# The 3-Packing Number of Complete Grid Graphs

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The $k$-packing number of an $m \times n$ checkerboard, $P_k(P_{m,n})$, is the maximum number of checkers that can be placed on an $m \times n$ board so at least $k$ squares separate each pair of checkers (Hare and Hare [**2**]). This is also called the $k$-packing number of an $m \times n$ complete grid graph. It is easy to show the 1-packing number of an $m \times n$ board is

$$P_1(P_{m,n}) = \lceil mn/2 \rceil . \tag{1}$$

Let $a \circ b$ be the number $l$ such that $l = a \pmod b$ with $0 \le l < b$. Fisher [**1**] showed the 2-packing number of an $m \times n$ board when $m \le n$ is:

$$P_2(P_{m,n}) = \begin{cases} \lceil (m+1)n/6 \rceil & \text{if } m \in \{1,2,3\} \\ \lceil 6n/7 \rceil & \text{if } m = 4 \text{ and } n \circ 7 \ne 1 \\ \lceil 6n/7 \rceil + 1 & \text{if } m = 4 \text{ and } n \circ 7 = 1 \\ 10 & \text{if } (m,n) = (7,7) \\ \lceil (mn+2)/5 \rceil & \text{if } m \in \{5,6,7\} \text{ and } (m,n) \ne (7,7) \\ 17 & \text{if } (m,n) = (8,10) \\ \lceil mn/5 \rceil & \text{if } m \ge 8 \text{ and } (m,n) \ne (8,10) \end{cases} \tag{2}$$

The paper examines the 3-packing number of an $m \times n$ board (see Figure 1). We find $P_3(P_{m,n})$ when $m \le 18$. We show that for $m \le n$ with $m \le 18$,

$$P_3(P_{m,n}) = \begin{cases} \left.\begin{cases} \left\lceil \frac{m}{8} \left\lceil \frac{(m+2)n}{m+1} \right\rceil \right\rceil & \text{if } n \circ (m+1) \text{ is odd} \\ \left\lceil \frac{m+2}{8} \left\lceil \frac{mn}{m+1} \right\rceil \right\rceil & \text{if } n \circ (m+1) \text{ is even} \end{cases} \right\} & \text{if } m \text{ is even} \\ \left.\begin{cases} \left\lceil \frac{m+1}{4} \left\lceil \frac{n}{2} \right\rceil \right\rceil + 1 & \begin{array}{l}\text{if } (m,n) = (11,16), (11,18), (13,20), \\ (15,20), (15,22), (15,24), (15,26), \\ (15,38), (15,40), (17,24), (17,26), \\ (17,28) \text{ or } (17,44)\end{array} \\ \left\lceil \frac{m+1}{4} \left\lceil \frac{n}{2} \right\rceil \right\rceil & \text{otherwise} \end{cases} \right\} & \text{if } m \text{ is odd.} \end{cases} \tag{3}$$

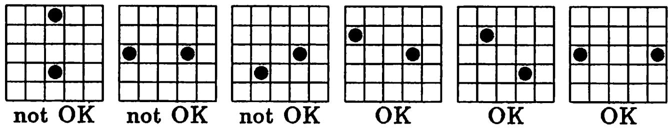


**Figure 1 – Legal 3-Packings.** Checkers in a 3-packing must be separated by at least three squares (horizontally or vertically).

Equations (2) and (3) qualitatively differ in two ways. First, the 3-packing number depends on the parity of $m$ and $n$. Second, there seems to be a growing (as $m$ grows) list of exceptions to the general rule for 3-packing. These exceptions prevented us from finding $P_3(P_{m,n})$ for all $m$ and $n$.

# 1 Preliminary Calculations

This section gives three results needed to prove (3).

**Lemma 1.** *For all $j$, $m$ and $n$ with $0 < j < n$ and $m > 0$, $P_3(P_{m,n}) \leq P_3(P_{m,n-j}) + P_3(P_{m,j})$.*

**Proof.** A maximal 3-packing of $P_{m,n}$ has at most $P_3(P_{m,n-j})$ checkers in the first $n-j$ columns and at most $P_3(P_{m,j})$ checkers in the last $j$ columns. □

**Lemma 2.** *For all $m > 0$ and $n > 0$,*

$$P_3(P_{m,n}) \geq \begin{cases} \left\{ \begin{array}{ll} \left\lceil \frac{m}{8} \left\lceil \frac{(m+2)n}{m+1} \right\rceil \right\rceil & \textit{if } n \circ (m+1) \textit{ is odd} \\ \left\lceil \frac{m+2}{8} \left\lceil \frac{mn}{m+1} \right\rceil \right\rceil & \textit{if } n \circ (m+1) \textit{ is even} \end{array} \right\} & \textit{if } m \textit{ is even} \\ \left\lceil \frac{m+1}{4} \left\lceil \frac{n}{2} \right\rceil \right\rceil & \textit{if } m \textit{ is odd.} \end{cases}$$

**Proof.** Since $P_3(P_{m,n})$ is the maximum number of checkers in a 3-packing, any 3-packing of$P_{m,n}$ is a lower bound for $P_3(P_{m,n})$. Figure 2 proves the result when $m$ is odd. Figure 3 proves the result when $m$ is even. □

We also needed $P_3(P_{m,n})$ for a number of small cases. Proving these "by hand" would have been very tedious. Instead, a computer program was used that found $P_3(P_{m,n})$ with a branch and bound algorithm which bounded with previous values of $P_3(P_{m,n})$. Lemma 3 summarizes these results.

**Lemma 3:** *Equation (3) holds when $m \leq 18$ and $n \leq 24$.*

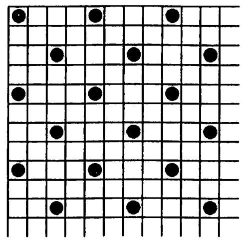

**Figure 2 – A 3-packing.** On $P_{m,n}$, this 3-packing has $\lceil m/2 \rceil \lceil n/2 \rceil /2$ checkers.

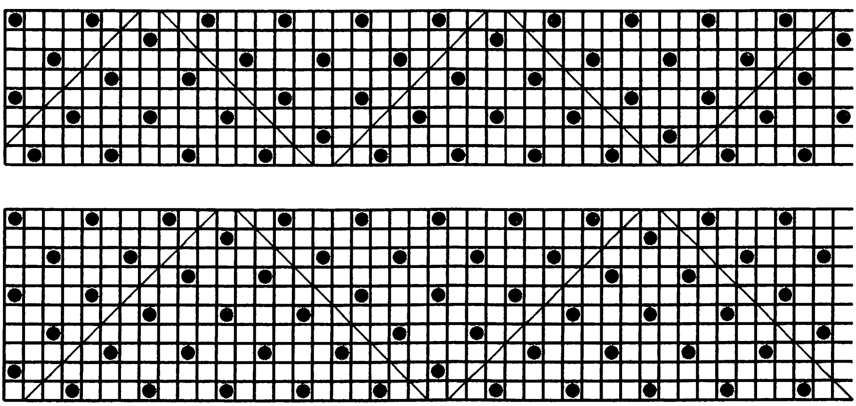

**Figure 3 – 3-packings of $P_{m,n}$ for Even $m$.** This shows 3-packings of $P_{m,n}$ when $m$ is even (only shown when $m$ is 8 and 10) which has more checkers than the 3-packing in Figure 2. It has $\left\lceil \frac{m}{8} \left\lceil \frac{(m+2)n}{m+1} \right\rceil \right\rceil$ checkers if $n \circ (m+1)$ is odd, and $\left\lceil \frac{m+2}{8} \left\lceil \frac{mn}{m+1} \right\rceil \right\rceil$ checkers if $n \circ (m+1)$ is even. The diagonal "fault lines" show boundaries between 3-packings like those in Figure 2.

# 2 Verification of (3)

For clarity, the proof of (3) is given as 18 theorems (one each for $m = 1, 2, 3, \ldots, 18$). When $m \leq 10$, the proofs are straightforward. When $m > 10$, there are "exceptions" where there is a 3-packing with more checkers than the 3-packings of Figure 2 or Figure 3.

**Theorem 1.** *For all $n > 0$, $P_3(P_{1,n}) = \lceil n/4 \rceil$.*

**Proof.** Lemma 3 gives the results for $n < 5$. For $n \geq 5$, assume it holds for $n - 4$. Then $P_3(P_{1,n}) \leq P_3(P_{1,4}) + P_3(P_{1,n-4}) = 1 + \lceil (n-4)/4 \rceil = \lceil n/4 \rceil$. Lemma 2 shows $P_3(P_{1,n}) \geq \lceil n/4 \rceil$. The result follows by induction. □

**Theorem 2.** *For all* $n > 0$, $P_3(P_{2,n}) = \lceil n/3 \rceil$.

**Proof.** Lemma 3 gives the results for $n < 4$. For $n \geq 4$, assume it holds for $n-3$. Then $P_3(P_{2,n}) \leq P_3(P_{2,3}) + P_3(P_{2,n-3}) = 1 + \lceil (n-3)/3 \rceil = \lceil n/3 \rceil$. Lemma 2 shows $P_3(P_{2,n}) \geq \lceil n/3 \rceil$. The result follows by induction. □

**Theorem 3.** *For all* $n > 0$, $P_3(P_{3,n}) = \lceil n/2 \rceil$.

**Proof.** Lemma 3 gives the results for $n < 3$. For $n \geq 3$, assume it holds for $n-2$. Then $P_3(P_{3,n}) \leq P_3(P_{3,2}) + P_3(P_{3,n-2}) = \lceil (n-2)/2 \rceil + 1 = \lceil n/2 \rceil$. Lemma 2 shows $P_3(P_{3,n}) \geq \lceil n/2 \rceil$. The result follows by induction. □

**Theorem 4.** *For all* $n > 0$, $P_3(P_{4,n}) = \lceil 3n/5 \rceil$.

**Proof.** Lemma 3 gives the results for $n < 6$. For $n \geq 6$, assume it holds for $n-5$. Then $P_3(P_{4,n}) \leq P_3(P_{4,5}) + P_3(P_{4,n-5}) = 3 + \lceil 3(n-5)/5 \rceil = \lceil 3n/5 \rceil$. Lemma 2 shows $P_3(P_{3,n}) \geq \lceil 3n/5 \rceil$. The result follows by induction. □

**Theorem 5.** *For all* $n > 0$, $P_3(P_{5,n}) = \lceil 1.5 \lceil n/2 \rceil \rceil$.

**Proof.** Lemma 3 gives the results for $n < 5$. For $n \geq 5$, assume it holds for $n-4$. Then $P_3(P_{5,n}) \leq P_3(P_{5,4}) + P_3(P_{5,n-4}) = 3 + \lceil 1.5 \lceil (n-4)/2 \rceil \rceil = \lceil 1.5 \lceil n/2 \rceil \rceil$. Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 6.** *For all* $n > 0$, $P_3(P_{6,n}) = \begin{cases} \lceil 6n/7 \rceil + 1 & \textit{if } n \circ 7 = 1 \\ \lceil 6n/7 \rceil & \textit{otherwise.} \end{cases}$

**Proof.** Lemma 3 gives the results for $n < 8$. For $n \geq 8$, assume it holds for $n-7$. Then

$$P_3(P_{6,n}) \leq P_3(P_{6,7}) + P_3(P_{6,n-7}) = 6 + \begin{cases} \lceil 6(n-7)/7 \rceil + 1 & \text{if } n \circ 7 = 1 \\ \lceil 6(n-7)/7 \rceil & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lceil 6n/7 \rceil + 1 & \text{if } n \circ 7 = 1 \\ \lceil 6n/7 \rceil & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 7.** *For all* $n > 0$, $P_3(P_{7,n}) = \begin{cases} n+1 & \textit{if } n = 2, 4, \textit{ or } n \textit{ is odd} \\ n & \textit{otherwise.} \end{cases}$

**Proof.** Lemma 3 gives the results for $n < 11$. For $n \geq 11$, assume it holds for $n-6$. Then

$$P_3(P_{7,n}) \leq P_3(P_{7,6}) + P_3(P_{7,n-6}) = 6 + \begin{cases} n-6+1 & \text{if } n \text{ is odd} \\ n-6 & \text{otherwise} \end{cases}$$

$$= \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 8.** *For all* $n > 0$, $P_3(P_{8,n}) = \begin{cases} \lceil 10n/9 \rceil + 1 & \textit{if } n \circ 9 = 6, 8 \\ \lceil 10n/9 \rceil & \textit{otherwise.} \end{cases}$

**Proof.** Lemma 3 gives the results for $n < 10$. For $n \geq 10$, assume it holds for $n - 9$. Then

$$\begin{aligned} P_3(P_{8,n}) &\leq P_3(P_{8,9}) + P_3(P_{8,n-9}) \\ &= 10 + \begin{cases} \lceil 10(n-9)/9 \rceil + 1 & \text{if } n \circ 9 = 6, 8 \\ \lceil 10(n-9)/9 \rceil & \text{otherwise} \end{cases} \\ &= \begin{cases} \lceil 10n/9 \rceil + 1 & \text{if } n \circ 9 = 6, 8 \\ \lceil 10n/9 \rceil & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 9.** *For all* $n > 0$, $P_3(P_{9,n}) = \begin{cases} 6 & \textit{if } n = 4 \\ \lceil 2.5 \lceil n/2 \rceil \rceil & \textit{otherwise.} \end{cases}$

**Proof.** Lemma 3 gives the results for $n < 13$. For $n \geq 13$, assume it holds for $n - 8$. Then $P_3(P_{9,n}) \leq P_3(P_{9,8}) + P_3(P_{9,n-8}) = 10 + \lceil 2.5 \lceil (n-8)/2 \rceil \rceil = \lceil 2.5 \lceil n/2 \rceil \rceil$. Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 10.** *For all* $n > 0$,

$$P_3(P_{10,n}) = \begin{cases} \lceil 15n/11 \rceil + 1 & \textit{if } n = 2 \textit{ or } n \circ 11 = 1, 8, 10 \\ \lceil 15n/11 \rceil & \textit{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n < 14$. For $n \geq 14$, assume it holds for $n - 11$. Then

$$\begin{aligned} P_3(P_{10,n}) &\leq P_3(P_{10,11}) + P_3(P_{10,n-11}) \\ &= 15 + \begin{cases} \lceil 15(n-11)/11 \rceil + 1 & \text{if } n \circ 11 = 1, 8, 10 \\ \lceil 15(n-11)/11 \rceil & \text{otherwise.} \end{cases} \\ &= \begin{cases} \lceil 15n/11 \rceil + 1 & \text{if } n \circ 11 = 1, 8, 10 \\ \lceil 15n/11 \rceil & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 11.** *For all* $n > 0$,

$$P_3(P_{11,n}) = \begin{cases} 3 \lceil n/2 \rceil + 1 & \textit{if } n = 2, 4, 6, 8, 16, 18 \\ 3 \lceil n/2 \rceil & \textit{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \leq 24$ (a maximal 3-packing for $n = 16$ and $n = 18$ is given by Figure 4). For $n = 26$ and $n = 28$, $P_3(P_{11,26}) \leq P_3(P_{11,12}) + P_3(P_{11,14}) = 18 + 21 = 39$ and $P_3(P_{11,28}) \leq$

$2P_3(P_{11,14}) = 2 \cdot 21 = 42$. Lemma 2 shows these are equalities. For $n > 24$ with $n \neq 26, 28$, assume the result holds for $n - 10$. Then $P_3(P_{11,n}) \leq P_3(P_{11,10}) + P_3(P_{11,n-10}) = 15 + 3\lceil (n-10)/2 \rceil = 3\lceil n/2 \rceil$. Lemma 2 shows $P_3(P_{11,n}) \geq 3\lceil n/2 \rceil$. The results follow by induction. □

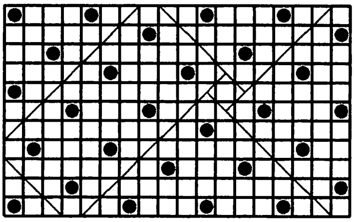

**Figure 4 – Maximal 3-packings of 11 × 16 and 11 × 18 Boards.** Above is a maximal 3-packing of an 11 × 18 board. The first 16 columns is a maximal 3-packing of an 11 × 16 board.

**Theorem 12.** *For all $n > 0$,*

$$P_3(P_{12,n}) = \begin{cases} \lceil 21n/13 \rceil + 1 & \text{if } n = 4 \text{ or } n \circ 13 = 1,3,6,8,10,12 \\ \lceil 21n/13 \rceil & \text{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n < 18$. For $n \geq 18$, assume it holds for $n - 13$. Then

$$P_3(P_{12,n}) \leq P_3(P_{12,13}) + P_3(P_{12,n-13})$$
$$= 21 + \begin{cases} \lceil 21(n-13)/13 \rceil + 1 & \text{if } n \circ 13 = 1,3,6,8,10,12 \\ \lceil 21(n-13)/13 \rceil & \text{otherwise} \end{cases}$$
$$= \begin{cases} \lceil 21n/13 \rceil + 1 & \text{if } n \circ 13 = 1,3,6,8,10,12 \\ \lceil 21n/13 \rceil & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 13.** *For all $n > 0$,*

$$P_3(P_{13,n}) = \begin{cases} \lceil 3.5 \lceil n/2 \rceil \rceil + 1 & \text{if } n = 2,4,6,8,20 \\ \lceil 3.5 \lceil n/2 \rceil \rceil & \text{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \leq 24$. Figure 5 give a maximal 3-packing for $n = 20$. For $n = 32$, $P_3(P_{13,32}) \leq 2P_3(P_{13,16}) = 2 \cdot 28 = 56$ and Lemma 2 gives $P_3(P_{13,32}) \geq 56$. For $n > 24$ with $n \neq 32$, assume the result holds for $n - 12$. Then

$$P_3(P_{13,n}) \leq P_3(P_{13,12}) + P_3(P_{13,n-12})$$
$$= 21 + \lceil 3.5 \lceil (n-12)/2 \rceil \rceil = \lceil 3.5 \lceil n/2 \rceil \rceil .$$

Lemma 2 shows this is equality. The result follows by induction. □

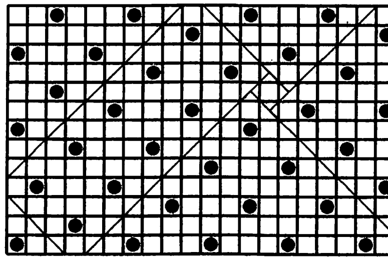

**Figure 5 – Maximal 3-packing of a 13 × 20 Board.**

**Theorem 14.** *For all* $n > 0$,

$$P_3(P_{14,n}) = \begin{cases} 2n+1 & \textit{if } n = 2,4 \\ \lceil 1.75 \lceil \frac{16n}{15} \rceil \rceil & \textit{if } n \circ 15 \textit{ is odd} \\ 2 \lceil \frac{14n}{15} \rceil & \textit{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n < 20$. For $n \geq 20$, assume it holds for $n - 15$. Then

$$P_3(P_{14,n}) \leq P_3(P_{14,15}) + P_3(P_{14,n-15})$$

$$= 28 + \begin{cases} \lceil 1.75 \lceil 16(n-15)/15 \rceil \rceil & \text{if } n \circ 15 \text{ is odd} \\ 2 \lceil 14(n-15)/15 \rceil & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lceil 1.75 \lceil 16n/15 \rceil \rceil & \text{if } n \circ 15 \text{ is odd} \\ 2 \lceil 14n/15 \rceil & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is an equality. The result follows by induction. □

**Theorem 15.** *For all* $n > 0$,

$$P_3(P_{15,n}) = \begin{cases} 2n+2 & \textit{if } n = 6,8 \\ 2n+1 & \textit{if } n = 2,4,10,12,20,22,24,26,38,40 \\ 4\lceil n/2 \rceil & \textit{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \leq 24$. We also have $P_3(P_{15,26}) \leq P_3(P_{15,10}) + P_3(P_{15,16}) = 21 + 32 = 53$, $P_3(P_{15,34}) \leq P_3(P_{15,16}) + P_3(P_{15,18}) = 32 + 36 = 68$, $P_3(P_{15,36}) \leq 2P_3(P_{15,18}) = 2 \cdot 36 = 72$, $P_3(P_{15,38}) \leq P_3(P_{15,16}) + P_3(P_{15,22}) = 32 + 45 = 77$, $P_3(P_{15,40}) \leq P_3(P_{15,16}) + P_3(P_{15,24}) = 32 + 49 = 81$, $P_3(P_{15,52}) \leq P_3(P_{15,16}) + 2P_3(P_{15,18}) = 32 + 2 \cdot 36 = 104$, and $P_3(P_{15,54}) \leq 3P_3(P_{15,18}) = 3 \cdot 36 = 108$. Lemma 2 or Figure 6 show these inequalities are equalities. For $n > 24$ with $n \notin \{26, 34, 36, 38, 40, 52, 54\}$, assume the result holds for $n - 14$. Then $P_3(P_{15,n}) \leq P_3(P_{15,14}) + P_3(P_{15,n-14}) = 28 + 4\lceil (n-14)/2 \rceil = 4\lceil n/2 \rceil$. Lemma 2 shows this is an equality. The result follows by induction. □

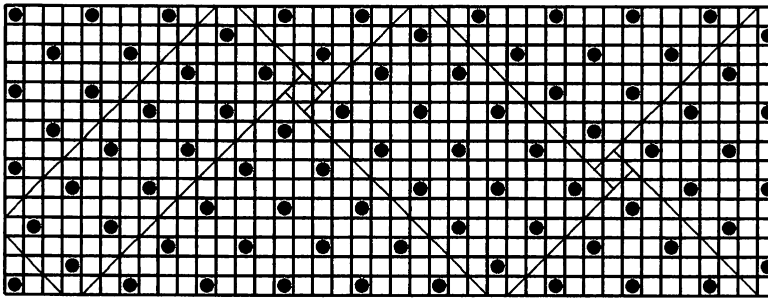

**Figure 6 – Maximal 3-packings of 15 × 20, 15 × 22, 15 × 24, 15 × 26, 15 × 38 and 15 × 40 Boards.** The first $n$ columns are a maximal 3-packing of a $15 \times n$ board where $n = 20, 22, 24, 26, 38$ and 40.

**Theorem 16.** *For all $n > 0$,*

$$P_3(P_{16,n}) = \begin{cases} 2\lceil 18n/17 \rceil & \text{if } n \circ 17 \text{ is odd} \\ \lceil 2.25 \lceil 16n/17 \rceil \rceil & \text{otherwise} \end{cases} + \begin{cases} 1 & \text{if } n = 2, 4, 11 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \le 24$. For $n = 28$, $P_3(P_{16,28}) \le P_3(P_{16,13}) + P_3(P_{16,15}) = 28 + 32 = 60$. Lemma 2 shows $P_3(P_{16,28}) \ge 60$. For $n > 24$ with $n \ne 28$, assume the result holds for $n - 17$. Then

$$P_3(P_{16,n}) \le P_3(P_{16,17}) + P_3(P_{16,n-17})$$

$$= 36 + \begin{cases} 2\lceil 18(n-17)/17 \rceil & \text{if } n \circ 17 \text{ is odd} \\ \lceil 2.25 \lceil 16(n-17)/17 \rceil \rceil & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2\lceil 18n/17 \rceil & \text{if } n \circ 17 \text{ is odd} \\ \lceil 2.25 \lceil 16n/17 \rceil \rceil & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is equality. The result follows by induction. □

**Theorem 17.** *For all $n > 0$,*

$$P_3(P_{17,n}) = \begin{cases} 2.25n + 2 & \text{if } n = 4, 8 \\ \lceil 2.25n \rceil + 1 & \text{if } n = 2, 6, 10, 12, 24, 28, 44 \\ \lceil 4.5 \lceil n/2 \rceil \rceil & \text{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \le 24$. We also have $P_3(P_{17,26}) \le P_3(P_{17,6}) + P_3(P_{17,20}) = 15 + 45 = 60$, $P_3(P_{17,28}) \le P_3(P_{17,12}) + P_3(P_{17,16}) = 28 + 36 = 64$, $P_3(P_{17,40}) \le 2P_3(P_{17,20}) = 2 \cdot 45 = 90$, $P_3(P_{17,42}) \le P_3(P_{17,20}) + P_3(P_{17,22}) = 45 + 50 = 95$, $P_3(P_{17,44}) \le P_3(P_{17,20}) + P_3(P_{17,24}) = 45 + 55 = 100$, and $P_3(P_{17,60}) \le 3P_3(P_{17,20}) = 3 \cdot 45 = 135$.

Lemma 2 or Figure 7 show these inequalities are equalities. For $n > 24$ with $n \notin \{26, 28, 40, 42, 44, 60\}$, assume the result holds for $n - 16$. Then

$$P_3(P_{17,n}) \leq P_3(P_{17,16}) + P_3(P_{17,n-16})$$

$$= 36 + \lceil 4.5 \lceil (n-16)/2 \rceil \rceil = \lceil 4.5 \lceil n/2 \rceil \rceil .$$

Lemma 2 shows this is equality. The result follows by induction. □

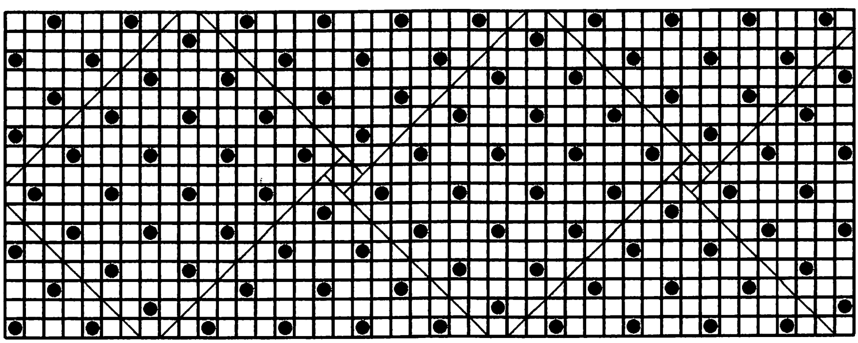

**Figure 7 – Maximal 3-packings of $17 \times 24$, $17 \times 26$, $17 \times 28$, and $17 \times 44$ Boards.** The first $n$ columns are a maximal 3-packing of a $17 \times n$ board where $n = 24$, 26, 28 and 44.

**Theorem 18.** *For all $n > 0$,*

$$P_3(P_{18,n}) = \begin{cases} \lceil 2.25 \lceil 20n/19 \rceil \rceil & \textit{if } n \circ 19 \textit{ is odd} \\ \lceil 2.5 \lceil 18n/19 \rceil \rceil & \textit{otherwise} \end{cases} + \begin{cases} 1 & \textit{if } n = 2, 4, 6, 11 \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof.** Lemma 3 gives the results for $n \geq 24$. For $n = 30$, $P_3(P_{18,30}) \leq 2P_3(P_{18,15}) = 2 \cdot 36 = 72$. Lemma 2 shows $P_3(P_{18,30}) = 72$. For $n > 24$ with $n \neq 30$, assume the result holds for $n - 19$. Then

$$P_3(P_{18,n}) \leq P_3(P_{18,19}) + P_3(P_{18,n-19})$$

$$= 45 + \begin{cases} \lceil 22.5 \lceil 20(n-19)/19 \rceil \rceil & \text{if } n \circ 19 \text{ is odd} \\ \lceil 2.5 \lceil 18(n-19)/19 \rceil \rceil & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lceil 2.25 \lceil 20n/19 \rceil \rceil & \text{if } n \circ 19 \text{ is odd} \\ \lceil 2.5 \lceil 18n/19 \rceil \rceil & \text{otherwise.} \end{cases}$$

Lemma 2 shows this is equality. The result follows by induction. □

## 3 Remarks

- In a 3-packing, squares with area 8 (rotated 45° from the checkerboard squares) can be centered on each checker so the squares do not overlap. It is then easy to show $P_3(P_{m,n}) \leq (m+1)(n+1)/8$. This together with the 3-packing of Figure 1 give $P_3(P_{m,n}) = (m+1)(n+1)/8$ when $m$ and $n$ are both odd.

- The program which prove Lemma 3 used a bound similar to Lemma 2 to eliminate unproductive branches. On $P_{m,n}$, let $r_{i,j}$ be the number of checkers used in a 3-packing of the first $i-1$ columns and the first $j$ squares of column $i$. Then

$$\begin{aligned} P_3(P_{m,n}) \quad \leq \quad & r_{i,j} + \min\big[P_3(P_{m-i,n}) + P_3(P_{1,n-j+1}), \\ & \min_{0 \leq k < j}(P_3(P_{m-i+1,n-k}) + P_3(P_{m-i,k}))\big]. \end{aligned}$$

- *Why did we stop at $m = 18$?* A Vax 8820 took 116 cpu-minutes to prove Lemma 3 (68% was for $m = 18$). The time needed seems to triple when $m$ is increased by one. Further, the number of exceptions seem to grow with $m$. This will force a higher upper limit on $n$ in Lemma 3 (higher than the 24), and would increase the complexity of the proofs. Still, a few more values of $m$ could probably be done without undue effort by the authors or the computer. However, this would probably have not qualitatively altered the paper, and we had to stop at some point.

- *How can the 3-packing number of an $m \times n$ board be found for all $m$ and $n$?* The 2-packing number of an $m \times n$ board also had irregularities that when $m \leq 8$ and $m \leq n$. However, for $m > 8$, a regular pattern emerged. Preliminary work by the first author shows the domination number of an $m \times n$ board seems to settle into a regular pattern for $m > 15$ when $m \leq n$. Hopefully, this also happens for 3-packing. If this is the case, then a useful approach might be to automate the approach developed in this paper and hope that a provable pattern appears before the computations become overwhelming.

### References


1. D.C. Fisher, *The 2-Packing Number of Complete Grid Graphs*, to appear in Ars Combinatoria.

2. E.O. Hare and W.R. Hare, *k-Packing of $P_m \times P_n$*, Congressus Numeratium (to appear).