Isomorphisms of Infinite Steiner Triple Systems

Frantisek Franek

Department of Computer Science and Systems
McMaster University
Hamilton, Ontario L8S 4K1 Canada

Abstract. An infinite countable Steiner triple system is called universal if any countable Steiner triple system can be embedded into it. The main result of this paper is the proof of non-existence of a universal Steiner triple system. The fact is proven by constructing a family S of size 2^{ω} of infinite countable Steiner triple systems so that no finite Steiner triple system can be embedded into any of the systems from S and no infinite countable Steiner triple system can be embedded into any two of the systems from S (it follows that the systems from S are pairwise non-isomorphic). A Steiner triple sytem is called rigid if the only automorphism it admits is the trivial one — the identity. An additional result presented in this paper is a construction of a family of size 2^{ω} of pairwise non-isomorphic infinite countable rigid Steiner triple systems.

Introduction.

There is a vast literature dealing with finite Steiner systems (see, for example, [LR]). There has been very little published on infinite Steiner systems (*cf*. [So], [Si], [GGP], [N], [HS], [PS]). In [GGP] a class of size 2^{ω} of non-isomorphic infinite countable Steiner triple systems is presented. In [N] a class of size 2^{κ} of Steiner systems of order κ, κ an infinite cardinal, is presented so that the systems are pairwise non-monomorphic. In [PS] a general theorem in category theory is presented, and one of its consequences is that for any infinite cardinal κ there are 2^{κ} non-isomorphic Steiner triple systems of order κ.

In the original version of this paper the thrust was in the construction of an uncountable family of rigid non-isomorphic countable Steiner triple systems carried in a general way so it could be extended to higher cardinals as well. The basic idea of the construction used in this paper comes from [GGP], but the modifications had to be provided in order to produce a whole family of non-isomorphic rigid Steiner triple systems. After discussing the method and results with J. Nešetřil, we realised that, in fact, something stronger had been proven, namely, the non-existence of a universal countable Steiner triple system. Thus, this paper was simplified in its approach (that is, made less general) and the emphasis was shifted.

If one considers two subsystems of a Steiner triple system, their intersection must be either empty, or one element, or a Steiner triple system, either a trivial one (that is, a single block), or a non-trivial one. Thus, the fundamental idea of our proof of non-existence of a universal Steiner triple system is to construct such an uncountable family of countable Steiner triple systems that its members can only have trivial pairwise intersections when embedded into a universal system.

Research supported by NSERC OGP0025112 grant

Throughout this paper, $\mathbf{Q}$ denotes the set of rational numbers, $\mathbf{Z}$ denotes the set of integers, and $\mathbf{N}$ denotes the set of non-negative integers. ω denotes the first infinite cardinal (elsewhere denoted also as $\aleph_0$ or ω_0). The term *countable* is used here strictly in the sense of *infinite countable*. By 2^ω we denote the cardinality of the continuum, that is, the cardinality of the set of all subsets of $\mathbf{N}$. If X is a set and n is a positive integer, then $[X]^n$ denotes the set of all subsets of X of size n. The symbol $|x|$ is used in two meanings, as the absolute value of a number, and the cardinality (size) of a set. Which meaning is used is always clear from the context. A **Steiner triple system** (or **STS** for short) S is defined as a pair (V, B), where V is a non-empty set of **elements** of S, and B is a non-empty set of **blocks** of S, these blocks are triples (that is, $B \subseteq [V]^3$) and B has the property that every pair $\{a, b\}$ of distinct elements from V occurs in exactly one of the blocks from B. The **order** of a Steiner triple system is the size of its set of elements. A Steiner triple system $S_1 = (V_1, B_1)$ is a **subsystem** of a Steiner triple system $S_2 = (V_2, B_2)$ if $V_1 \subseteq V_2$ and $B_1 \subseteq B_2$. For an overview of Steiner systems see, for example, [**LR**].

Methods and results.

Let $\sim$ be a relation defined on $\mathbf{Q} - \{0\}$ by $a \sim b$ iff $a = \pm(-2)^z b$ for some $z \in \mathbf{Z}$. $\sim$ is a relation of equivalence.

First a simple observation about $\sim$.

Lemma 1. *Let r and p be integers so that $1 \leq r < p$. Let p be odd and let $x \in \mathbf{Q} - \{0\}$. Then $px \notin [rx]$.*

Proof: By the way of contradiction let us assume that $px \in [rx]$. Then $px = \pm(-2)^z rx$, so $p = \pm(-2)^z r$, and, thus, either $p = -(-2)^{2k+1} r$ for some $k \in \mathbf{N}$, and so $p = 2^{2k+1} r$, or $p = (-2)^{2k} r$ for some $k \in \mathbf{N}$, and so $p = 2^{2k} r$. It follows that $p = 2^n r$ for some $n \in \mathbf{N} - \{0\}$, a contradiction as p is odd. ∎

It follows from Lemma 1, that there are countably many classes of equivalence of $\sim$. Let us choose and enumerate class representatives, that is, $\mathbf{Q} - \{0\} = \bigcup_{n \in \mathbf{N}} [q_n]$ (where $[q_n]$ denotes the so-called class of equivalence of $\sim$ determined by q_n, that is, the set of all elements $\mathbf{Q} - \{0\}$ that are in the relation $\sim$ with q_n).

$A \subseteq \mathbf{N}$ is called **good** iff $\bigcup_{n \in A} [q_n] \cup \{0\}$ is closed under the operation $+$.

Lemma 2. *If $A \subseteq \mathbf{N}$ is good, then A is an infinite set.*

Proof: By the way of contradiction let us assume that A is finite. Pick $x \in \bigcup_{n \in A} [q_n]$. Since $2x = -(-2)^1 x$, $3x = x + 2x \in \bigcup_{n \in A} [q_n]$, $5x = 3x + 2x \in \bigcup_{n \in A} [q_n], \ldots, (2k+1)x \in \bigcup_{n \in A} [q_n]$ for every $k \in \mathbf{N}$. Since A is finite, there must be $k_0 < k_1$ and $n \in A$ so that $(2k_0 + 1)x \in [q_n]$ and $(2k_1 + 1)x \in [q_n]$, so $(2k_1 + 1)x \in [(2k_0 + 1)x]$, which contradicts Lemma 1. ∎

If $x \in \mathbf{Q} - \{0\}$ we say that the **type** of x is **even** if $x = \pm(-2)^z q_n$ and $|z|$ is even. Otherwise the type of x is **odd**.

For $A \subseteq \mathbf{N}$ good, let $S(A) = (V(A), B(A))$ be a countable STS defined as follows:

$V(A) = \bigcup_{n\in A}[q_n] \cup \{0, +\infty, -\infty\}$ is the set of elements of $S(A)$.

$B(A)$ is the set of blocks of $S(A)$. These blocks are of three types:

Type I: $\{x, y, z\}$ where $x \neq y$, $x \neq z$, $y \neq z$, $x \neq \pm\infty$, $y \neq \pm\infty$, $z \neq \pm\infty$, and $x + y + z = 0$.

Type II: $\{x, -2x, \infty\}$ where $x \neq 0$, $x \neq \pm\infty$. The sign of ∞ is determined by the type of x; if the type of x is even, then the sign of ∞ is $+$, otherwise it is $-$.

Type III: $\{0, +\infty, -\infty\}$.

It is easy to check that $S(A)$ is, indeed, a countable STS, so it is left to the reader.

Let $S = (V, B)$ be a STS. A configuration of four blocks b_1, b_2, b_3, b_4, from B is called a **quadrilateral** (or **Pasch configuration**) if each two blocks have exactly one element in common. It follows that the four blocks of a quadrilateral have exactly 6 different elements. Thus, a quadrilateral looks like this: $\{a_1, a_2, a_3\}$, $\{a_3, a_4, a_5\}$, $\{a_2, a_5, a_6\}$, $\{a_1, a_4, a_6\}$. A **complement** of a quadrilateral consists of a complement of each block relative to the 6 elements of the quadrilateral, that is, the complement of $\{a_1, a_2, a_3\}$, $\{a_3, a_4, a_5\}$, $\{a_2, a_5, a_6\}$, $\{a_1, a_4, a_6\}$ consists of $\{a_4, a_5, a_6\}$, $\{a_1, a_2, a_6\}$, $\{a_1, a_3, a_4\}$, $\{a_2, a_3, a_5\}$. We shall call the process of replacing a quadrilateral by its complement **inversion.** It is easy to see that if a quadrilateral of a STS $S = (V, B)$ is inverted, the result is again a STS with the same elements (but different blocks).

Lemma 3. *Let $A \subseteq \mathbf{N}$ be good. Then every quadrilateral of $S(A) = (V(A), B(A))$ has form $\{-x, x, 0\}, \{-2x, 2x, 0\}, \{-x, 2x, \infty\}, \{x, -2x, \infty\}$, where the sign of ∞ depends on the type of x.*

Proof: (1) *No quadrilateral contains 3 blocks of type I.*

By the way of contradiction let us assume that some quadrilateral does. Let $\{x, y, -(x+y)\}$, $\{x, z, -(x+z)\}$, and $\{y, s, -(x+z)\}$, be the 3 blocks of type I. Then $s + y - x - z = 0$. Also $\{z, s, -(x+y)\}$ must be a block (and, consequently, of type I), so $s + z - x - y = 0$. It follows that $y - x - z = z - x - y$ yielding $z = y$, a contradiction.

(2) *No quadrilateral contains 3 blocks of type II.*

By the way of contradiction let us assume that some quadrilateral does. Two of the 3 blocks of type II must share the same ∞. Assume it is $+\infty$ and the two blocks of type II are $\{x, -2x, +\infty\}$ and $\{y, -2y, +\infty\}$ (that is, the type of x is even, as well as the type of y). Then either $\{x, -2y, -\infty\}$ and $\{y, -2x, -\infty\}$ are the remaining blocks, or $\{x, y, -\infty\}$ and $\{-2x, -2y, -\infty\}$ are the remaining blocks. In the former case, $\{x, -2y, -\infty\}$ forces $-2y = -\frac{x}{2}$ as the type of x is even, and $\{y, -2x, -\infty\}$ forces $-2x = -\frac{y}{2}$ as the type of y is even, which leads to $x = y = 0$, a contradiction. The latter case $\{x, y, -\infty\}$ and $\{-2x, -2y, -\infty\}$ is impossible as the type of both x and y is even.

(3) *No quadrilateral contains the block* $\{0, +\infty, -\infty\}$.

By the way of contradiction let us assume that some quadrilateral does. Then $\{-x, x, 0\}$ must be one of the blocks of the quadrilateral. Assume that the type of x is even. Then $\{x, -2x, +\infty\}$ must be another block. The remaining block must be $\{-x, -2x, -\infty\}$ which is impossible since $\{-x, -2x, 3x\}$ is a block.

So a quadrilateral must consist of 2 blocks of type I and 2 blocks of type II.

Thus, there are two possibilities:

Either the two blocks of type II are $\{x, -2x, +\infty\}$ and $\{-2x, 4x, -\infty\}$, which forces the remaining two blocks of type I to be either $\{s, 4x, +\infty\}$ and $\{s, x, -\infty\}$ or $\{s, x, 4x\}$ and $\{s, -\infty, +\infty\}$, for some s, a contradiction.

Or the two blocks of type II are $\{x, -2x, \infty\}$ and $\{y, -2y, \infty\}$, which forces the remaining two blocks of type I to be either $\{s, x, -2y\}$ and $\{s, y, -2x\}$, or $\{s, x, y\}$ and $\{-2x, -2y, s\}$. The former case is not possible as it leads to $x = y = s$. The latter case is possible if and only if $s = 0$, $x = -y$, and $y = -x$. Thus, the quadrilateral has form $\{x, -x, 0\}$ $\{2x, -2x, 0\}$, $\{x, -2x, \infty\}$, $\{-x, 2x, \infty\}$. ∎.

Let $f: \mathbf{Z} \to \{0, 1\}$. If $f(z) = 1$, we shall call z a peek of f. In the following we shall define a certain set $\mathcal{F}$ of functions from $\mathbf{Z} \to \{0, 1\}$:

$f \in \mathcal{F}$ if $f: \mathbf{Z} \to \{0, 1\}$, and 0 is its peek, the first positive peek of f is 4 and the gaps between peeks are strictly increasing as $z \to +\infty$, the first negative peek of f is -4 and the gaps between peeks are strictly increasing as $z \to -\infty$. Moreover, we require that $f(n) \neq f(-n)$ for some $n > 4$, and for two distinct functions f, $g \in \mathcal{F}$, for some $n > 4$, $f(n) \neq g(n)$, for some $m > 4$, $f(-m) \neq g(m)$, and for some $k > 4$, $f(-k) \neq g(-k)$.

Note that if $f \in \mathcal{F}$, then the function has a well-defined "centre" at 0 (it is the only peek where the gap to the next peek on the left-hand side is the same as the gap on the right-hand side), f is not "symmetric", that is, the part of f for negative integers is not a mirror image of the part of f for positive integers, and if $g \in \mathcal{F}$ is distinct from f, then neither the positive nor the negative part of f is identical either with the positive part of g, or with the negative part of g.

The size of $\mathcal{F}$ is 2^ω. (To show that, first get a set of size 2^ω of distinct strictly increasing sequences of positive integers starting with $0, 4$. Then partition the set into two disjoint sets of size 2^ω and enumerate them: $\{s_\alpha : \alpha < 2^\omega\}$ and $\{s'_\alpha : \alpha < 2^\omega\}$. Each pair $\{s_\alpha, s'_\alpha\}$ defines one function g_α in the following way: let $s_\alpha = \{s_{\alpha,n} : n \in \mathbf{N}\}$, and let $s'_\alpha = \{s'_{\alpha,n} : n \in \mathbf{N}\}$. Define sequences $\{a_{\alpha,n} : n \in N\}$ and $\{a'_{\alpha,n} : n \in N\}$ by induction $a_{\alpha,0} = 0$, $a_{\alpha,n+1} = a_{\alpha,n} + s_{\alpha,n}$, and $a'_{\alpha,0} = 0$, $a'_{\alpha,n+1} = a'_{\alpha,n} - s'_{\alpha,n}$. Then $a_{\alpha,n} < \frac{a_{\alpha,n+1} + a_{\alpha,n-1}}{2}$ and $a'_{\alpha,n} > \frac{a'_{\alpha,n-1} + a'_{\alpha,n+1}}{2}$, $n \in \mathbf{N}$,

$n \geq 1$. For $z \in \mathbf{Z}$ define

$$g_\alpha(z) = \begin{cases} 1, & \text{if } z \geq 0 \text{ and } z = a_{\alpha,n} \text{ for some } n \in \mathbf{N}; \\ 1, & \text{if } z \leq 0 \text{ and } z = a'_{\alpha,n} \text{ for some } n \in \mathbf{N}; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that each g_α belongs to $\mathcal{F}$.)

We can partition $\mathcal{F}$ into 2^ω countable pairwise disjoint sets, that is, $\mathcal{F} = \bigcup_{\alpha<2^\omega} \mathcal{F}_\omega$ where $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ whenever $\alpha \neq \beta$, and for each α, $\mathcal{F}_\alpha$ is countable. Let $\mathcal{F}_\alpha = \{f_{\alpha,n} : n \in \mathbf{N}\}$ be a fixed enumeration of $\mathcal{F}_\alpha$.

In the following, we shall describe how to construct a countable STS $S(A)_\alpha = (V(A), B(A)_\alpha)$ for a good A and $\alpha < 2^\omega$.

Consider $S(A) = (V(A), B(A))$. Let $n \in A$. From the definition of $V(A)$ it follows that $[q_n] \subseteq V(A)$. Let $z \in \mathbf{Z}$. By $Q_n^A(z)$ we shall denote the quadrilateral $\{-x, x, 0\}, \{-2x, 2x, 0\}, \{-x, 2x, \infty\}, \{x, -2x, \infty\}$, where $x = |(-2)^z q_n|$ and where the sign of ∞ depends on the type of x. It is clear that $Q_n^A(z)$ and $Q_n^A(z+1)$ have exactly one block in common, $\{-2x, 2x, 0\}$. Thus, each class $[q_n]$ for $n \in A$ determines a **quadrilateral chain** $\{Q_n^A(z) : z \in \mathbf{Z}\}$ in which every quadrilateral $Q_n^A(z)$ is sharing with its left-hand side neighbour $Q_n^A(z-1)$ a block, and a different block with its right-hand side neighbour $Q_n^A(z+1)$. All quadrilaterals of $S(A)$ are located in these quadrilateral chains.

Below is a fragment of such quadrilateral chain:

$$\begin{array}{llllll}
.. \left\{-\frac{x}{4}, \frac{x}{4}, 0\right\} & .. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, 0\} & .. \{-2x, 2x, 0\} & .. \{-4x, 4x, 0\} & .. \\
.. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, 0\} & .. \{-2x, 2x, 0\} & .. \{-4x, 4x, 0\} & .. \{-8x, 8x, 0\} & .. \\
.. \left\{-\frac{x}{4}, \frac{x}{2}, +\infty\right\} & .. \left\{-\frac{x}{2}, x, -\infty\right\} & .. \{-x, 2x, +\infty\} & .. \{-2x, 4x, -\infty\} & .. \{-4x, 8x, +\infty\} & .. \\
.. \left\{\frac{x}{4}, -\frac{x}{2}, +\infty\right\} & .. \left\{\frac{x}{2}, -x, -\infty\right\} & .. \{x, -2x, +\infty\} & .. \{2x, -4x, -\infty\} & .. \{4x, -8x, +\infty\} & ..
\end{array}$$

The blocks of $S(A)_\alpha$ are constructed from the blocks of $S(A)$. Let $n \in A$, and let $f_{\alpha,n} \in \mathcal{F}_\alpha$. For every quadrilateral $Q_n^A(z)$ such that $f_{\alpha,n}(z) = 1$, we shall invert the quadrilateral $Q_n^A(z)$. It is clear that $S(A)_\alpha$ is again a countable STS with the same elements as $S(A)$, but different blocks.

Let us see what happens when a quadrilateral is inverted. In the quadrilateral chain displayed above, let us invert the quadrilateral $\{-x, x, 0\}$, $\{-2x, 2x, 0\}$, $\{-x, 2x, +\infty\}$, $\{x, -2x, +\infty\}$.

$$\begin{array}{llllll}
.. \left\{-\frac{x}{4}, \frac{x}{4}, 0\right\} & .. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, +\infty\} & .. \{-2x, 2x, +\infty\} & .. \{-4x, 4x, 0\} & .. \\
.. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, +\infty\} & .. \{-2x, 2x, +\infty\} & .. \{-4x, 4x, 0\} & .. \{-8x, 8x, 0\} & .. \\
.. \left\{-\frac{x}{4}, \frac{x}{2}, +\infty\right\} & .. \left\{-\frac{x}{2}, x, -\infty\right\} & .. \{-x, 2x, 0\} & .. \{-2x, 4x, -\infty\} & .. \{-4x, 8x, +\infty\} & .. \\
.. \left\{\frac{x}{4}, -\frac{x}{2}, +\infty\right\} & .. \left\{\frac{x}{2}, -x, -\infty\right\} & .. \{x, -2x, 0\} & .. \{2x, -4x, -\infty\} & .. \{4x, -8x, +\infty\} & ..
\end{array}$$

The middle quadrilateral is inverted, the leftmost and the rightmost quadrilaterals are preserved (we shall refer to them as old quadrilaterals), but the left-hand side neighbour and the right-hand side neighbour of the inverted quadrilateral were destroyed, that is, they do not form quadrilaterals any more (we shall refer to them as destroyed quadrilaterals). Thus, inverting a quadrilateral punches two holes in the quadrilateral chain, as the inverted quadrilateral has no block in common with any other quadrilateral from the quadrilateral chain. It is possible that some new quadrilaterals outside of quadrilateral chains may be introduced (and indeed they are — we shall refer to them as new quadrilaterals). The three lemmas following are presented in order to make sure that we know the structure of all possible quadrilaterals in $S(A)_\alpha$.

Lemma 4. *Let $A \subseteq \mathrm{N}$ be good and let $\alpha < 2^\omega$. No quadrilateral of $S(A)_\alpha$ can contain as elements both $+\infty$ and $-\infty$.*

Proof: By the way of contradiction let us assume that we have such a quadrilateral Q.

(1) Assume that $\{0,+\infty,-\infty\}$ is a part of Q. Q has form $\{0,+\infty,-\infty\}$, $\{a,b,0\}$, $\{a,c,-\infty\}$, $\{c,b,+\infty\}$. Since $\{a,b,O\}$ is a block, there are three possibilities: (i) $a=-b$, (ii) $a=-2b$, and (iii) $a=-\frac{b}{2}$.

(i) $a=-b$.
Then Q has form $\{0,+\infty,-\infty\}, \{-b,b,0\}, \{-b,c,-\infty\}, \{c,b,+\infty\}$. Since $\{-b,b,0\}$ is a block, the quadrilateral determined by b is either old or destroyed. In any case $\{-b,2b,\infty\}$, $\{b,-2b,\infty\}$ are blocks. If the type of b is even, then $\{-b,2b,+\infty\}$, $\{b,-2b,+\infty\}$ are blocks. Then b and $+\infty$ determine $c = -2b$, and so $\{-b,c,-\infty\} = \{-b,-2b,-\infty\}$, a contradiction as $\{-b,-2b,3b\}$ is a block. So the type of b must be odd. Then $\{-b,-2b,-\infty\}$ $\{b,-2b,-\infty\}$ are blocks. Then $-b$ and $-\infty$ determine $c=2b$, and so $\{c,b,+\infty\} = \{b,2b,+\infty\}$, a contradiction as $\{b,2b,-3b\}$ is a block.

(ii) $a=-2b$.
Q has form $\{0,+\infty,-\infty\}, \{b,-2b,0\}, \{c,-2b,-\infty\}, \{c,b,+\infty\}$. Since $\{b,-2b,0\}$ is a block, the quadrilateral determined by b must be inverted. If the type of b is even, then the inverted quadrilateral determined by b is $\{-b,b,+\infty\}, \{-2b,2b,+\infty\}, \{-b,2b,0\}, \{b,-2b,0\}$. Hence, b and $+\infty$ determine $c = -b$, and so $\{c,-2b,-\infty\} = \{-b, -2b, -\infty\}$, which is a contradiction as $\{-b,-2b,3b\}$ is a block. Thus, the type of b must be odd. So the inverted quadrilateral determined by b is $\{-b,b,-\infty\}$, $\{-2b,2b,-\infty\}$, $\{-b,2b,0\}$, $\{b,-2b,0\}$. Hence, $-2b$ and $-\infty$ determine $c = 2b$, and so $\{0,+\infty,-\infty\}$, $\{b,-2b,0\}$, $\{c,b,+\infty\} = \{b,2b,+\infty\}$, which is a contradiction since $\{b,2b,-3b\}$ is a block.

(iii) $a=-\frac{b}{2}$.
Then $b=-2a$ and we can proceed as in case (ii).

(2) Hence, Q has form $\{a, b, +\infty\}, \{c, d, +\infty\}, \{c, b, -\infty\}, \{a, d, -\infty\}$, where $a \neq 0$, $b \neq 0$, $c \neq 0$ and $d \neq 0$. Since $\{a, b, +\infty\}$ is a block, there are three possibilities: (i) $a = -b$, (ii) $a = -2b$, and (iii) $a = \frac{b}{2}$.

(i) $a = -b$.

Then Q has form $\{-b, b, +\infty\}, \{c, d, +\infty\}, \{c, b, -\infty\}, \{-b, d, -\infty\}$. If the type of b is even, then b determines an inverted quadrilateral, while $\frac{b}{2}$ determines a destroyed quadrilateral:

$$\begin{array}{llll} .. & \{-\frac{b}{2}, \frac{b}{2}, 0\} & .. \{-b, b, +\infty\} & .. \\ .. & \{-b, b, +\infty\} & .. \{-2b, 2b, +\infty\} & .. \\ .. & \{-\frac{b}{2}, b, -\infty\} & .. \{-b, 2b, 0\} & .. \\ .. & \{\frac{b}{2}, -b, -\infty\} & .. \{b, -2b, 0\} & .. \end{array}$$

It follows that b and $-\infty$ determine $c = -\frac{b}{2}$, and $-b$ and $-\infty$ determine $d = \frac{b}{2}$. So $\{c, d, +\infty\} = \{-\frac{b}{2}, \frac{b}{2}, +\infty\}$, a contradiction as $\{-\frac{b}{2}, \frac{b}{2}, 0\}$ is a block. Thus, the type of b must be odd. It follows that the quadrilateral determined by $\frac{b}{2}$ is inverted, while the quadrilateral determined by b is destroyed:

$$\begin{array}{llll} .. & \{-\frac{b}{2}, \frac{b}{2}, +\infty\} & .. \{-b, b, +\infty\} & .. \\ .. & \{-b, b, +\infty\} & .. \{-2b, 2b, 0\} & .. \\ .. & \{-\frac{b}{2}, b, 0\} & .. \{-b, 2b, -\infty\} & .. \\ .. & \{\frac{b}{2}, -b, 0\} & .. \{b, -2b, -\infty\} & .. \end{array}$$

Then b and $-\infty$ determine $c = -2b$, and $-b$ and $-\infty$ determine $d = 2b$. So $\{c, d, +\infty\} = \{-2b, 2b, +\infty\}$, a contradiction as $\{-2b, 2b, 0\}$ is a block.

(ii) $a = -2b$.

Then Q has form $\{b, -2b, +\infty\}$, $\{c, d, +\infty\}$, $\{c, b, -\infty\}$, $\{-2b, d, -\infty\}$. Since $\{b, -2b, +\infty\}$ is a block, the type of b is even and b determines either an old quadrilateral or a destroyed one. If b determines an old quadrilateral, it has to be $\{-b, b, 0\}, \{-2b, 2b, 0\}, \{-b, 2b, +\infty\}, \{b, -2b, +\infty\}$. $\frac{b}{2}$ determines either a destroyed quadrilateral or an old one, but in either case $\{-\frac{b}{2}, b, -\infty\}$ and $\{\frac{b}{2}, -b, -\infty\}$ are blocks. $2b$ determines either a destroyed quadrilateral or an old one, but in either case $\{-2b, 4b, -\infty\}$ and $\{2b, -4b, -\infty\}$ are blocks. Hence, b and $-\infty$ determine $c = -\frac{b}{2}$, and $-2b$ and $-\infty$ determine $d = 4b$. So $\{c, d, +\infty\} = \{-\frac{b}{2}, 4b, +\infty\}$, a contradiction as $\{-\frac{b}{2}, 4b, -\frac{7b}{2}\}$ is a block. Thus, b must determine a destroyed quadrilateral. Then $2b$ must determine a quadrilateral. If it is an old one:

$$\begin{array}{llll} .. & \{-b, b, -\infty\} & .. \{-2b, 2b, 0\} & .. \\ .. & \{-2b, 2b, 0\} & .. \{-4b, 4b, 0\} & .. \\ .. & \{-b, 2b, +\infty\} & .. \{-2b, 4b, -\infty\} & .. \\ .. & \{b, -2b, +\infty\} & .. \{2b, -4b, -\infty\} & .. \end{array}$$

Then b and $-\infty$ determine $c = -b$, and $-2b$ and $-\infty$ determine $d = 4b$. So $\{c, d, +\infty\} = \{-b, 4b, +\infty\}$, a contradiction as $\{-b, 4b, -3b\}$ is a block. Hence, $2b$ must determine an inverted quadrilateral and, consequently, $\frac{b}{2}$ must determine an old quadrilateral:

$$\begin{array}{llll} .. \{-\frac{b}{2}, \frac{b}{2}, 0\} & .. \{-b, b, 0\} & .. \{-2b, 2b, -\infty\} & .. \\ .. \{-b, b, 0\} & .. \{-2b, 2b, -\infty\} & .. \{-4b, 4b, -\infty\} & .. \\ .. \{-\frac{b}{2}, b, -\infty\} & .. \{-b, 2b, +\infty\} & .. \{-2b, 4b, 0\} & .. \\ .. \{\frac{b}{2}, -b, -\infty\} & .. \{b, -2b, +\infty\} & .. \{2b, -4b, 0\} & .. \end{array}$$

Then b and $-\infty$ determine $c = -\frac{b}{2}$, and $-2b$ and $-\infty$ determine $d = 2b$. So $\{c, d, +\infty\} = \{-\frac{b}{2}, 2b, +\infty\}$, a contradiction as $\{-\frac{b}{2}, 2b, -\frac{5b}{2}\}$ is a block.

(iii) $a = -\frac{b}{2}$.
Then $b = -2a$ and we can proceed as in (ii).

This completes the proof of Lemma 4. ∎

Lemma 5. *Let $A \subseteq \mathbb{N}$ be good and let $\alpha < 2^\omega$. Then no new quadrilateral of $S(A)_\alpha$ can contain as elements both 0 and ∞.*

Proof: By Lemma 4, 0 and ∞ cannot be in the same block of a quadrilateral. Thus, by the way of contradiction let us assume the existence of a new quadrilateral Q of the form $\{a, b, 0\}$, $\{c, d, 0\}$, $\{a, d, +\infty\}$, $\{c, b, +\infty\}$. Since $\{a, b, 0\}$, is a block, there are three possibilities: $a = -b$, $a = -2b$, and $a = -\frac{b}{2}$.

(i) $a = -b$.
So Q has form $\{-b, b, 0\}$, $\{c, d, 0\}$, $\{-b, d, +\infty\}$, $\{c, b, +\infty\}$. Since $\{-b, b, 0\}$ is a block, the quadrilateral determined by b is either old or destroyed.

If b determines an old quadrilateral, $\{-b, b, 0\}$, $\{-2b, 2b, 0\}$, $\{-b, 2b, +\infty\}$, $\{b, -2b, +\infty\}$ (that is, if the type of b is even), then b and $+\infty$ determine $c = -2b$, and $-b$ and $+\infty$ determine $d = 2b$, hence, Q has form $\{-b, b, 0\}$, $\{-2b, 2b, 0\}$, $\{-b, 2b, +\infty\}$, $\{-2b, b, +\infty\}$ which is an old quadrilateral, a contradiction. If b determines an old quadrilateral $\{-b, b, 0\}$, $\{-2b, 2b, 0\}$, $\{-b, 2b, -\infty\}$, $\{b, -2b, -\infty\}$ (that is, if the type of b is odd), then $\{-\frac{b}{2}, b, +\infty\}$ and $\{\frac{b}{2}, -b, +\infty\}$ are blocks and b and $+\infty$ determine $c = -\frac{b}{2}$, and $-b$ and $+\infty$ determine $d = \frac{b}{2}$, hence, Q has form $\{-\frac{b}{2}, \frac{b}{2}, 0\}$, $\{-b, b, 0\}$ $\{-\frac{b}{2}, b, +\infty\}$ $\{-\frac{b}{2}, b, +\infty\}$, which is an old quadrilateral, a contradiction.

If b determines a destroyed quadrilateral $\{-b, b, 0\}$, $\{-2b, 2b, -\infty\}$, $\{-b, 2b, +\infty\}$, $\{b, -2b, +\infty\}$ (that is, if the type of b is even), then b and $+\infty$ determine $c = -2b$ and $-b$ and $+\infty$ determine $d = 2b$, so Q has form $\{-b, b, 0\}$, $\{-2b, 2b, 0\}$, $\{-b, 2b, +\infty\}$, $\{b, -2b, +\infty\}$, which is an old quadrilateral, a contradiction.

If b determines a destroyed quadrilateral $\{-b,b,0\}$, $\{-2b,2b,+\infty\}$, $\{-b,2b,-\infty\}$, $\{b,-2b,-\infty\}$ (that is, if the type of b is odd), then $\frac{b}{2}$ must determine an old quadrilateral $\{-\frac{b}{2},\frac{b}{2},0\}$, $\{-b,b,0\}$, $\{-\frac{b}{2},b,+\infty\}$, $\{\frac{b}{2},-b,+\infty\}$. Then b and $+\infty$ determine $c=-\frac{b}{2}$ and $-b$ and $+\infty$ determine $d=\frac{b}{2}$. So Q has form $\{-\frac{b}{2},\frac{b}{2},0\}$, $\{-b,b,0\}$, $\{-\frac{b}{2},b,+\infty\}$, $\{\frac{b}{2},-b,+\infty\}$, which is an old quadrilateral, a contradiction.

(ii) $a=-2b$.
So Q has form is $\{b,2b,0\}$, $\{c,d,0\}$, $\{c,b,+\infty\}$, $\{d,-2b,+\infty\}$. Since $\{b,-2b,0\}$ is a block, b must determine an inverted quadrilateral.

If b determines the quadrilateral $\{-b,b,+\infty\}$, $\{-2b,2b,+\infty\}$, $\{-b,2b,0\}$, $\{b,-2b,0\}$ (that is, if the type of b is even), then b and $+\infty$ determine $c=-b$, and $-2b$ and $+\infty$ determine $d=2b$. So Q has form $\{-b,b,+\infty\}$, $\{-2b,2b,+\infty\}$, $\{-b,2b,0\}$, $\{b,-2b,0-\}$, which is an inverted quadrilateral, a contradiction.

If b determines the quadrilateral $\{-b,b,-\infty\}$, $\{-2b,2b,-\infty\}$, $\{-b,2b,0\}$, $\{b,-2b,0\}$ (that is, if the type of b is odd), then $\{-\frac{b}{2},b,+\infty\}$ and $\{\frac{b}{2},-b,+\infty\}$ are blocks as well as $\{-2b,2b,+\infty\}$ and $\{2b,-2b,+\infty\}$. Thus, b and $+\infty$ determine $c=-\frac{b}{2}$ and $-2b$ and $+\infty$ determine $d=4b$. Hence, $\{c,d,0\}=\{-\frac{b}{2},4b,0\}$, a contradiction as $\{-\frac{b}{2},4b,-\frac{7b}{2}\}$ is a block.

(iii) $a=-\frac{b}{2}$.
Then $b=-2a$ and we can proceed as in (ii).

This completes the proof of Lemma 5. ∎

Lemma 6. *Let $A\subseteq \mathrm{N}$ be good and let $\alpha<2^{\omega}$. Then every new quadrilateral of $S(A)_\alpha$ has one of the following forms: either $\{-x,x,\infty\}$, $\{x,-\frac{x}{3},-\frac{2x}{3}\}$, $\{-x,-\frac{x}{3},\frac{4x}{3}\}$, $\{-\frac{2x}{3},\frac{4x}{3},\infty\}$ or $\{-x,2x,0\}$, $\{2x,-\frac{x}{2},-\frac{3x}{2}\}$, $\{-x,-\frac{x}{2},\frac{3x}{2}\}$, $\{-\frac{3x}{2},\frac{3x}{2},0\}$ for some $x\in V(A)-\{0,+\infty,-\infty\}$.*

Proof: Let Q be a new quadrilateral in $S(A)_\alpha$. Then one of the blocks of Q must be new. There are two possibilities:

(1) Q contains a block of type $\{-x,x,\infty\}$ for some $x\in V(A)-\{0,+\infty,-\infty\}$.

Then Q has form $\{-x,x,\infty\}$, $\{x,a,b\}$, $\{-x,a,c\}$, $\{c,b,\infty\}$. By Lemma 4 and Lemma 5, $a,b,c\in V(A)-\{0,+\infty,-\infty\}$. Since $\{c,b,\infty\}$ is a block, it must be one of two types: (i) $\{-y,y,\infty\}$, or (ii) $\{-y,2y,\infty\}$ for some $y\in V(A)-\{0,+\infty,-\infty\}$.

(i) Then Q has form $\{-x,x,\infty\}$, $\{x,a,-y\}$, $\{-x,a,y\}$, $\{-y,y,\infty\}$. It follows that $x+a-y=-x+a+y$, and so $y=x$, a contradiction.

(ii) Then Q either has form $\{-x,x,\infty\}$, $\{x,a,-y\}$, $\{-x,a,2y\}$, $\{-y,2y,\infty\}$, or $\{-x,x,\infty\}$, $\{x,a,2y\}$, $\{-x,a,-y\}$, $\{-y,2y,\infty\}$. In the former case, it follows that $x+a-y=-x+a+2y$, and so

$y = \frac{2x}{3}, a = -\frac{x}{3}$. Henceforth, Q has form $\{-x, x, \infty\}, \{x, -\frac{x}{3}, -\frac{2x}{3}\}$, $\{-x, -\frac{x}{3}, \frac{4x}{3}\}, \{-\frac{2x}{3}, \frac{4x}{3}, \infty\}$. In the latter case, it follows that $x + a - 2y = -x + a - y$, and so $y = -\frac{2x}{3}$, $a = \frac{x}{3}$. Henceforth, Q has form $\{-x, x, \infty\}, \{x, \frac{x}{3}, -\frac{4x}{3}\}$ $\{-x, \frac{x}{3}, \frac{2x}{3}\}, \{\frac{2x}{3}, -\frac{4x}{3}, \infty\}$.

(2) Q contains a block of type $\{-x, 2x, 0\}$ for some $x \in V(A) - \{0, +\infty, -\infty\}$.
Then Q has form $\{-x, 2x, 0\}, \{2x, a, b\}, \{-x, a, c\}, \{c, b, 0\}$. By Lemma 4 and Lemma 5, $a, b, c \in V(A) - \{0, +\infty, -\infty\}$. Since $\{c, b, 0\}$ is a block, it must be one of two types: (i) $\{-y, y, 0\}$, or (ii) $\{-y, 2y, 0\}$ for some $y \in V(A) - \{0, +\infty, -\infty\}$.
   (i) Then Q has form $\{-x, 2x, 0\}, \{2x, a, -y\}, \{-x, a, y\}, \{-y, y, 0\}$. It follows that $2x + a - y = -x + a + y$, and so $y = \frac{3x}{2}$ and $a = -\frac{x}{2}$. Henceforth, Q has form $\{-x, 2x, 0\}, \{2x, -\frac{x}{2}, -\frac{3x}{2}\}, \{-x, -\frac{x}{2}, \frac{3x}{2}\}$, $\{-\frac{3x}{2}, \frac{3x}{2}, 0\}$.
   (ii) Then Q either has form $\{-x, 2x, 0\}$, $\{2x, a, -y\}$, $\{-x, a, 2y\}$, $\{-y, 2y, 0\}$, or $\{-x, 2x, 0\}, \{2x, a, 2y\}, \{-x, a, -y\}, \{-y, 2y, 0\}$. In the former case, it follows that $2x + a - y = -x + a + 2y$, and so $y = x$, a contradiction. In the latter case, it follows that $2x + a + 2y = -x + a - y$, and so $y = -x$, $a = 0$, a contradiction.

This completes the proof of Lemma 6. ∎

Lemma 7. *Let $A \subseteq \mathbf{N}$ be good and let $\alpha < 2^{\omega}$. Let $S = (V, B)$ be a non-trivial subsystem of $S(A)_\alpha$. Then there is a good $A' \subseteq A$ so that $S = S(A')_\alpha$.*

In particular, $S(A)_\alpha$ does not have a finite subsystem other than a block.

Proof: The proof will be carried through a sequence of claims.

(1) *Let $a, b, c \in V - \{0, +\infty, -\infty\}$ be all positive or negative so that $0 < |a| < |b| < |c|$. Then $-2a \in V$.*

Assume that $0 < a < b < c$.
Since $a, b \in V$, the block $\{a, b, -(a + b)\} \in B$, hence $-(a + b) \in V$.
Since $a, c \in V$, the block $\{a, c, -(a + c)\} \in B$, hence $-(a + c) \in V$.
Since $b, c \in V$, the block $\{b, c, -(b + c)\} \in B$, hence $-(b + c) \in V$.
Since $-(a+b), -(a+c) \in V$, the block $\{-(a+b), -(a+c), 2a+b+c\} \in B$, hence $2a + b + c \in V$.
If $2(b+c) = 2a+b+c$ then $a = \frac{b+c}{2}$, which is impossible as $a = \frac{b+c}{2}$ yields that $c < b$.
If $2(2a+b+c) = b+c$ then $4a = -b-c$, which is impossible as $a > 0$.

Hence, the pair $\{2a + b + c, -(b + c)\}$ (both elements are in V) must be covered by the block $\{-(2a + b + c), -(b + c), -2a\}$ which must be in B, and so $-2a \in V$.

(2) *Let* $a, b, c \in V - \{0, +\infty, -\infty\}$ *be all positive or negative. Then* $\{0, +\infty, -\infty\} \subseteq V$.

Assume that $0 < a < b < c$.

By (1), $-2a \in V$. As in (1) we can show that $-(a+c), -(a+b) \in V$. Since $-(a+c) < -(a+b) < -2a < 0$, applying (1) again we get that $4a \in V$.

Thus, $a, -2a, 4a \in V$. Let us assume that the type of a is even. There are five possibilities.

(a) Both a and $2a$ determine old quadrilaterals in $S(A)_\alpha$.

$$\begin{array}{llll} .. \{-a, a, 0\} & .. \{-2a, 2a, 0\} & .. \\ .. \{-2a, 2a, 0\} & .. \{-4a, 4a, 0\} & .. \\ .. \{-a, 2a, +\infty\} & .. \{-2a, 4a, -\infty\} & .. \\ .. \{a, -2a, +\infty\} & .. \{2a, -4a, -\infty\} & .. \end{array}$$

Since $a, -2a \in V$, the block $\{a, -2a, +\infty\} \in B$, and so $+\infty \in V$.
Since $-2a, 4a \in V$, the block $\{-2a, 4a, -\infty\} \in B$, and so $-\infty \in V$.
Since $-\infty, +\infty \in V$, the block $\{0, +\infty, -\infty\} \in B$, and so $0 \in V$.

(b) a determines an old quadrilateral in $S(A)_\alpha$ while $2a$ determines a destroyed quadrilateral.

$$\begin{array}{llll} .. \{-a, a, 0\} & .. \{-2a, 2a, 0\} & .. \\ .. \{-2a, 2a, 0\} & .. \{-4a, 4a, +\infty\} & .. \\ .. \{-a, 2a, +\infty\} & .. \{-2a, 4a, -\infty\} & .. \\ .. \{a, -2a, +\infty\} & .. \{2a, -4a, -\infty\} & .. \end{array}$$

Since $a, -2a \in V$, the block $\{a, -2a, +\infty\} \in B$, and so $+\infty \in V$.
Since $-2a, 4a \in V$, the block $\{-2a, 4a, -\infty\} \in B$, and so $-\infty \in V$.
Since $-\infty, +\infty \in V$, the block $\{0, +\infty, -\infty\} \in B$, and so $0 \in V$.

(c) a determines an old quadrilateral in $S(A)_\alpha$, while $2a$ determines an old quadrilateral.

$$\begin{array}{llll} .. \{-a, a, -\infty\} & .. \{-2a, 2a, 0\} & .. \\ .. \{-2a, 2a, 0\} & .. \{-4a, 4a, 0\} & .. \\ .. \{-a, 2a, +\infty\} & .. \{-2a, 4a, -\infty\} & .. \\ .. \{a, -2a, +\infty\} & .. \{2a, -4a, -\infty\} & .. \end{array}$$

Since $a, -2a \in V$, the block $\{a, -2a, +\infty\} \in B$, and so $+\infty \in V$.
Since $-2a, 4a \in V$, the block $\{-2a, 4a, -\infty\} \in B$, and so $-\infty \in V$.
Since $-\infty, +\infty \in V$, the block $\{0, +\infty, -\infty\} \in B$, and so $0 \in V$.

(d) a determines an inverted quadrilateral in $S(A)_\alpha$, while $2a$ determines a destroyed quadrilateral.

$$\begin{array}{llll} .. \{-a, a, +\infty\} & .. \{-2a, 2a, +\infty\} & .. \\ .. \{-2a, 2a, +\infty\} & .. \{-4a, 4a, 0\} & .. \\ .. \{-a, 2a, 0\} & .. \{-2a, 4a, -\infty\} & .. \\ .. \{a, -2a, 0\} & .. \{2a, -4a, -\infty\} & .. \end{array}$$

Since $a, -2a \in V$, the block $\{a, -2a, 0\} \in B$, and so $0 \in V$.
Since $-2a, 4a \in V$, the block $\{-2a, 4a, -\infty\} \in B$, and so $-\infty \in V$.
Since $-\infty, 0 \in V$, the block $\{0, +\infty, -\infty\} \in B$, and so $+\infty \in V$.

(e) a determines a destroyed quadrilateral in $S(A)_\alpha$, while $2a$ determines an inverted quadrilateral.

$$\begin{array}{llll} .. \{-a, a, 0\} & .. \{-2a, 2a, -\infty\} & .. \\ .. \{-2a, 2a, -\infty\} & .. \{-4a, 4a, -\infty\} & .. \\ .. \{-a, 2a, +\infty\} & .. \{-2a, 4a, 0\} & .. \\ .. \{a, -2a, +\infty\} & .. \{2a, -4a, 0\} & .. \end{array}$$

Since $a, -2a \in V$, the block $\{a, -2a, +\infty\} \in B$, and so $+\infty \in V$.
Since $-2a, 4a \in V$, the block $\{-2a, 4a, 0\} \in B$, and so $0 \in V$.
Since $+\infty, 0 \in V$, the block $\{0, +\infty, -\infty\} \in B$, and so $-\infty \in V$.

(3) *Let* $\{0, +\infty, -\infty\} \subseteq V$. *Then for every* $x \in V - \{0, +\infty, -\infty\}$, $[x] \subseteq V$.
It suffices to show that $-x, \frac{x}{2}, -\frac{x}{2}, 2x, -2x \in V$. Let us assume that the type of x is even. There are five possibilities.

(a) Both $\frac{x}{2}$ and x determine old quadrilaterals in $S(A)_\alpha$.

$$\begin{array}{llll} .. \{-\frac{x}{2}, \frac{x}{2}, 0\} & .. \{-x, x, 0\} & .. \\ .. \{-x, x, 0\} & .. \{-2x, 2x, 0\} & .. \\ .. \{-\frac{x}{2}, x, -\infty\} & .. \{-x, 2x, +\infty\} & .. \\ .. \{\frac{x}{2}, -x, -\infty\} & .. \{x, -2x, +\infty\} & .. \end{array}$$

Since $x, 0 \in V$, the block $\{-x, x, 0\} \in B$, and so $-x \in V$.
Since $-x, -\infty \in V$, the block $\{\frac{x}{2}, -x, -\infty\} \in B$, and so $\frac{x}{2} \in V$.
Since $x, -\infty \in V$, the block $\{-\frac{x}{2}, x, -\infty\} \in B$, and so $-\frac{x}{2} \in V$.
Since $-x, +\infty \in V$, the block $\{-x, 2x, +\infty\} \in B$, and so $2x \in V$.
Since $x, +\infty \in V$, the block $\{x, -2x, +\infty\} \in B$, and so $-2x \in V$.

(b) $\frac{x}{2}$ determines an old quadrilateral in $S(A)_\alpha$, while x determines a destroyed quadrilateral.

$$\begin{array}{llll} .. \{-\frac{x}{2}, \frac{x}{2}, 0\} & .. \{-x, x, 0\} & .. \\ .. \{-x, x, 0\} & .. \{-2x, 2x, -\infty\} & .. \\ .. \{-\frac{x}{2}, x, -\infty\} & .. \{-x, 2x, +\infty\} & .. \\ .. \{\frac{x}{2}, -x, -\infty\} & .. \{x, -2x, +\infty\} & .. \end{array}$$

Since $x, 0 \in V$, the block $\{-x, x, 0\} \in B$, and so $-x \in V$.
Since $-x, -\infty \in V$, the block $\{\frac{x}{2}, -x, -\infty\} \in B$, and so $\frac{x}{2} \in V$.
Since $x, -\infty \in V$, the block $\{-\frac{x}{2}, x, -\infty\} \in B$, and so $-\frac{x}{2} \in V$.
Since $-x, +\infty \in V$, the block $\{-x, 2x, +\infty\} \in B$, and so $2x \in V$.
Since $x, +\infty \in V$, the block $\{x, -2x, +\infty\} \in B$, and so $-2x \in V$.

(c) $\frac{x}{2}$ determines a destroyed quadrilateral in $S(A)_\alpha$, while x determines an old quadrilateral.

$$\begin{array}{llllll} .. & \{-\frac{x}{2}, \frac{x}{2}, +\infty\} & .. & \{-x, x, 0\} & .. \\ .. & \{-x, x, 0\} & .. & \{-2x, 2x, 0\} & .. \\ .. & \{-\frac{x}{2}, x, -\infty\} & .. & \{-x, 2x, +\infty\} & .. \\ .. & \{\frac{x}{2}, -x, -\infty\} & .. & \{x, -2x, +\infty\} & .. \end{array}$$

Since $x, 0 \in V$, the block $\{-x, x, 0\} \in B$, and so $-x \in V$.
Since $-x, -\infty \in V$, the block $\{\frac{x}{2}, -x, -\infty\} \in B$, and so $\frac{x}{2} \in V$.
Since $x, -\infty \in V$, the block $\{-\frac{x}{2}, x, -\infty\} \in B$, and so $-\frac{x}{2} \in V$.
Since $-x, +\infty \in V$, the block $\{-x, 2x, +\infty\} \in B$, and so $2x \in V$.
Since $x, +\infty \in V$, the block $\{x, -2x, +\infty\} \in B$, and so $-2x \in V$.

(d) $\frac{x}{2}$ determines an inverted quadrilateral in $S(A)_\alpha$, while x determines a destroyed quadrilateral.

$$\begin{array}{llllll} .. & \{-\frac{x}{2}, \frac{x}{2}, -\infty\} & .. & \{-x, x, -\infty\} & .. \\ .. & \{-x, x, -\infty\} & .. & \{-2x, 2x, 0\} & .. \\ .. & \{-\frac{x}{2}, x, 0\} & .. & \{-x, 2x, +\infty\} & .. \\ .. & \{\frac{x}{2}, -x, 0\} & .. & \{x, -2x, +\infty\} & .. \end{array}$$

Since $x, -\infty \in V$, the block $\{-x, x, -\infty\} \in B$, and so $-x \in V$.
Since $-x, 0 \in V$, the block $\{\frac{x}{2}, -x, 0\} \in B$, and so $\frac{x}{2} \in V$.
Since $x, 0 \in V$, the block $\{-\frac{x}{2}, x, 0\} \in B$, and so $-\frac{x}{2} \in V$.
Since $-x, +\infty \in V$, the block $\{-x, 2x, +\infty\} \in B$, and so $2x \in V$.
Since $x, +\infty \in V$, the block $\{x, -2x, +\infty\} \in B$, and so $-2x \in V$.

(e) $\frac{x}{2}$ determines a destroyed quadrilateral in $S(A)_\alpha$, while x determines an inverted quadrilateral.

$$\begin{array}{llllll} .. & \{-\frac{x}{2}, \frac{x}{2}, 0\} & .. & \{-x, x, +\infty\} & .. \\ .. & \{-x, x, +\infty\} & .. & \{-2x, 2x, +\infty\} & .. \\ .. & \{-\frac{x}{2}, x, -\infty\} & .. & \{-x, 2x, 0\} & .. \\ .. & \{\frac{x}{2}, -x, -\infty\} & .. & \{x, -2x, 0\} & .. \end{array}$$

Since $x, +\infty \in V$, the block $\{-x, x, +\infty\} \in B$, and so $-x \in V$.
Since $-x, -\infty \in V$, the block $\{\frac{x}{2}, -x, -\infty\} \in B$, and so $\frac{x}{2} \in V$.
Since $x, -\infty \in V$, the block $\{-\frac{x}{2}, x, -\infty\} \in B$, and so $-\frac{x}{2} \in V$.
Since $-x, 0 \in V$, the block $\{x, -2x, 0\} \in B$, and so $-2x \in V$.
Since $x, 0 \in V$, the block $\{x, -2x, 0\} \in B$, and so $-2x \in V$.

(4) *There are* $a,b,c \in V - \{0,+\infty,-\infty\}$ *so that either they are all positive, or they are all negative.*

Let $V^- = \{x \in V - \{0,+\infty,-\infty\}: x < 0\}$ and let $V^+ = \{x \in V - \{0,+\infty,-\infty\}: x > 0\}$. Since $S = (V,B)$ is a non-trivial subsystem of $S(A)_\alpha$, $|V| \geq 7$.

(a) Assume that both $+\infty, -\infty \in V$.
Then $\{0,+\infty,-\infty\} \in B$, and so $0 \in V$. Since $|V - \{0,+\infty,-\infty\}| \geq 4$, by (3) $[x] \subseteq V$ for any $x \in V - \{0,+\infty,-\infty\}$. It follows that either V^- is infinite, or V^+ is infinite.

(b) Either $-\infty$ or $+\infty \notin V$.
It follows that $|V - \{0,+\infty,-\infty\}| \geq 5$. Then either $|V^-| \geq 3$, or $|V^+| \geq 3$.

Now we are ready to conclude the proof of the lemma. By (4) there are $a, b, c \in V - \{0, +\infty, -\infty\}$, so that either all of them are positive, or all of them are negative. By (2) $\{0, +\infty, -\infty\} \subseteq V$. By (3) $[x] \subseteq V$ for every $x \in V - \{0,+\infty,-\infty\}$. Define $A' = \{n \in A: [q_n] \cap V \neq \emptyset\} = \{n \in A: [q_n] \subseteq V\}$. It follows that $V = \bigcup_{n \in A'} [q_n] \cup \{0,+\infty,-\infty\} = V(A')$. Since V is closed under $+$, A' is good. It is also clear that $B = B(A')_\alpha$. ∎

Let us recall the definition of isomorphism of STS's. Let $S_1 = (V_1, B_1)$ and $S_2 = (V_2, B_2)$ be STS's. $\phi: V_1 \to V_2$ is an **isomorphism** from S_1 onto S_2 if ϕ is a bijection and $\{a,b,c\} \in B_1$ *iff* $\{\phi(a),\phi(b),\phi(c)\} \in B_2$ for any $\{a,b,c\} \in [V_1]^3$. $\phi: V_1 \to V_2$ is an **embedding** of S_1 and S_2 if ϕ is $1-1$ and $\{a,b,c\} \in B_1$ implies that $\{\phi(a),\phi(b),\phi(c)\} \in B_2$ for any $\{a,b,c\} \in [V_1]^3$.

Lemma 8. *Let* $A_1, A_2 \subseteq \mathbb{N}$ *be good and let* $\alpha \neq \beta < 2^\omega$. *Then* $S(A_1)_\alpha$ *and* $S(A_2)_\beta$ *are not isomorphic.*

Proof: By the way of contradiction let us assume that $\phi: V(A_1) \to V(A_2)$ is an isomorphism from $S(A_1)_\alpha$ onto $S(A_2)_\beta$.

First, note that $\phi(0) = 0$, $\phi(\pm\infty) = \pm\infty$.

$\{0,+\infty,-\infty\}$ is the only block in $S(A_1)_\alpha$ that has an element in common with every quadrilateral of $S(A_1)_\alpha$. The same is true for $\{0,+\infty,-\infty\}$ in $S(A_2)_\beta$, and so ϕ must map $\{0,+\infty,-\infty\}$ onto $\{0,+\infty,-\infty\}$.

Let us assume that ϕ moves every element of $\{0,+\infty,-\infty\}$: for example, $\phi(0) = +\infty$, $\phi(+\infty) = -\infty$, and $\phi(-\infty) = 0$. Choose some $x \in V(A_1) - \{0,+\infty,-\infty\}$ of even type so that x determines an old quadrilateral in $S(A_1)_\alpha$: $\{-x,x,0\}$, $\{-2x,2x,0\}$, $\{-x,2x,+\infty\}$, $\{x,-2x,+\infty\}$. Then ϕ will map this quadrilateral onto $\{\phi(-x),\phi(x),-\infty\}$, $\{\phi(-2x),\phi(2x),-\infty\}$, $\{\phi(-x),\phi(2x),+\infty\}$, $\{\phi(x),\phi(-2x),+\infty\}$, and such quadrilateral does not occur in $S(A_2)_\beta$ (by Lemma 5), a contradiction.

So let us assume that ϕ swaps 0 and some ∞: for example, $\phi(0) = +\infty$, $\phi(+\infty) = 0$, and $\phi(-\infty) = -\infty$. Choose some $x \in V(A_1) - \{0,+\infty,-\infty\}$ of even type so that $2x$ determines an old quadrilateral in $S(A_1)_\alpha$: $\{-2x, 2x, 0\}$, $\{-4x, 4x, 0\}, \{-2x, 4x, -\infty\}, \{2x, -4x, -\infty\}$. Then ϕ will map this quadrilateral onto $\{\phi(-2x), \phi(2x), +\infty\}$, $\{\phi(-4x), \phi(4x), +\infty\}$, $\{\phi(-2x), \phi(4x), -\infty\}$, $\{\phi(2x), \phi(-4x), -\infty\}$ and such quadrilateral does not occur in $S(A_2)_\beta$ (by Lemma 5), a contradiction.

It follows that $\phi(0) = 0$ and $\phi(\pm\infty) = \pm\infty$.

According to Lemma 5, ϕ must map every old or inverted quadrilateral from $S(A_1)_\alpha$ onto an old or inverted quadrilateral from $S(A_2)_\beta$ (as no new quadrilateral can have both 0 and ∞). Since only an inverted quadrilateral has no block in common with any other quadrilateral in the quadrilateral chain, it follows that ϕ must map an inverted quadrilateral of $S(A_1)_\alpha$ onto an inverted quadrilateral of $S(A_2)_\beta$ and, consequently, an old quadrilateral of $S(A_1)_\alpha$ onto an old quadrilateral of $S(A_2)_\beta$. Since ϕ must map two quadrilaterals that have a block in common onto two quadrilaterals that have a block in common, it must map a quadrilateral chain of $S(A_1)_\alpha$ onto a quadrilateral chain of $S(A_2)_\beta$. Since a quadrilateral chain of $S(A_1)_\alpha$ has a well-defined "centre" (it is the only inverted quadrilateral that has on the left-hand side a destroyed quadrilateral and an old quadrilateral and a destroyed quadrilateral, and the same on the right-hand side), ϕ must map a quadrilateral chain of $S(A_1)_\alpha$ onto a quadrilateral chain of $S(A_2)_\beta$ so that the "centre" is mapped onto the "centre". But since $f_{\alpha,n}$ is completely "distinct" from $f_{\beta,m}$ for any n, m and, consequently, every quadrilateral chain of $S(A_1)_\alpha$ is completely "distinct" from any quadrilateral chain of $S(A_2)_\beta$, the mapping is not possible. ∎

Lemma 9. *Let $A_1, A_2 \subseteq \mathrm{N}$ be good and let $\alpha \neq \beta < 2^\omega$. There is no non-trivial STS $S = (V, B)$ which can be embedded into both $S(A_1)_\alpha$ and $S(A_2)_\beta$.*

Proof: By the way of contradiction let us assume that we have a system $S = (V, B)$ and embeddings ϕ_1 and ϕ_2 of S into $S(A_1)_\alpha$ and $S(A_2)_\beta$, respectively. Denote by S_1 the image of S by ϕ_1, so S_1 is a subsystem of $S(A_1)_\alpha$ isomorphic to S. Denote by S_2 the image of S by ϕ_2, so S_2 is a subsystem of $S(A_2)_\beta$ isomorphic to S.

By Lemma 7, $S_1 = S(A'_1)_\alpha$ for some good $A'_1 \subseteq A_1$, and $S_2 = S(A'_2)_\beta$ for some good $A'_2 \subseteq A_2$. It follows that $S(A'_1)_\alpha$ is isomorphic to $S(A'_2)_\beta$, which contradicts Lemma 8. ∎

A countable STS $S = (V, B)$ is called **universal** if any finite or countable STS can be embedded into S. It is clear from Fraïssé's theorem (see [F], or for more on universal objects in general [KP]) that the class of countable STS's does not have a homogeneous universal element, as the class of finite partial STS's does not posses the amalgamation property. This, of course, does not rule out the existence of a universal STS. Nevertheless, the following theorem shows that no universal STS can exist.

Theorem 10. *There is no univeral STS.*

Proof: Let us first prove a little claim.

(1) *Let X be a countable set and let Y be an uncountable family of countable subsets of X. Then for every $n \in \mathbf{N}$ there are $A, B \in Y$ so that $|A \cap B| \geq n$.*

By the way of contradiction let us assume the opposite, that is, there are an uncountable family Y of countable subsets of X and an $n \in \mathbf{N}$ so that $|A \cap B| < n$ for every $A, B \in Y$. Let X' be the family of all finite subsets of X of cardinality at least n. For $a \in X'$ let $T_a = \{A \in Y : a \subseteq A\}$. Since $Y = \bigcup_{a \in X'} T_a$ and X' is countable, there must be an $a \in X'$ so that T_a is uncountable. Let $A, B \in T_a$. Then $|A \cap B| < n$, and also $a \subseteq A \cap B$ and, consequently, $|A \cap B| \geq |a| \geq n$, a contradiction. This completes the proof of the claim.

By the way of contradiction let us assume that $S = (V, B)$ is a universal STS. Choose any good $A \subseteq \mathbf{N}$. For every $\alpha < 2^{\omega}$, $S(A)_{\alpha}$ can be embedded into S. Let $S_{\alpha} = (V_{\alpha}, B_{\alpha})$ be a subsystem of S isomorphic to $S(A)_{\alpha}$.

Consider $V_{\alpha} \cap V_{\beta}$ for $\alpha \neq \beta < 2^{\omega}$. If $|V_{\alpha} \cap V_{\beta}| \geq 2$, then for every $x \neq y \in V_{\alpha} \cap V_{\beta}$ there is exactly one $z \in V$ so that $\{x, y, z\} \in B$, consequently, $\{x, y, z\} \in B_{\alpha} \cap B_{\beta}$, so $z \in V_{\alpha} \cap V_{\beta}$. It follows that the intersection of S_{α} and S_{β} is a STS. By Lemma 9 the intersection of S_{α} and S_{β} must be a trivial STS. Thus, V_{α} and V_{β} are either disjoint, or they share a single element, or three elements (a block). Then $\{V_{\alpha} : \alpha < 2^{\omega}\}$ is an uncountable family of countable subsets of a countable set V so that $|V_{\alpha} \cap V_{\beta}| < 4$ whenever $\alpha \neq \beta < 2^{\omega}$, which contradicts (1). ∎

For our next task to construct a family of size 2^{ω} of countable rigid mutually non-isomorphic STS's we shall modify the construction of $S(A)$. First we shall fix a good $A \subseteq \mathbf{N}$ for the rest of this paper. Let $a \in A$ be the least element of A. We shall construct $S(A)$ in the same way as before with one exception: when $x \in [q_a]$, we shall determine the sign of ∞ in a different way: type II block for $x \in [q_a]$ will be $\{x, -2x, \infty\}$, and the

$$\text{sign of } \infty = \begin{cases} +, & \text{if the type of } x \text{ is even and } x > 0; \\ +, & \text{if the type of } x \text{ is odd and } x < 0; \\ -, & \text{if the type of } x \text{ is even and } x < 0; \\ -, & \text{if the type of } x \text{ is odd and } x > 0. \end{cases}$$

This results in the fact that $[q_a]$ does not determine a quadrilateral chain in $S(A)$:

$$\begin{array}{llllll}
.. \left\{-\frac{x}{4}, \frac{x}{4}, 0\right\} & .. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, 0\} & .. \{-2x, 2x, 0\} & .. \{-4x, 4x, 0\} & .. \\
.. \left\{-\frac{x}{2}, \frac{x}{2}, 0\right\} & .. \{-x, x, 0\} & .. \{-2x, 2x, 0\} & .. \{-4x, 4x, 0\} & .. \{-8x, 8x, 0\} & .. \\
.. \left\{-\frac{x}{4}, \frac{x}{2}, +\infty\right\} & .. \left\{-\frac{x}{2}, \frac{x}{2}, -\infty\right\} & .. \{-x, 2x, +\infty\} & .. \{-2x, 4x, -\infty\} & .. \{-4x, 8x, +\infty\} & .. \\
.. \left\{\frac{x}{4}, -\frac{x}{2}, -\infty\right\} & .. \left\{\frac{x}{2}, -\frac{x}{2}, +\infty\right\} & .. \{x, -2x, -\infty\} & .. \{2x, -4x, +\infty\} & .. \{4x, -8x, -\infty\} & ..
\end{array}$$

$S(A)_{\alpha}$ is then produced from $S(A)$ in the same way as before by inverting some of the quadrilaterals of $S(A)$.

Lemma 11. *$S(A)_\alpha$ and $S(A)_\beta$ are not isomorphic whenever $\alpha \neq \beta < 2^\omega$.*

Proof: So similar to the proof of Lemma 8, that we are leaving it to the reader. ∎

Let S be a STS. An isomorphism from S onto S is called **automorphism.** The **trivial automorphism** is the identity function. S is called **rigid** if it admits only the trivial automorphism.

Lemma 12. *For every $\alpha < 2^\omega$, $S(A)_\alpha$ is rigid.*

Proof: Fix an $\alpha < 2^\omega$. Let ϕ be an automorphism of $S(A)_\alpha$. As in the proof of Lemma 8 one can show that $\phi(0) = 0$, $\phi(\pm\infty) = \pm\infty$, ϕ has to map a quadrilateral chain onto itself, and, consequently, each quadrilateral onto itself. It follows that $\phi(+\infty) = +\infty$, $\phi(-\infty) = -\infty$, and $\phi(x) = \begin{cases} -x \\ x \end{cases}$ for every $x \in V(A) - \{0,+\infty,-\infty\} - [q_\alpha]$.

(1) *Assume that for some $x \in V(A) - \{0,+\infty,-\infty\} - [q_\alpha]$, $\phi(x) = -x$. Then $\phi(x) = -x$ for every $x \in V(A) - \{0,+\infty,-\infty\}$.*

It is obvious that $\phi(y) = -y$ for every $y \in [x]$. Thus, $\phi(x) = -x$ and $\phi(2x) = -2x$. ϕ must map the block $\{x, 2x, -3x\}$ onto $\{-x, -2x, 3x\}$, and so $\phi(-3x) = 3x$ and $\phi(3x) = -3x$.

Let $y \in V(A) - \{0,+\infty,-\infty\} - [x] - [q_\alpha]$. By the way of contradiction let us assume that $\phi(y) = y$. It follows that $\phi(z) = z$ for every $z \in [y]$. ϕ must map the block $\{y, 2y, -3y\}$ onto $\{y, 2y, -3y\}$, and so $\phi(3y) = 3y$ and $\phi(-3y) = -3y$. If $-(x+y) \notin [q_\alpha]$ then ϕ must map the block $\{x, y, -(x+y)\}$ onto $\{-x, y, \pm(x+y)\}$. So either $-x + y = x + y$ (which gives $x = 0$, a contradiction), or $-x + y = -x - y$ (which gives $y = 0$, a contradiction). If $-(x+y) \in [q_\alpha]$, then by Lemma 1,$-(3x+3y) \notin [q_\alpha]$. ϕ must map the block $\{3x, 3y, -(3x+3y)\}$ onto $\{-3x, 3y, \pm(3x+3y)\}$. So either $-3x + 3y = 3x + 3y$ (which gives $x = 0$, a contradiction), or $-3x + 3y = -3x - 3y$ (which gives $y = 0$, a contradiction). Thus, $\phi(y) = -y$. We have shown that $\phi(y) = -y$ for every $y \in V(A) - \{0+\infty, -\infty\} - [q_\alpha]$.

Let $z \in [q_\alpha]$. By Lemma 1, $-\frac{3z}{2}, \frac{5z}{2} \notin \left[\frac{z}{2}\right] = [q_\alpha]$. ϕ must map the block $\{-\frac{3z}{2}, \frac{5z}{2}, z\}$ onto $\{\frac{3z}{2}, \frac{5z}{2} - z\}$, and so $\phi(z) = -z$.

(2) *Assume that for some $x \in V(A) - \{0,+\infty,-\infty\} - [q_\alpha]$, $\phi(x) = x$. Then $\phi(x) = x$ for every $x \in V(A) - \{0,+\infty,-\infty\}$.*

The proof is analogical to the proof of claim (1), so we are leaving it to the reader.

Now back to the proof of the lemma. We showed that for any $x \in V(A) - \{0,+\infty,-\infty\} - [q_\alpha]$, $\phi(x) = \pm x$. Let us assume that for some $x \in V(A) - \{0,+\infty,-\infty\} - [q_\alpha]$, $\phi(x) = -x$. By (1) $\phi(x) = -x$ for every $x \in V(A) - \{0,+\infty,-\infty\}$. Pick $x \in [q_\alpha]$. Assume that the type of x is even and that $x > 0$. Then $\{-x, 2x, +\infty\}$ and $\{x, -2x, -\infty\}$ are blocks. ϕ will map them onto

$\{x, -2x, +\infty\}$ and $\{-x, 2x, -\infty\}$, respectively, and such blocks do not exist, a contradiction. Thus, $\phi(x) = x$ for every $x \in V(A) - \{0, +\infty, -\infty\}$ by (2). Hence, $\phi(x) = x$ for every $x \in V(A)$. ∎

Theorem 13. *There is a family of size 2^ω of countable rigid mutually non-isomorphic STS.*

Proof: The family $\{S(A)_\alpha : \alpha < 2^\omega\}$ is such. From Lemma 11 it follows that the members of the family are mutually non-isomorphic. From Lemma 12 it follows that each is a rigid STS. ∎

A few notes at the conclusion of the paper:

The only reason that q_a for the least $a \in A$ does not determine a quadrilateral chain in our modified construction of $S(A)$ is to supress the possibility of an automorphism ϕ such that $\phi(x) = -x$ for all $x \in V(A)$, which, in turn, makes $S(A)_\alpha$ rigid.

The whole proof of Theorem 10 goes through with the modified construction of $S(A)$ (that is, when $[q_a]$ for the least element $a \in A$ does not determine a quadrilateral chain). Thus, there is a family $\mathcal{H}$ of size 2^ω of pairwise non-isomorphic countable rigid Steiner triple systems such that no finite Steiner triple system can be embedded into any system from $\mathcal{H}$, and no countable Steiner triple system can be embedded into any two systems from $\mathcal{H}$.

By a careful selection of which $[q_n]$'s determine quadrilateral chains and which do not, the number of automorphisms of $S(A)$ can be controlled. It is possible for $S(A)$ not to have any quadrilaterals, that is, to be a so-called **anti-Pasch** system.

References

[F] R. Fraïssé, *Sur l' extension aux relations de quelques propriétés des orders*, Ann. Sci. École Norm. Sup. 71 (1954), 361–388.

[HS] Z. Hedrín, J. Sichler, *Any Boundable Binding Category Contains 4 Proper Class of Mutually Disjoint Copies of Itself*, Algebra Universalis **1** (1971), 97–103.

[KP] P. Komjáth, J. Pach, *Universal elements and the complexity of certain classes of infinite graphs*, Disc. Math. **95** (1991), 255–270.

[LR] C.C. Lindner, A. Rosa (editors), *Topics on Steiner systems*, Ann. Discrete Math. 7 (1980).

[N] J. Nešetřil, *On infinite precise objects*, Mathematica Slovaca **28** (1978), 253–260.

[PS] D. Pigozzi, J. Sichler, *Homomorphisms of Partial and of Complete Steiner Triple Systems and Quasigroups*, in "Lecture Notes in Mathematics", Springer, Berlin - New York, 1985, pp. 224–237.

[Si] W. Sierpiński, *Sur un problème de triádes*, C. R. Soc. Sci. Varsovie 33–38 (1946), 13–16.

[So] B. Sobocinski, *A theorem of Sierpinski on triads and the axiom of choice*, Notre Dame J. Formal Logic **5** (1964), 51–58.
[Vu] V. Vucković, *Note on a theorem of W. Sierpiński*, Notre Dame J. Formal Logic **6** (1985), 180–182.
[GGP] M.J. Grannell, T.S. Griggs, J.S. Phelan, *Countably infinite Steiner triple systems*, Ars Combinatoria **24 B** (1987), 189–216.