolutely clique-Helly pseudograph M a *$\mathcal{P}^*$ pseudograph* whenever M^* satisfies $\mathcal{P}$. Call $\mathcal{P}$ a *selfdual property* whenever $\mathcal{P}$ and $\mathcal{P}^*$ are equivalent. In spite of the preceding comments on intervals satisfying both the Helly property and its hypergraph dual, the pseudo dual of an interval pseudograph need not be interval. (A counterexample is exhibited in the next paragraph.) Therefore, being interval is not a selfdual property. Furthermore, neither "unit interval pseudographs" (i.e., intersection pseudographs of congruent subpaths of a path) nor "proper interval pseudographs" (i.e., intersection pseudographs of annotations of subpaths of a path) correspond to selfdual properties. (These are pseudograph versions of "indifference graphs" which enter the theory of iterated clique graphs in [13].)

A good deal of interest surrounds the question of when "competition graphs" (i.e., intersection graphs of out-neighborhoods of acyclic digraphs) are interval graphs; see for instance [14] and [23]. We can similarly look at interval competition multigraphs (see [16]), and so pseudographs. If we discard isolated, loopless vertices (which necessarily exist, corresponding to "sinks" in the digraph), then interval competition pseudographs have pseudo duals by Corollary 2. But being an interval competition pseudograph is still not a selfdual property; the interval competition pseudograph of the at the bottom of page 485 in [24] is a counterexample. However being interval does confer an advantage to competition pseudographs: If M is an interval competition pseudograph of a digraph, then M^* (whether interval or not) is the competition pseudograph of the converse of the digraph (i.e., what is called the "common enemy" pseudograph of the original digraph).

The intersection pseudograph of the family of all *upsets* $I_x = \{y\colon x \leq y\}$ of a poset (partially ordered set) is called an *upper bound pseudograph*. This generalizes the notions of upper bound graph from McMorris & Zaslavsky [21] (in which the vertices of the graph are the elements of the posett with two adjacent whenever they have a common upper bound) and upper bound multigraphs from [15]. (Warning: rM is defined somewhat differently in [15].)

Corollary 3. *Every reducible upper bound pseudograph has a pseudo dual.*

Proof: Suppose M is a reducible upper bound pseudograph of a poset $(P, \leq)$. The proof of Lemma 3 of [15], modified for pseudographs, shows the equivalence of the following:

(1) M everywhere satisfies $\mu(v_i, v_j) \geq \kappa(v_i, v_j)$;

(2) The family of upsets of P satisfies the Helly property;

(3) Each maxclique of the underlying looped graph of M contains a simplicial vertex.

Observe that M is the reducible lower bound pseudograph of the poset P^c converse to P. The members of C are precisely the residual cliques of M

(by Theorem 1 of [15] and (1) $\Longrightarrow$ (3)) and they correspond to the downsets $J_x = \{w \colon w \le x\}$ of P^c (by the note three sentences before Proposition 2 in [15]). This set of downsets satisfies the Helly property (by (1) $\Longrightarrow$ (2) applied to P^c), and C also satisfies the Helly property.

Since sM is reducible and is the upper bound pseudograph of the poset of nonminimal elements of P, the above can be repeated to show that the residual cliques of sM satisfy the Helly property. Repeating this eventually shows that M is absolutely clique-Helly. □

Indeed, if M is a reducible upper bound pseudograph of a poset P, then rM (and sM) are the upper bound pseudographs of (respectively) the nonmaximal (nonminimal) elements of P, and M^* is the lower bound pseudograph of P, i.e., the upper bound pseudograph of P^c; in fact, $(V(M), \lhd) \cong (P^c, \le)$. Therefore the family of all reducible upper bound pseudographs is closed under the clique pseudograph operator K, and so being a reducible upper bound pseudograph is a selfdual property. (Also, interval upper bound pseudographs, see section 4 of [15], form a selfdual subcategory of interval pseudographs.)

As another example, the intersection pseudograph of the family of all root-to-leaf paths in a rooted tree is called a *component-reducible pseudograph* (or *co-pseudograph*). This generalizes the notion of a complement-reducible graph (or cograph) from [4] and component-reducible multigraph from [17].

Corollary 4. *Every co-pseudograph has a pseudo dual.*

Proof: Co-pseudographs are reducible by Theorem 2 of [17] with residual cliques corresponding to subtrees induced by internal vertices in the rooted tree representation. Also, C satisfies the Helly property since the root-to-leaf paths satisfy the nonformal property. Each $r^i M$ is also a co-pseudograph by Corollary 2.1 of [17]. Therefore repeating the above argument shows that M is absolutely clique-Helly. □

Being a co-pseudograph is not selfdual; Example 2 of [17] is a counterexample. When M is the co-pseudograph of a rooted tree T, the underlying graph of M^* will be the comparability graph of T (but M^* is not the comparability pseudograph of T in the sense of [20]).

4 Connections with Hypergraphs and Simple Graphs

Just as a graph can be viewed as a binary relation (with the unordered related pairs determining the edges of the graph), a *hypergraph* is a general relation (with the related sets determining the "edges," which we call *hyperedges* of the hypergraph). The *dual hypergraph* interchanges the roles of the elements and hyperedges (generalizing the familiar notion of line graph of a graph). See [3] for a general treatment and [19] for connections with multigraphs and pseudographs; [5] also develops intersection graph theory within hypergraph theory.

Hence when we consider a pseudograph M together with a family of subgraphs, as we do in Figure 1, we are describing a hypergraph on the same vertex set with the members of $\mathcal{C}$ as the hyperedges. The proof of Theorem 1 shows that the hypergraph determined by M^* (with hyperedges from $\mathcal{C}^*$) is the hypergraph dual of M (with hyperedges from $\mathcal{C}$); i.e., the vertices of M correspond to the residual cliques of M^* and the vertices of M^* correspond to the residual cliques of M. Thus what we are calling pseudo duality is a special case of hypergraph duality.

But what we are doing is not just hypergraph theory. We are considering how certain pseudographs (namely, the reducible ones) induce a natural hypergraph (by means of the residual cliques); the hyperedges are completely determined by the pseudograph structure, rather than being assigned independently. Hence, we are properly working within pseudograph theory (although in a manner describe within hypergraph theory).

Theorem 7.4 of [5] (credited to [7]) is an interesting connection between pseudographs and hypergraphs: Whenever $\{E_i : i \in I\}$ is a family of subpaths of a path and $\{F_i : i \in I\}$ is any family of sets such that $|E_i \cap E_j| = |F_i \cap F_j|$ for all $i, j \in I$, then the two families are isomorphic. (In other words, an interval pseudograph determines a unique interval hypergraph.) Let M_E and M_F be the isomorphic intersection pseudographs of the two families. Then each hyperedge E_i of the interval hypergraph corresponds uniquely to a residual clique of M_E, so to a vertex of M_E^*, so to a vertex of the isomorphic pseudograph M_F^*, so to a residual clique of M_F, and so finally to an F_j; this correspondence is an isomorphism.

Finally, for the reader weary of pseudographs, we observe that a vestige of pseudo duals exists within "simple graph theory." Suppose G is any simple graph and we append a multiloop at each vertex v with $\mu(v, v)$ equal to the "clique degree" of v in G (i.e., the number of maxcliques of G which contain v). In order for this multi-looped graph G to be reducible, it must be edge-partitionable into maxcliques. This occurs automatically when each *block* (i.e., maximal 2-connected subgraph) is complete, and so by [11] or Theorem 3.5 of [12] whenever G is a *block pseudogragh*: i.e., the intersection pseudograph $B(H)$ of all the blocks of some graph H. (The underlying loopless graph of $B(H)$ is the traditional *block graph* of H in [12].)

Suppose $B(H)$ is any block pseudograph. Then $B(H)$ has a pseudo dual by Theorem 1, and $B(H)^* \cong H$. This pseudo duality of $B(H)$ and H corresponds to the *ad hoc* construction in the final paragraph of the proof of Theorem 3.5 in [12] (with the multiloops now keeping track of the noncutpoints in the corresponding blocks, as in that proof). This also relates to Corollary 1, since block graphs are tribally clique-Helly.

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