A Note on the Compositions of a Positive Integer

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Abstract. In this paper, a composition result *viz.*, the number of r-compositions of n dominated by the r-compositions of m ($m \geq n$) subject to certain restrictions, has been derived by the method of induction.

1. Introduction.

Narayana (1955) has considered a generalized occupancy problem which can be viewed as a problem in compositions of integers. Narayana and Fulton (1958) considered the r-composition (or r-partition) of a positive integer n ($1 \leq r \leq n$) and discussed its various properties. Also, they discussed the relation of 'domination' defined on the r-compositions of n, which is reflexive, transitive, and antisymmetric. Thus, it represents a 'partial order' defined on the r-compositions of n. Narayana (1959) discussed the same domination principle and the partial order defined on the compositions of a positive integer and gave some of its applications in probability theory. Some definitions are quoted below from Narayana (1959).

Definition 1: $(t_1, t_2, \ldots, t_r)$ represents an r-composition of a positive integer n if, and only if,

$$\sum_{i=1}^{r} t_i = n \text{ and } t_i \geq 1, \quad i = 1, 2, \ldots, r.$$

We remark that, in general, we shall consider $(t_1, t_2, \ldots, t_r)$ and $(t_2, t_1, \ldots, t_r)$, where $t_1 + t_2 + \ldots + t_r = n$, as distinct r-compositions of n, unless $t_1 = t_2$. If r is an integer such that $1 \leq r \leq n$, we have, obviously, $\binom{n-1}{r-1}$ distinct r-compositions of n.

Definition 2: An r-composition $(t_1, t_2, \ldots, t_r)$ of n 'dominates' another r-composition $(t'_1, t'_2, \ldots, t'_r)$ of n if, and only if, the following conditions hold:

$$\begin{aligned} t_1 &\geq t'_1 \\ t_1 + t_2 &\geq t'_1 + t'_2 \\ t_1 + t_2 + t_3 &\geq t'_1 + t'_2 + t'_3 \\ &\vdots \\ t_1 + t_2 + \ldots + t_{r-1} &\geq t'_1 + t'_2 + \ldots + t'_{r-1} \text{ and} \\ t_1 + t_2 + \ldots + t_r &= t'_1 + t'_2 + \ldots + t'_r = n. \end{aligned} \tag{1}$$

Definition 3: An r-composition $(t_1, t_2, \dots, t_r)$ of m 'dominates' an r-composition $(t'_1, t'_2, \dots, t'_r)$ of n ($m > n$) if, and only if,

$$\sum_{i=1}^{j} t_i \geq \sum_{i=1}^{j} t'_i, \quad j = 1, 2, \dots, r-1. \tag{2}$$

Let us suppose we number the $\binom{n-1}{r-1}$ r-compositions of n, taken in some order, using the symbols $p_1, p_2, \dots, p_{\binom{n-1}{r-1}}$. Taking the composition p_i, let x_i be the number of compositions dominated by p_i in the set $p_1, p_2, \dots, p_{\binom{n-1}{r-1}}$; $i = 1, 2, \dots, \binom{n-1}{r-1}$. The total

$$(n; r) = x_1 + x_2 + \dots + x_{\binom{n-1}{r-1}}$$

obviously does not depend upon the particular ordering chosen for numbering the r-compositions of n and denotes the number of r-compositions of n dominated by the r-compositions of n.

Narayana (1959), on p. 93, gave a geometric representation of the r-compositions of n and proved in Lemma 1, on p. 92, that the number of r-compositions of n dominated by the r-compositions of n is given by

$$\begin{aligned} (n; r) &= \binom{n-1}{r-1}\binom{n}{r-1} - \binom{n}{r}\binom{n-1}{r-2} \\ &= \frac{1}{n}\binom{n}{r}\binom{n}{r-1}. \end{aligned} \tag{3}$$

According to the above mentioned geometric representation, an r-composition $(t'_1, t'_2, \dots, t'_r)$ of n dominated by another r-composition $(t_1, t_2, \dots, t_r)$ of n can be represented by a 'lattice path' from $(0, 0)$ to (n, n) not rising above the diagonal $y = x$ and having exactly r horizontal and r vertical components, by plotting the points $(0, 0), (t_1, 0), (t_1, t'_1), (t_1 + t_2, t'_1), (t_1 + t_2, t'_1 + t'_2), (t_1 + t_2 + t_3, t'_1 + t'_2), (t_1 + t_2 + t_3, t'_1 + t'_2 + t'_3), \dots, (t_1 + \dots + t_r, t'_1 + \dots + t'_{r-1})$ and $(t_1 + \dots + t_r, t'_1 + \dots + t'_r) \equiv (n, n)$ on an x-y plane and joining each one of them with the next one (see Figure 1). Clearly, both horizontal and vertical components represent an r-composition of n. Hence, $(n; r)$, as given in (3), is equivalent to the total number of lattice paths from $(0, 0)$ to (n, n) starting with a horizontal step and never crossing the line $y = x$, each path having exactly r horizontal and r vertical components.

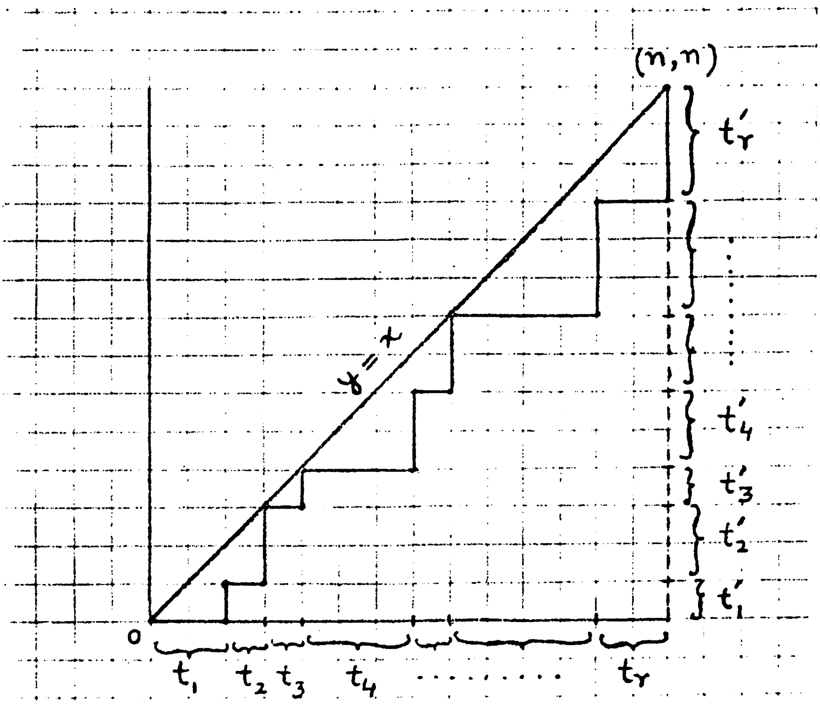

Figure 1
A lattice path from $(0,0)$ to $(n,n) \equiv (14,14)$ representing an $r(\equiv 7)$-composition $(t_1,\dots,t_r) \equiv (2,1,1,3,1,4,2)$ of $n \equiv 14$ dominating another $r(\equiv 7)$-composition $(t'_1,\dots,t'_r) \equiv (1,2,1,2,2,3,3)$ of $n \equiv 14$.

In this paper, we derive a formula by using the method of mathematical induction, for the number of r-compositions of n dominated by the r-compositions of n, subject to certain additional restrictions, which in turn becomes a proper subset of the set of elements in $(n; r)$. We also give a similar formula for the number of r-compositions of n dominated by the r-compositions of m $(m > n)$.

2. The composition result.

In what follows, we shall denote by $N_H(x, y; r, p; t)$, where $x \geq y - t$, the number of lattice paths from $(0,0)$ to (x, y) not crossing the line $y = x + t$, starting with a horizontal step, having exactly r horizontal and r vertical components and touching the line $y = x + t$ exactly p times.

We shall use in the sequel the following result on 'strict domination'. By 'strict domination' we mean that the $(r-1)$ inequalities in (1) are all strict inequalities.

Result on strict domination: The number of r-compositions of n 'strictly dominated' by the r-compositions of n is given by

$$\begin{aligned} N_H(n,n;r,1;0) &= \binom{n-2}{r-1}\binom{n-1}{r-1} - \binom{n-1}{r}\binom{n-2}{r-2} \\ &= \frac{1}{r}\binom{n-2}{r-1}\binom{n-1}{r-1}, \end{aligned} \tag{4}$$

which follows from (3) by replacing n by $n-1$. In other words, (4) is the number of lattice paths from $(0,0)$ to (n,n) lying entirely below the line $y = x$, never touching it in-between except at the end points, each path having exactly r horizontal and r vertical components.

A summation formula needed in the sequel is quoted from Feller (1968; Ch. II (12.8), p. 64):

$$\sum_{i=0}^{r} \binom{i+k-1}{i} = \binom{r+k}{k}, \tag{5}$$

where r and k are positive integers.

Theorem 1. *The number of r-compositions of n dominated by the r-compositions of n subject to the restriction that any $p-1$ relationships out of the first $r-1$ in (1) are equalities (so that the last relationship in (1) becomes the pth equality) and the rest are strict inequalities is given by*

$$\begin{aligned} N_H(n,n;r,p;0) &= \binom{n-1}{r-1}\binom{n-p}{r-p} - \binom{n}{r}\binom{n-p-1}{r-p-1} \\ &= \frac{p}{r}\binom{n-p-1}{r-p}\binom{n-1}{r-1}. \end{aligned} \tag{6}$$

Proof: For proving the theorem we make use of the method of induction on r and p. According to the geometric representation of Narayana (1959), the right-hand side of (6) is equivalent to the number of lattice paths from $(0,0)$ to (n,n) starting with a horizontal step, never rising above the line $y = x$, having exactly r horizontal and r vertical components and having exactly p contacts with $y = x$ including the last one at (n,n).

It is easy to see that

$$
\begin{aligned}
N_H(n,n;1,1;0) &= 1,\\
N_H(n,n;1,2;0) &= 0,\\
N_H(n,n;2,1;0) &= \sum_{y=1}^{1}\sum_{x=2}^{n-1} N_H(x,y;1,0;0)\\
&\quad + \sum_{y=2}^{n-2}\sum_{x=y+1}^{n-1} N_H(x,y;1,0;0), \text{ where } x > y\\
&= \sum_{y=1}^{1}\sum_{x=2}^{n-1} 1 + \sum_{y=2}^{n-2}\sum_{x=y+1}^{n-1} 1\\
&= (n-2) + \sum_{y=2}^{n-2}(n-y-1)\\
&= \binom{n-1}{2},
\end{aligned}
$$

$$
\begin{aligned}
N_H(n,n;2,2;0) &= \sum_{x=1}^{n-1} N_H(x,x;1,1;0)\\
&= \sum_{x=1}^{n-1} 1 = (n-1).
\end{aligned}
$$

Assuming that the theorem holds true for $r-1$ compositions and $p-1$ equalities, we have

$$
\begin{aligned}
&N_H(n,n;r,p;0)\\
&= \sum_{x=p-1}^{n-r+p-2} N_H(x,x;p-1,p-1;0)\cdot N_H(n-x,n-x;r-p+1,1;0)\\
&\quad + \sum_{q=p}^{r-2}\sum_{x=q}^{n-r+q-1} N_H(x,x;q,p-1;0)\cdot N_H(n-x,n-x;r-q,1;0)\\
&\quad + \sum_{x=r-1}^{n-1} N_H(x,x;r-1,p-1;0)\cdot N_H(n-x,n-x;1,1;0),
\end{aligned}
$$

as we break the requisite path at the point where it touches the line $y = x$ for the

$(p-1)$th time. Thus, by (4) and (6),

$$N_H(n,n;r,p;0)$$
$$= \sum_{x=p-1}^{n-r+p-2} \binom{x-1}{p-2} \frac{1}{r-p+1} \binom{n-x-2}{r-p} \binom{n-x-1}{r-p}$$
$$+ \sum_{q=p}^{r-2} \sum_{x=q+1}^{n-r+q-1} \frac{p-1}{q} \binom{x-p}{q-p+1} \binom{x-1}{q-1} \frac{1}{r-q} \binom{n-x-2}{r-q-1} \binom{n-x-1}{r-q-1}$$
$$+ \sum_{x=r}^{n-1} \frac{p-1}{r-1} \binom{x-p}{r-p} \binom{x-1}{r-2} \cdot 1$$

$$= \frac{1}{r-p+1} \binom{n-p-1}{r-p} \binom{n-p}{r-p}$$
$$+ \sum_{x=p}^{n-r+p-2} \binom{x-1}{p-2} \frac{1}{r-p+1} \binom{n-x-2}{r-p} \binom{n-x-1}{r-p}$$
$$+ \sum_{q=p}^{r-2} \sum_{x=q+1}^{n-r+q-1} \frac{p-1}{q} \binom{x-p}{q-p+1} \binom{x-1}{q-1} \frac{1}{r-q} \binom{n-x-2}{r-q-1} \binom{n-x-1}{r-q-1}$$
$$+ \frac{p-1}{r-1} \binom{n-p-1}{r-p} \binom{n-2}{r-2} + \sum_{x=r}^{n-2} \frac{p-1}{r-1} \binom{x-p}{r-p} \binom{x-1}{r-2}$$

$$= \frac{1}{r-p+1} \binom{n-p-1}{r-p} \binom{n-p}{r-p} + \frac{p-1}{r-1} \binom{n-p-1}{r-p} \binom{n-2}{r-2}$$
$$+ \sum_{q=p-1}^{r-1} \sum_{x=q+1}^{n-r+q-1} \frac{p-1}{q} \binom{x-p}{q-p+1} \binom{x-1}{q-1} \frac{1}{r-q} \binom{n-x-2}{r-q-1} \binom{n-x-1}{r-q-1}, \tag{7}$$

which on simplification leads to (6). The empirical equivalence of the expressions in (6) and (7) have been shown in the following table for different values of n, r and p.

Table I

	Values of n, r and p	Value of the R.H.S. of (6)	Value of the R.H.S. of (7)
(A)	$n=5, r=3, p=2$	8	8
(B)	$n=5, r=4, p=3$	3	3
(C)	$n=6, r=3, p=2$	20	20
(D)	$n=6, r=5, p=3$	3	3
(E)	$n=6, r=4, p=2$	15	15
(F)	$n=7, r=4, p=2$	60	60
(G)	$n=7, r=4, p=3$	45	45
(H)	$n=8, r=5, p=2$	140	140
(I)	$n=8, r=4, p=3$	105	105
(J)	$n=8, r=4, p=2$	175	175
(K)	$n=10, r=5, p=3$	1134	1134
(L)	$n=12, r=8, p=5$	4125	4125
(M)	$n=16, r=10, p=8$	84084	84084
(N)	$n=16, r=10, p=6$	378378	378378

∎

Deductions:

(i) Putting $p = 1$ in (6), it reduces to the result (4) of strict domination.

(ii) Summing (6) over p from 1 to r and using the summation formula in Feller (1968; Ch. II (12.16), p. 65), it verifies (3).

Theorem 2. *The number of r-compositions of n dominated by the r-compositions of m $(m > n)$ subject to the restriction that exactly p inequalities out of the $(r-1)$ in (2) are equalities and the rest are strict inequalities is given by*

$$N_H(m, n; r, p; 0) = \binom{m-p-2}{r-p-1}\binom{n-1}{r-1} - \binom{m-p-2}{r-p-2}\binom{n-1}{r}, m > n. \tag{8}$$

Proof: We again use the method of induction. It is easy to see that, for $m > n$,

$$\begin{aligned}
N_H(m, n; 1, 1; 0) &= 0, \\
N_H(m, n; 2, 1; 0) &= (n-1), \\
N_H(m, n; 2, 2; 0) &= 0, \\
N_H(m, n; 3, 2; 0) &= \sum_{x=2}^{n-1} N_H(x, x; 2, 2; 0) = \sum_{x=2}^{n-1}(x-1), \text{ by (6)}, \\
&= \binom{n-1}{2}, \\
N_H(m, n; 3, 3; 0) &= 0.
\end{aligned}$$

Assuming that the theorem holds true for $r-1$ and $p-1$, we have

$$
\begin{aligned}
&N_H(m,n;r,p;0)\\
&\quad=\sum_{x=r-2}^{n-2} N_H(x,x;r-2,p-1;0)\cdot N_H(m-x,n-x;2,1;0)\\
&\qquad+\sum_{y=r-2}^{r-2}\sum_{x=r-1}^{m-2} N_H(x,y;r-2,p-1;0)\cdot N_H(m-x,n-y;2,1;0)\\
&\qquad+\sum_{y=r-1}^{n-2}\sum_{x=y+1}^{m-2} N_H(x,y;r-2,p-1;0)\cdot N_H(m-x,n-y;2,1;0),
\end{aligned}
$$

where $x > y$. Now by (6) and (8), we have

$$
\begin{aligned}
&N_H(m,n;r,p;0)\\
&\quad=\sum_{x=r-2}^{n-2}\frac{p-1}{r-2}\binom{x-p}{r-p-1}\binom{x-1}{r-3}\cdot(n-x-1)\\
&\qquad+\sum_{y=r-2}^{r-2}\sum_{x=r-1}^{m-2}\left[\binom{x-p-1}{r-p-2}\binom{y-1}{r-3}-\binom{x-p-1}{r-p-3}\binom{y-1}{r-2}\right]\cdot(n-y-1)\\
&\qquad+\sum_{y=r-1}^{n-2}\sum_{x=y+1}^{m-2}\left[\binom{x-p-1}{r-p-2}\binom{y-1}{r-3}-\binom{x-p-1}{r-p-3}\binom{y-1}{r-2}\right]\cdot(n-y-1)\\
&\quad=I_1+I_2+I_3,
\end{aligned}
\tag{9}
$$

where

$$
I_1=\sum_{x=r-1}^{n-2}(n-x-1)\frac{p-1}{r-2}\binom{x-p}{r-p-1}\binom{x-1}{r-3},
$$

since $x = r-2$ term is zero,

$$
\begin{aligned}
I_2&=\sum_{x=r-1}^{m-2}\sum_{y=r-2}^{r-2}\left[\binom{x-p-1}{r-p-2}\binom{y-1}{r-3}-\binom{x-p-1}{r-p-3}\binom{y-1}{r-2}\right](n-y-1)\\
&=\sum_{x=r-1}^{m-2}\binom{x-p-1}{r-p-2}(n-r+1)\\
&=\binom{m-p-2}{r-p-1}(n-r+1),
\end{aligned}
$$

by (5), and

$$I_3 = \sum_{y=r-1}^{n-2} \sum_{x=y+1}^{m-2} \left[\binom{x-p-1}{r-p-2} \binom{y-1}{r-3} - \binom{x-p-1}{r-p-3} \binom{y-1}{r-2} \right] (n-y-1)$$

$$= \sum_{y=r-1}^{n-2} (n-y-1) \left[\binom{y-1}{r-3} \left\{ \binom{m-p-2}{r-p-1} - \binom{y-p}{r-p-1} \right\} \right.$$

$$\left. - \binom{y-1}{r-2} \left\{ \binom{m-p-2}{r-p-2} - \binom{y-p}{r-p-2} \right\} \right],$$

by (5). Further,

$$I_3 = \binom{m-p-2}{r-p-1} \sum_{y=r-1}^{n-2} (n-y-1) \binom{y-1}{r-3}$$

$$- \binom{m-p-2}{r-p-2} \sum_{y=r-1}^{n-2} (n-y-1) \binom{y-1}{r-2}$$

$$- \sum_{y=r-1}^{n-2} (n-y-1) \left[\binom{y-1}{r-3} \binom{y-p}{r-p-1} - \binom{y-1}{r-2} \binom{y-p}{r-p-2} \right],$$

where

$$\sum_{y=r-1}^{n-2} (n-y-1) \binom{y-1}{r-3} = (n-1) \sum_{y=r-1}^{n-2} \binom{y-1}{r-3} - (r-2) \sum_{y=r-1}^{n-2} \binom{y}{r-2}$$

$$= (n-1) \left[\binom{n-2}{r-2} - 1 \right] - (r-2) \left[\binom{n-1}{r-1} - 1 \right]$$

$$= (n-1) \binom{n-2}{r-2} - (r-2) \binom{n-1}{r-1} - (n-r+1)$$

$$= \binom{n-1}{r-1} - (n-r+1), \text{ and}$$

$$\sum_{y=r-1}^{n-2} (n-y-1) \binom{y-1}{r-2} = \binom{n-1}{r},$$

by (5). Thus,

$$I_3 = \binom{m-p-2}{r-p-1} \left[\binom{n-1}{r-1} - (n-r+1) \right] - \binom{m-p-2}{r-p-2} \binom{n-1}{r}$$

$$- \sum_{y=r-1}^{n-2} (n-y-1) \frac{p-1}{r-2} \binom{y-p}{r-p-1} \binom{y-1}{r-3}.$$

Upon substituting these expressions for I_1, I_2, I_3, in equation (9) and then simplifying, it leads to (8). ∎

Alternative Proof of Theorem 1: An alternative proof of Theorem 1 can now be given by using the result of Theorem 2 as follows. Assuming that Theorem 1 holds true for $r-1$ and $p-1$ and breaking the requisite path at the point, say (x, y), $x \geq y$, where it completes its $(r-1)$ components in both the directions, we have

$$\begin{aligned} N_H(n,n;r,p;0) &= \sum_{x=r-1}^{n-1} N_H(x,x;r-1,p-1;0) \\ &+ \sum_{y=r-1}^{r-1}\sum_{x=r}^{n-1} N_H(x,y;r-1,p-1;0) \\ &+ \sum_{y=r}^{n-2}\sum_{x=y+1}^{n-1} N_H(x,y;r-1,p-1;0), \end{aligned}$$

where $x > y$. Now on using (6) and (8), we have

$$\begin{aligned} N_H(n,n;r,p;0) &= \sum_{x=r-1}^{n-1} \frac{p-1}{r-1}\binom{x-p}{r-p}\binom{x-1}{r-2} \\ &+ \sum_{y=r-1}^{r-1}\sum_{x=r}^{n-1}\left[\binom{x-p-1}{r-p-1}\binom{y-1}{r-2} - \binom{x-p-1}{r-p-2}\binom{y-1}{r-1}\right] \\ &+ \sum_{y=r}^{n-2}\sum_{x=y+1}^{n-1}\left[\binom{x-p-1}{r-p-1}\binom{y-1}{r-2} - \binom{x-p-1}{r-p-2}\binom{y-1}{r-1}\right] \\ &= I_4 + I_5 + I_6, \end{aligned} \tag{10}$$

where

$$I_4 = \sum_{x=r}^{n-1} \frac{p-1}{r-1}\binom{x-p}{r-p}\binom{x-1}{r-2},$$

since $x = r-1$ term is zero,

$$I_5 = \sum_{x=r}^{n-1} \sum_{y=r-1}^{r-1} \left[\binom{x-p-1}{r-p-1} \binom{y-1}{r-2} - \binom{x-p-1}{r-p-2} \binom{y-1}{r-1} \right]$$
$$= \sum_{x=r}^{n-1} \binom{x-p-1}{r-p-1} = \binom{n-p-1}{r-p},$$

by (5), and

$$I_6 = \sum_{y=r}^{n-2} \left[\binom{y-1}{r-2} \left\{ \binom{n-p-1}{r-p} - \binom{y-p}{r-p} \right\} \right.$$
$$\left. - \binom{y-1}{r-1} \left\{ \binom{n-p-1}{r-p-1} - \binom{y-p}{r-p-1} \right\} \right]$$
$$= \binom{n-p-1}{r-p} \sum_{y=r}^{n-2} \binom{y-1}{r-2} - \binom{n-p-1}{r-p-1} \sum_{y=r}^{n-2} \binom{y-1}{r-1}$$
$$- \sum_{y=r}^{n-2} \left[\binom{y-1}{r-2} \binom{y-p}{r-p} - \binom{y-1}{r-1} \binom{y-p}{r-p-1} \right]$$
$$= \binom{n-p-1}{r-p} \left[\binom{n-2}{r-1} - 1 \right]$$
$$- \binom{n-p-1}{r-p-1} \binom{n-2}{r} - \sum_{y=r}^{n-2} \frac{p-1}{r-1} \binom{y-p}{r-p} \binom{y-1}{r-2},$$

by (5). Upon substituting these expressions for I_4, I_5, I_6, in (10) and then simplifying, we obtain

$$N_H(n, n; r, p; 0)$$
$$= \frac{p-1}{r-1} \binom{n-p-1}{r-p} \binom{n-2}{r-2}$$
$$+ \binom{n-p-1}{r-p} \binom{n-2}{r-1} - \binom{n-p-1}{r-p-1} \binom{n-2}{r}$$
$$= \binom{n-p-1}{r-p} \binom{n-1}{r-1} \left[\frac{p-1}{n-1} + \frac{n-r}{n-1} - \frac{(r-p)(n-r-1)}{r(n-1)} \right],$$

which leads to (6). This completes the alternative proof of Theorem 1. ∎

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