A note on the edge reconstruction conjecture

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Abstract. We proved that if a graph G of minimum valency $\delta = 6\alpha + 5$, with α non-negative integer, can triangulate a surface Σ with $\chi(\Sigma) = -\alpha n + \beta$, where $\beta \in \{0, 1, 2\}$, then G is edge reconstructible.

In this note, all graphs $G = (V(G), E(G))$ considered will be finite and simple. Let $n = |V(G)|$, $m = |E(G)|$. For a vertex $v \in V(G)$, let $d(v)$ be the degree of v. If $d(v) = k$, we call v a k-vertex and we use n_k to denote the number of k-vertices in G.

Surfaces denoted by Σ in this note are understood to be compact and connected 2-manifolds without boundary. If a graph G with n vertices and m edges has an embedding in a surface Σ of characteristic $\chi(\Sigma)$, and if f is the number of faces of the embedding, the Euler's formula states that

$$n - m + f = \chi(\Sigma).$$

This gives that $m \leq 3n - 3\chi(\Sigma)$, with equality holding if and only if G triangulates Σ.

A graph G is *edge reconstructible* if its isomorphic class is uniquely determined by the collection $D(G) = \{G - e;\ e \in E(G)\}$ of single-edge-deleted subgraphs of G. The edge form of the reconstruction conjecture states that every graph with at least four edges is edge reconstructible. In this note, we give a very simple proof of the following theorem.

Theorem. Let G be a graph of minimum valency $\delta = 6\alpha + 5$ with α non-negative integer. If G can triangulate a surface Σ with $\chi(\Sigma) = -\alpha n + \beta$, where $\beta \in \{0, 1, 2\}$, then G is edge reconstructible.

Before we prove our theorem, we need two lemmas. The first lemma is due to Hoffman[1].

Lemma 1. Let G be a graph of minimum valency δ. Suppose that, for some $k \geq 0$, there is a vertex in G of degree $\delta + k$ adjacent to $k + 1$ or more vertices of degree δ. Then G is edge reconstructible.

We define $E_{\delta,i} = \{e \in E(G);$ one end of e is incident to a δ-vertex and the other end of e is incident to an i-vertex$\}$ and $t_{\delta,i} = |E_{\delta,i}|$, $i > \delta$. By Lemma 1, if a graph G of minimum valency δ is not edge reconstructible, then $t_{\delta,i} \leq (i - \delta)n_i$.

The second lemma comes from Lemma 1.4. of [2].

Lemma 2. If graph G of minimum valency δ contains a triangle which is incident to one δ-vertex and two $(\delta + 1)$-vertices, then G is edge reconstructible.

Throughout the rest of this note, G always stands for a graph of minimum valency δ which can triangulate a surface Σ with $\chi(\Sigma) = -\alpha n + \beta$. Since regular graphs and bidegreed graphs [3] are edge reconstructible, we assume that G is neither a regular graph nor a bidegreed graph. Since G is trivially edge reconstructible from G_{uv} for any two adjacent δ-vertices u and v in G, we will assume that there do not exist any two adjacent δ-vertices in G.

Proof of Theorem. By Lemma 1, we can assume $t_{\delta,i} \leq (i-\delta)n_i$, $i \geq \delta + 1$. Since G triangulates a surface Σ, then $m = 3n - 3\chi(\Sigma) = 3(1+\alpha)n - 3\beta$. Since $2m = \sum_{i=\delta}^{\Delta} i n_i$ and $n = \sum_{i=\delta}^{\Delta} n_i$, where Δ is the maximum degree of G, we have

$$n_\delta = n_{\delta+2} + \ldots + [\Delta - (\delta + 1)]n_\Delta + 6\beta.$$

By $\delta n_\delta = \sum_{i=\delta+1}^{\Delta} t_{\delta,i}$, we have

$$t_{\delta,\delta+1} = \delta n_\delta - t_{\delta,\delta+2} - \ldots - t_{\delta,\Delta}$$
$$= \delta n_{\delta+2} + \ldots + \delta[\Delta - (\delta + 1)]n_\Delta - t_{\delta,\delta+2} - \ldots - t_{\delta,\Delta} + 6\delta\beta$$
$$= (2n_{\delta+2} - t_{\delta,\delta+2}) + \ldots + [(\Delta - \delta)n_\Delta - t_{\delta,\Delta}] + (\delta - 2)n_{\delta+2} + \ldots + [(\delta - 1)\Delta - \delta^2]n_\Delta + 6\delta\beta$$
$$\geq \frac{\delta+1}{2} n_{\delta+2} + \ldots + \frac{\delta+1}{2}[\Delta - (\delta+1)]n_\Delta + 3(\delta+1)\beta + \frac{\delta-5}{2} n_{\delta+2} + \ldots + \frac{1}{2}[\Delta(\delta - 3) - \delta^2 + 2\delta + 1]\, n_\Delta + 3(\delta - 1)\beta$$
$$\geq \frac{\delta+1}{2} n_\delta + \frac{\delta-5}{2} n_{\delta+2} + \ldots + \frac{1}{2}[\Delta(\delta - 3) - \delta^2 + 2\delta + 1]\, n_\Delta + 3(\delta - 1)\beta.$$

By $\beta \geq 0$, it is clear that $t_{\delta,\delta+1} \geq \frac{\delta+1}{2} n_\delta$. Hence G has at least one δ-vertex which is adjacent to at least $\frac{\delta+1}{2}$ $(\delta + 1)$-vertices. Therefore G has a triangle which is incident to one δ-vertex and two $(\delta + 1)$-vertices. By Lemma 2, G is edge reconstructible. #

As corollary of our theorem, we have

Corollary. Let G be a graph of minimum valency 5. If G can triangulate a surface Σ with $\chi(\Sigma) \geq 0$, then G is edge reconstructible.

REFERENCES

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[3] W. J. Myrvold, M. N. Ellingham and D. G. Hoffman, Bidegreed graphs are edge reconstructible, J. Graph Theory 11 (1987), 281-302.